

A thermodynamic theory of isotropic elastic-plastic materials

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THE CLASSICAL theory of thermodynamics is unable to provide a sound thermodynamic foundation for elastic-plastic materials. In a recent paper by BREE and BEEVERS [1] a general non-equilibrium theory of thermodynamics is developed, based on a statement of the second law of thermodynamics which modifies the classical version used by CARATHÉODORY [2]. In order to emphasize some of the main features of [1], the present paper examines a constitutive theory for isotropic elastic-plastic materials at finite strain. The existence of the absolute temperature and an entropy function follows from the modified statement of the second law of thermodynamics. A global entropy inequality is also constructed. This inequality is a sufficient but not a necessary thermodynamic requirement. One of the most important consequences of the thermodynamic theory is that the plastic work rate need not always be positive. This is consistent with a large Bauschinger effect and the fact that the yield surface does not always enclose the origin in stress space.

Klasyczna termodynamika nie daje wystarczających podstaw do opisu sprężysto-plastycznych materiałów. W niedawno opublikowanej pracy BREE'a i BEEVERSA [1] zbudowana została teoria termodynamiczna stanów nierównowagi na podstawie drugiej zasady termodynamiki. Zasada ta jest zmodyfikowana w stosunku do klasycznej wersji zaproponowanej przez CARATHÉODORY [2]. Rozwijając koncepcję podaną w pracy [1], zbadano teorię konstytutywną dla sprężysto-plastycznego materiału przy skończonych odkształceniach. Podany jest dowód istnienia funkcji absolutnej temperatury i funkcji entropii wykorzystując zmodyfikowaną wersję drugiej zasady termodynamiki. Zbudowana jest również globalna nierówność entropii. Nierówność ta jest dostatecznym, ale nie koniecznym warunkiem termodynamicznym. Jednym z najważniejszych wniosków wynikających z obecnej teorii jest fakt, że moc plastyczna niekoniecznie musi być dodatnia. Spostrzeżenie to jest zgodne z efektem Bauschingera oraz zachowaniem się powierzchni płynięcia w przestrzeni naprężeń, nie obejmującej początku układu.

Классическая термодинамика не дает достаточных основ для описания упруго-пластических материалов. В недавно опубликованной работе, Бри и Биверс [1], построена термодинамическая теория неравновесных состояний, опираясь на второй закон термодинамики. Этот закон модифицирован по отношению к классическому варианту, предложенному Каратеодори [2]. Развивая концепцию приведенную в работе [1], исследуется определяющая теория для упруго-пластического материала при конечных деформациях. Дается доказательство существования функции абсолютной температуры и функции энтропии, используя модифицированный вариант второго закона термодинамики. Построено тоже глобальное неравенство энтропии. Это неравенство является достаточным, но не необходимым термодинамическим условием. Одним из самых важных выводов, вытекающих из настоящей теории, является факт, что пластическая мощность не обязательно должна быть положительной. Это замечание consistently с эффектом Баушингера и поведением поверхности течения в пространстве напряжений не охватывающей начала системы.

1. Introduction

IN THE PAST twenty years many non-equilibrium theories of thermodynamics, both local and global, have appeared but few authors have attempted a rigorous derivation of a non-equilibrium theory of thermodynamics from realistic physical assumptions. However, BREE and BEEVERS [1] have recently developed a non-equilibrium theory of thermodynamics

based on a statement of the second law of thermodynamics which is a modification of that used by CARATHÉODORY [2] in classical thermodynamics. There are several new ideas in [1] which warrant further exposure. In order, then, to highlight some of the most important features of [1], it is appropriate to consider a constitutive model for a material for which the classical theory of thermodynamics is inapplicable. It is well-known that classical thermodynamics cannot be applied rigorously to materials in which the internal dissipation effects do not vanish during quasi-static transitions. This point is carefully explained by BUCHDAHL [3]. It is stated in [1] that an example of such a material is one which is capable of experiencing elastic-plastic deformations. For this reason, then, the present paper will emphasize some of the main points in [1] by discussing a constitutive theory for isotropic elastic-plastic materials at finite strain.

Since history-dependent effects on the state variables have not yet been included in the thermodynamic theory, it is necessary to assume that the elastic properties of the material are not influenced by the plastic deformation. Consequently, it will be assumed in this paper that plastic deformations do not contribute to volume changes since this would manifest itself as a history-dependent effect on the elastic properties. In the near future it is hoped to develop a thermodynamic theory capable of dealing with the full effects of history-dependence. For the present it is fortunate that the above assumptions for plastic deformation are not unrealistic, particularly for materials such as metals. These assumptions imply that the elastic and heat conduction properties of the material are preserved when referenced to the local plastically deformed reference configuration. If the yield surface encloses the origin in stress space, then this local configuration is the stress-free reference configuration at the reference temperature. Otherwise, this local configuration must be determined by using the constitutive postulates.

Following a brief statement of the mechanical balance laws, the notion of state and process variable is introduced in Sect. 3. Although in [1] shock waves are included in the general theory, this is not done here. A modified version of Carathéodory's statement of the second law of thermodynamics is presented in Sect. 4. It is from this that the existence of the absolute temperature and an entropy function can be established. A global entropy inequality is then deduced and this is sufficient to ensure that all possible transitions of an isotropic elastic-plastic material are consistent with the thermodynamic theory.

In the final section the consequences of the thermodynamic theory for a constitutive model of an isotropic elastic-plastic material are stated. One of the most important results is that the work rate due to plastic deformations need not necessarily be positive. This is consistent with the experimentally observed effect known as the Bauschinger effect. If this effect is large enough, then the yield surface in stress space may be such that it does not always enclose the origin so that a negative plastic work rate becomes possible.

2. Mechanical balance laws

It is convenient to give a brief presentation of the usual mechanical equations of motion. With respect to a fixed set of rectangular Cartesian axes the Cartesian coordinates of a typical particle in a continuous medium at time t are denoted by x_k . With respect

to the same set of axes the Cartesian coordinates of the particle in some reference configuration of the medium, at time $t = 0$ say, are given by X_K . Latin upper and lower case indices take the values 1, 2, 3, unless specifically stated otherwise, and the usual index notation operates throughout. The equation of motion is represented by

$$(2.1) \quad x_k = x_k(X_K, t).$$

Then, the velocity v and the deformation tensor F have the components

$$(2.2) \quad v_k = \dot{x}_k, \quad F_{kK} = x_{k,K},$$

where a superposed dot denotes the material time derivative, and a comma followed by the index K indicates partial differentiation with respect to X_K .

The local form of the mass and momentum balance are

$$(2.3) \quad \det(F_{kK}) = \frac{\rho_0}{\rho} > 0,$$

and

$$(2.4) \quad \sigma_{ij,j} + \rho b_i = \rho \dot{v}_i,$$

respectively, where ρ and ρ_0 are the densities of the medium in its current and reference configuration, σ is the Cauchy stress, b is the body force per unit mass and a comma followed by the index j denotes partial differentiation with respect to x_j .

Conservation of angular momentum requires that the Cauchy stress be symmetric so that

$$(2.5) \quad \sigma_{ij} = \sigma_{ji}.$$

3. State variables, process variables and the kinematics of isotropic elastic-plastic materials

A thermodynamic system Σ is determined by any part P of a continuous medium which consists always of the same material particles. The region of three-dimensional Euclidean space occupied by P at time t is denoted by B . Of the variables needed to describe an equilibrium configuration of P only those which determine the forces holding the material in equilibrium are called the state variables. These variables form a finite set of independent physical quantities $\bar{\xi} = (\xi_1, \xi_2, \dots, \xi_\alpha)$. These quantities include the empirical temperature θ and so are also denoted by $(\bar{\xi}, \theta)$ with $\xi_\alpha = \theta$. The variables $\bar{\xi} = (\xi_1, \xi_2, \dots, \xi_{\alpha-1})$ are called the deformation state variables and they possess the property of being freely adjustable by mechanical means only. Thus the state variables are a subset of those variables needed to describe the equilibrium configuration of the material. For example, for a material which has experienced elastic-plastic strains its equilibrium configuration is determined by the elastic strain, the plastic strain and the empirical temperature. However, if, as it is assumed here, the plastic strains do not influence the elastic properties of the material, then only the elastic strain and the temperature are required to determine the stress acting on the material. So in this case the elastic strain and the empirical temperature form the independent set of state variables. Since

the thermodynamic state is completely characterized by the independent state variables, any other state variable is related to ξ by an equation called an equation of state. Thus, if u is an additional state variable, then an equation of state can be written in the form

$$(3.1) \quad f(\xi, u) = 0.$$

Further variables are required to describe the processes which occur in the non-equilibrium system. These may be gradients or rates of change of the state variables or any other quantity necessary to specify the processes which are occurring in the material. Such quantities are called process variables. The processes are governed by equations which can include both state and process variables and these are called process equations. Hence, the constitutive equations which model material behaviour consist of two types: equations of state and process equations.

To illustrate these ideas it is now convenient to consider the kinematics of isotropic elastic-plastic materials. By the polar decomposition theorem the deformation gradient tensor \mathbf{F} may be expressed as

$$(3.2) \quad F_{kK} = R_{kM} U_{MK},$$

where \mathbf{R} is a proper orthogonal tensor and \mathbf{U} is a positive definite, symmetric tensor. The strain tensor \mathbf{E} is then

$$(3.3) \quad U_{KM} U_{ML} = C_{KL} = 2E_{KL} + \delta_{KL},$$

where δ_{KL} are the components of the Kronecker delta function.

It is supposed that the material is initially in an annealed condition in a stress-free state at some reference temperature. This is adopted as the reference configuration of the material. When a stress is applied to the material the initial deformation is elastic so that

$$(3.4) \quad E_{KL} = E_{KL}^{(e)}, \quad F_{kK} = F_{kK}^{(e)} = R_{kM}^{(e)} U_{MK}^{(e)},$$

where $\mathbf{E}^{(e)}$ and $\mathbf{F}^{(e)}$ denote the elastic strain and the elastic deformation respectively.

The symmetric Piola-Kirchhoff stress tensor σ_{KL} is defined by

$$(3.5) \quad \sigma_{ij} = \frac{\rho}{\rho_0} F_{iK}^{(e)} F_{jL}^{(e)} \sigma_{KL}$$

and given by the thermoelastic equation

$$(3.6) \quad \sigma_{KL} = \sigma_{KL}(E_{MN}^{(e)}, \theta).$$

The deformation remains elastic provided the point (σ_{KL}, θ) in stress-temperature space does not cross the yield surface. On crossing the yield surface plastic strains occur which alter the local stress-free reference configuration. It is assumed that elastic properties are not changed by plastic deformation, therefore Eq. (3.6) still holds provided σ_{KL} is referenced to the local plastically-deformed reference configuration and $\mathbf{E}^{(e)}$ is interpreted as the strain measured relative to that configuration.

Now the deformation tensor can be written as

$$(3.7) \quad F_{kK} = R_{kM}^{(e)} U_{MN}^{(e)} U_{NK}^{(p)} = R_{kM} U_{MK},$$

where $\mathbf{U}^{(p)}$ is a positive definite, symmetric plastic stretch tensor and the orientation of the plastically deformed reference configuration is chosen to be that of the original reference configuration. The plastic strains $\mathbf{C}^{(p)}$ and $\mathbf{E}^{(p)}$ are defined by

$$(3.8) \quad U_{KM}^{(p)} U_{ML}^{(p)} = C_{KL}^{(p)} = 2E_{KL}^{(p)} + \delta_{KL}.$$

Thus, $\mathbf{U}^{(p)}$, $\mathbf{C}^{(p)}$ and $\mathbf{E}^{(p)}$ provide some local strain measures of the local plastically deformed reference configuration measured with respect to the original reference configuration. If $F_{kN}^{(e)} = R_{kM}^{(e)} U_{MN}^{(e)}$ and $F_{Nk}^{(p)} = U_{Nk}^{(p)}$ then

$$(3.9) \quad F_{kk} = F_{kN}^{(e)} F_{Nk}^{(p)}$$

and $\mathbf{F}^{(e)}$ and $\mathbf{F}^{(p)}$ can be called respectively the elastic and plastic deformation tensors.

Since it is assumed that plastic deformation does not contribute to volume changes, then

$$(3.10) \quad \det(F_{Nk}^{(p)}) = 1, \quad \det(F_{kN}^{(e)}) = \frac{\rho_0}{\rho}.$$

At the onset of plastic flow

$$(3.11) \quad U_{KL}^{(p)} = C_{KL}^{(p)} = \delta_{KL}, \quad E_{KL}^{(p)} = 0, \quad R_{kK}^{(e)} = R_{kK}, \quad U_{KL}^{(e)} = U_{KL}$$

and these provide initial conditions which must be used in conjunction with Eq. (3.6) and constitutive equations for the plastic deformation rate to calculate the subsequent values of these quantities.

Since σ_{KL} depends only on $E_{MN}^{(e)}$ and θ , these are the independent state variables and Eq. (3.6) is an equation of state. The additional equations needed to determine the plastic deformation rate are process equations.

The rate at which work is done per unit mass in addition to that which contributes to the kinetic energy is

$$(3.12) \quad \dot{w} = \frac{1}{\rho} \sigma_{ij} d_{ij} = \frac{1}{\rho_0} \sigma_{KL} \dot{E}_{KL}^{(e)} + \frac{1}{\rho_0} \sigma_{KM} C_{ML}^{(e)} \dot{F}_{LN}^{(p)} F_{KN}^{(p)-1},$$

where

$$2d_{ij} = v_{i,j} + v_{j,i}.$$

The first and second terms on the right hand side of Eq. (3.12) can be interpreted as the work rate per unit mass due to elastic and plastic deformations respectively. Further, since the material is isotropic, the Cauchy stress σ_{ij} and the stretch tensor $V_{ij}^{(e)} = R_{iK}^{(e)} R_{jL}^{(e)} U_{KL}^{(e)}$ have the same principal directions, so it can be shown that

$$(3.13) \quad s_{KL} = \sigma_{KM} C_{ML}^{(e)} = \frac{\rho_0}{\rho} R_{iK}^{(e)} R_{jL}^{(e)} \sigma_{ij}.$$

In this case \mathbf{s} is symmetric, so the plastic work rate per unit mass \dot{w}_p can be written as

$$(3.14) \quad \dot{w}_p = \frac{1}{\rho_0} s_{KL} D_{KL}^{(p)},$$

where

$$(3.15) \quad D_{KL}^{(p)} = \frac{1}{2} (\dot{U}_{KN}^{(p)} U_{LN}^{(p)-1} + \dot{U}_{LN}^{(p)} U_{KN}^{(p)-1}).$$

Also, since plastic deformations do not contribute to volume changes, $D_{kk}^{(p)} = 0$. Hence, the hydrostatic part of \mathbf{s} does not contribute to the plastic work rate and so \mathbf{s} can be replaced by its deviatoric part \mathbf{s}' defined by

$$(3.16) \quad s'_{KL} = s_{KL} - \frac{1}{3} s_{MM} \delta_{KL}.$$

Then, the plastic work rate per unit mass is given by

$$(3.17) \quad \dot{w}_p = \frac{1}{\rho_0} s'_{KL} D_{KL}^{(p)} = \frac{1}{\rho} \sigma'_{ij} d_{ij}^{(p)},$$

where σ' is the deviatoric part of the Cauchy stress and

$$d_{ij}^{(p)} = R_{iK}^{(e)} R_{jL}^{(e)} D_{KL}^{(p)}.$$

4. The thermodynamic theory

It is assumed that there exists a specific internal energy function u which is dependent on the state variables only. The total internal energy is given by

$$(4.1) \quad U = \int_B \rho u dV.$$

Let r be the rate of supply of heat energy per unit mass due to radiation and take \mathbf{q} as the heat flux vector. The energy balance equation can be expressed as

$$(4.2) \quad \frac{d}{dt} \left\{ \int_B \rho \left(u + \frac{1}{2} v_k v_k \right) dV \right\} = \int_B \rho (r + b_k v_k) dV + \int_{\partial B} (\sigma_{ij} v_i - q_j) n_j dS,$$

where ∂B is the closed, regular boundary of B with the area element dS and the unit outward normal \mathbf{n} . Under the usual assumptions and employing the momentum equation (2.4) it can be shown that

$$(4.3) \quad \dot{u} = r + \frac{1}{\rho} \sigma_{ij} d_{ij} - \frac{1}{\rho} q_{k,k}.$$

Since u is a state variable and recalling the kinematics of isotropic elastic-plastic materials given in the previous section, it follows that

$$(4.4) \quad \dot{u}(\bar{\xi}, \theta) - \mathbf{P}(\bar{\xi}, \theta) \cdot \dot{\bar{\xi}} = r + \dot{w}_p - \frac{1}{\rho} \operatorname{div} \mathbf{q} = \dot{h},$$

where

$$(4.5) \quad \mathbf{P}(\bar{\xi}, \theta) \cdot \dot{\bar{\xi}} = \frac{1}{\rho_0} \sigma_{KL} (E_{MN}^{(e)}, \theta) \dot{E}_{KL}^{(e)}, \quad \dot{w}_p = \frac{1}{\rho_0} s'_{KL} D_{KL}^{(p)}.$$

For the system Σ which occupies the region B

$$(4.6) \quad r = r' + \omega,$$

where r' is the rate of exchange of radiative heat per unit mass between Σ and its surroundings and ω is the rate of exchange of radiative heat per unit mass between any particle of Σ and the remainder of Σ . Thus, ω represents the internal radiation and is a space functional depending on \mathbf{x} and B so that

$$(4.7) \quad \omega = \omega(x_i, B).$$

Clearly, the net rate of exchange of internal radiation is zero giving

$$(4.8) \quad \int_B \rho \omega dV = 0.$$

The value of ω is a property of the material and must be specified by a constitutive equation whereas r' (and hence r) can be chosen arbitrarily. Also, as B collapses to a point ω approaches zero and r' tends to r .

It is now necessary to introduce certain types of thermodynamic transitions: an adiabatic or a -transition is one for which

$$(4.9) \quad r' = 0 \text{ in } B, \quad \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \partial B.$$

An i_1 -transition is one for which

$$(4.10) \quad \dot{h} = 0 \text{ in } B.$$

An i_2 -transition is one for which $\theta = \theta(t)$, $\mathbf{q} \cdot \mathbf{n} = 0$ on ∂B and r' is chosen so that

$$(4.11) \quad \int_B \rho \dot{h} dV = 0.$$

Finally, an (a, i) -transition is such that any part of it is an a -transition, an i_1 -transition or an i_2 -transition.

The statement of the second law of thermodynamics presented in [1] modifies the classical statement of the second law given by CARATHÉODORY [2] and is as follows:

In every neighbourhood of a given state there exist states which cannot be reached from the given state by an (a, i) -transition.

A number of results from this statement of the second law have already been proved in [1] and some of these will now be stated. It can be shown that at any point in the non-equilibrium system there exists $T = T(\theta) > 0$ and $s = s(\bar{\xi}, \theta)$ such that

$$(4.12) \quad \dot{h} = T \dot{s},$$

where T is the absolute temperature and s is the specific entropy. The total entropy S may be defined by

$$(4.13) \quad S = \int_B \rho s dV.$$

It follows immediately from the energy equation and Eq. (4.12) that

$$(4.14) \quad T = \left(\frac{\partial u}{\partial s} \right)_{\bar{\xi}}, \quad P_\gamma = \frac{\partial \psi}{\partial \xi_\gamma}, \quad \gamma = 1, \dots, \alpha - 1,$$

where $\psi = u - Ts$ is the Helmholtz free energy. It is now assumed that θ can be replaced by T as a state variable. For the elastic-plastic material

$$(4.15) \quad s = - \frac{\partial \psi}{\partial T}, \quad \sigma_{KL} = \rho_0 \frac{\partial \psi}{\partial E_{KL}}.$$

A further consequence of the second law is that

$$(4.16) \quad \dot{S} \geq 0$$

for all (a, i) -transitions. It is also explained in [1] that if at least one of the rates of change of the state variables does not appear in the constitutive equations, then a necessary and

sufficient condition for material processes to comply with the thermodynamic theory is that the global entropy inequality

$$(4.17) \quad \int_B \rho \dot{s} dV \geq \int_B \frac{\rho r'}{T} dV - \int_{\partial B} \frac{\mathbf{q} \cdot \mathbf{n}}{T} dS$$

holds for all conceivable processes.

Using the energy equation, the divergence theorem and the thermodynamic relations (4.14), this is equivalent to the global inequality

$$(4.18) \quad \int_B \frac{\rho \omega}{T} dV + \int_B \frac{1}{T} \left(\rho \dot{w}_p - \frac{\mathbf{q} \cdot \mathbf{g}}{T} \right) dV \geq 0,$$

where \mathbf{g} is the temperature gradient with the components $g_k = T_{,k}$.

Since ω approaches zero as B collapses to a point, then a local inequality can be obtained from Eq. (4.18) in the form

$$(4.19) \quad \rho \dot{w}_p - \frac{\mathbf{q} \cdot \mathbf{g}}{T} \geq 0.$$

For a material which has constitutive equations containing the rates of change of all the state variables it is sufficient but not necessary that the inequalities (4.17)–(4.19) hold for all possible processes. This is true of materials which can experience elastic-plastic deformation.

5. A constitutive theory for isotropic elastic-plastic materials

The kinematics of isotropic elastic-plastic materials presented in Sect. 3 suggests that an appropriate stress measure with which to work in the equations for yielding and plastic flow is the deviatoric part s' of the stress s . The yield surface is given by

$$(5.1) \quad f(s'_{KL}, T, \kappa) = g(s'_{KL}) - c(T, \kappa) = 0,$$

where $\kappa = \kappa(\eta_A)$, $A = 1, 2, \dots$ and

$$(5.2) \quad \dot{\kappa} = 0 \quad \text{when} \quad D_{KL}^{(p)} = 0.$$

The parameters η_A depend on the plastic strain history and two possible choices for these parameters are

$$(5.3) \quad \eta_1 = \int \{ D_{KL}^{(p)} D_{KL}^{(p)} \}^{1/2} dt, \quad \eta_2 = \int s'_{KL} D_{KL}^{(p)} dt.$$

The constitutive postulate for $\mathbf{D}^{(p)}$ is as follows:

$$(5.4) \quad D_{KL}^{(p)} = G_{KL}(s'_{MN}, T, \kappa, \dot{s}'_{MN}, \dot{T}) \quad \text{when} \quad \frac{\partial f}{\partial s'_{MN}} \dot{s}'_{MN} + \frac{\partial f}{\partial T} \dot{T} > 0, \quad f = 0,$$

$$(5.5) \quad D_{KL}^{(p)} = 0 \quad \text{when} \quad \begin{cases} \frac{\partial f}{\partial s'_{MN}} \dot{s}'_{MN} + \frac{\partial f}{\partial T} \dot{T} = 0, & f = 0, \\ \text{or } f < 0, \end{cases}$$

where the conditions (5.4) and (5.5) are said to hold during loading, neutral loading and unloading, respectively. Clearly, the hydrostatic stress does not affect yielding or plastic flow directly but it does have an indirect effect for large elastic strains. This indirect effect vanishes in the limiting case of small elastic strains.

It is usually assumed that the plastic deformations are independent of the time scale and therefore G_{KL} is a homogeneous tensor function of degree one in \dot{s}'_{KL} and \dot{T} . Moreover, if the constitutive equations for plastic flow are to be continuous in state space, then G vanishes during neutral loading. Thus,

$$(5.6) \quad D_{KL}^{(p)} = H_{KL}(s'_{MN}, T, \kappa) \left(\frac{\partial g}{\partial s'_{MN}} \dot{s}'_{MN} - \frac{\partial c}{\partial T} \dot{T} \right),$$

during loading.

When f and κ have been specified H is not arbitrary since it is necessary to satisfy the condition $\dot{f} = 0$ and in this case $D_{KL}^{(p)} = 0$. Further, the independence of the particular time scale and condition (5.2) require that the $\dot{\eta}_A$ be homogeneous of degree one in $D_{KL}^{(p)}$. It can therefore be shown that

$$(5.7) \quad \lambda_{KL} H_{KL} = 1, \quad H_{KK} = 0,$$

where

$$(5.8) \quad \lambda = \frac{\lambda_{KL} D_{KL}^{(p)}}{\partial c / \partial \kappa}.$$

The constitutive equation for heat flux is

$$(5.9) \quad Q_K = Q_K(E_{MN}^{(e)}, T, G_N),$$

where Q_K and G_K are defined by

$$(5.10) \quad q_k = \frac{\rho}{\rho_0} F_{kK}^{(e)} Q_K, \quad G_K = F_{kK}^{(e)} g_k.$$

In this model of elastic-plastic materials the constitutive equations which hold during loading contain the rates of change of all the state variables and so it is not necessary that the entropy inequality (4.17) be satisfied for all conceivable processes. Nevertheless, if the material properties are restricted in this way, then this is sufficient to ensure that all possible processes of the material comply with the thermodynamic theory.

The global entropy inequality (4.18) now gives

$$(5.11) \quad \int_B \frac{\rho \omega}{T} dV + \int_B \frac{1}{T} \left(\rho_0 \dot{w}_P - \frac{\mathbf{Q} \cdot \mathbf{G}}{T} \right) \frac{\rho}{\rho_0} dV \geq 0.$$

The local entropy inequality (4.19), Eqs. (4.5)₂ and (5.6) together yield

$$(5.12) \quad s'_{KL} H_{KL} \left(\frac{\partial g}{\partial s'_{MN}} \dot{s}'_{MN} - \frac{\partial c}{\partial T} \dot{T} \right) - \frac{Q_K G_K}{T} \geq 0$$

during loading. Thus, by considering arbitrary uniform distributions of temperature for which $G_K = 0$, then

$$(5.13) \quad s'_{KL} H_{KL} \geq 0.$$

This is the condition for positive plastic work. It also follows from the inequality (5.12) that

$$(5.14) \quad Q_{\mathbf{k}} G_{\mathbf{k}} \leq 0.$$

It should be emphasized that the inequalities (5.11)–(5.14) are sufficient conditions for the constitutive postulates during loading to be consistent with the thermodynamic theory but they are not necessary. So, this thermodynamic theory does not impose the restriction that the plastic work rate be positive. This is consistent with the experimentally observed Bauschinger effect and the fact that the yield surface need not always enclose the origin in stress space.

Other authors have been able to include a negative plastic work rate within a thermodynamic theory (see for example GREEN and NAGHDI [4] and KESTIN and RICE [5].) However, unlike the thermodynamic theory presented in this paper, the theory in [4, 5] only permits a negative plastic work rate for elastic-plastic materials exhibiting history-dependent elastic properties.

During neutral loading and unloading the global entropy inequality provides

$$(5.15) \quad \int_B \frac{\rho \omega}{T} dV - \int_B \frac{\mathbf{Q} \cdot \mathbf{G}}{T^2} \frac{\rho}{\rho_0} dV \geq 0$$

and the local inequality implies that

$$(5.16) \quad Q_{\mathbf{k}} G_{\mathbf{k}} \leq 0.$$

The conditions (5.15) and (5.16) are both necessary and sufficient.

If the yield function is adopted as a plastic potential, then

$$(5.17) \quad D_{\mathbf{KL}}^{(p)} = k \frac{\partial g}{\partial s'_{MN}} \left(\frac{\partial g}{\partial s'_{MN}} \dot{s}'_{MN} - \frac{\partial c}{\partial T} \dot{T} \right)$$

during loading.

Further, if

$$(5.18) \quad \lambda_{\mathbf{KL}} = \alpha(T) \frac{\partial c}{\partial \kappa} s'_{\mathbf{KL}},$$

where $\alpha(T)$ is an increasing function of T and g is supposed to be homogeneous of degree n in $s'_{\mathbf{KL}}$, then it can be shown that

$$(5.19) \quad D_{\mathbf{KL}}^{(p)} = \frac{1}{\alpha(T) \frac{\partial c}{\partial \kappa} n g} \frac{\partial g}{\partial s'_{\mathbf{KL}}} \left(\frac{\partial g}{\partial s'_{MN}} \dot{s}'_{MN} - \frac{\partial c}{\partial T} \dot{T} \right)$$

during loading.

It is easily deduced from Eq. (3.13) that for isotropic elastic-plastic materials

$$(5.20) \quad s_{\mathbf{KL}} = R_{\mathbf{K}}^{(e)} R_{\mathbf{L}}^{(e)} p_{ij},$$

where $\rho p_{ij} = \rho_0 \sigma_{ij}$ and if \mathbf{p}' is the deviatoric part of \mathbf{p} , then the yield surface becomes

$$(5.21) \quad g(\mathbf{p}'_{ij}) - c(T, \kappa) = 0.$$

Here g is an isotropic function of \mathbf{p}' and can therefore be expressed as a function of J_2' and J_3' , the two principal invariants of \mathbf{p}' . Also, during loading the plastic flow rule becomes

$$(5.22) \quad d_{ij}^{(p)} = \frac{1}{\alpha(T) \frac{\partial c}{\partial \kappa} n g} \frac{\partial g}{\partial p'_{ij}} \left(\frac{\partial g}{\partial p'_{mn}} \dot{p}'_{mn} - \frac{\partial c}{\partial T} \dot{T} \right).$$

This theory is then equivalent to that proposed by LEE [6] for isotropic elastic-plastic materials. It is perhaps worthy of note that the indirect effect of the hydrostatic stress on yielding and plastic flow when finite elastic strains occur can be removed if in the constitutive equations for yielding and plastic flow \mathbf{s}' is replaced by \mathbf{t}' where \mathbf{t}' is the deviatoric part of the tensor \mathbf{t} with components which satisfy

$$(5.23) \quad \sigma_{ij} = R_{iK}^{(e)} R_{jL}^{(e)} t_{KL}.$$

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