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# Viscoelastic properties in axially symmetric squeeze-film flows

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IN THIS PAPER, being a continuation of the paper [10], we consider the problem of axially symmetric squeeze-film flows by means of convected coordinates. The flows discussed are treated as instantaneous motions with superposed proportional stretch histories (cf. [11]). The corresponding constitutive equations of an incompressible simple fluid are simplified for the case of low Deborah numbers. Two types of approximate solutions are obtained either for slightly viscoelastic fluids or in the forms valid in the vicinity of an arbitrarily chosen instant of time. The conditions of improved lubrication, leading to inequalities imposed on material constants and kinematic quantities, are discussed in greater detail. A theoretical possibility for elastic "bouncing" behaviour, observed experimentally at more severe loading conditions (cf. [6]) is also considered.

W niniejszej pracy, będącej kontynuacją poprzedniej pracy [10], rozważono zagadnienie osiowo-symetrycznych przepływów wyciskających opisanych współrzędnymi konwekcyjnymi. Omawiane przepływy potraktowano jako chwilowe ruchy z nałożonymi proporcjonalnymi historiami deformacji (por. [11]). Odpowiednie równania konstytutywne nieściśliwej cieczy prostej zostały uproszczone dla przypadku małych liczb Debory. Otrzymano dwa typy przybliżonych rozwiązań: dla cieczy nieznacznie lepkosprężystych oraz w postaci określonej w otoczeniu dowolnie wybranej chwili czasu. Przedyskutowano bardziej szczegółowo warunki lepszego smarowania, prowadzące do pewnych nierówności zawierających stałe materiałowe i wielkości kinematyczne. Rozważono również teoretyczną możliwość występowania zjawiska «podskoków» zaobserwowanego doświadczalnie przy większych obciążeniach (por. [6]).

В настоящей работе, будучей продолжением предыдущей работы [10], рассмотрена задача осесимметрических выдавливающих течений, описанных конвекционными координатами. Обсуждаемые течения трактуются как мгновенные движения с наложенными пропорциональными историями деформаций (ср. [11]). Соответствующие определяющие уравнения несжимаемой простой жидкости упрощены для случая малых чисел Деборы. Получены два типа приближенных решений: для незначительно вязкоупругих жидкостей и в виде определенном в окрестности произвольно избранного момента времени. Обсуждены более подробно условия лучшей смазки, приводящие к некоторым неравенствам содержавшим материальные постоянные и кинематические величины. Рассмотрена тоже теоретическая возможность выступания явления „прыжков", наблюдаемого экспериментально при больших нагрузках (ср. [6]).

## 1. Introduction

VARIOUS problems connected with squeeze-film flows of viscous and viscoelastic fluids, i.e. flows in which a test fluid contained between two horizontal plates or discs was squeezed out under a constant vertical load, have attracted the attention of many authors [1, 2, 3, 4, 5]. The most systematic studies of squeezing flows were recently presented by Walters and his collaborators [6, 7, 8, 9]. Viscoelastic effects, being in evidence for heavier loading conditions (or higher Deborah numbers), could not be explained satisfactorily by means of a simplified theoretical analysis, on the contrary to numerous experiments according to which viscoelastic fluids behaved as better lubricants than purely viscous, inelastic fluids.

In our previous paper [10] on plane squeeze-film flows, treated in a convected coordinates system as instantaneous motions with a proportional stretch history (cf. [11]), the theoretical predictions were in general accordance with the experimental data. An attempt was also undertaken to determine condition under which a certain "bouncing" behaviour of viscoelastic fluids may appear during flows even at lighter loadings (cf. [6]).

In the present paper similar problems are considered for the case of axially-symmetric squeezing flows of viscoelastic fluids. To this end the formulation used in [10] has been properly modified.

## 2. Squeeze-film flows described in convected coordinates

Let the flow considered be described by the equation of motion:

$$\boldsymbol{\xi}(\tau) = \boldsymbol{\chi}(\mathbf{X}, \tau), \quad X \in \mathcal{B}_R, \quad \tau \in (-\infty, \infty), \tag{2.1}$$

where $\boldsymbol{\xi}$ is the position vector of a particle at any time $\tau$, $\mathbf{X}$ — the position vector of the same particle in a reference configuration, $\boldsymbol{\chi}$ denotes the continuously differentiable mapping of $\mathcal{B}_R \times (-\infty, \infty)$ into the three-dimensional Euclidean space.

We assume the reference configuration to be chosen in such a way that

$$\mathbf{x} \equiv \boldsymbol{\xi}(t) = \boldsymbol{\chi}(\mathbf{X}, t), \tag{2.2}$$

where $\mathbf{x}$ denotes the position vector at present time $t$ $(\tau \leqslant t)$; the coordinates of vector $\mathbf{X}$ are called the material coordinates (cf. [12]). If, moreover, the system of material coordinates $X^\alpha (\alpha = 1, 2, 3)$ moves and deforms together with a fluid, we shall use the name of convected coordinates (cf. [13, 14]).

Kinematical characteristics of a flow result from Eq. (2.1). In particular, the velocity field $\mathbf{v}$, the velocity gradient $\mathbf{L}$ with respect to the reference configuration at time $t$, the deformation gradient $\mathbf{F}$, and the Cauchy–Green deformation tensor $\mathbf{C}$ are defined as follows (cf. [12]):

$$\mathbf{v}(t) \equiv \dot{\boldsymbol{\xi}}(t) = \frac{\partial}{\partial \tau} \boldsymbol{\chi}(\mathbf{X}, \tau)|_{\tau=t}, \quad \mathbf{L}(t) \equiv \nabla \mathbf{v}(t), \tag{2.3}$$

$$\mathbf{F}(\tau) = \nabla \boldsymbol{\chi}(\mathbf{X}, \tau), \quad \mathbf{C}(\tau) = \mathbf{F}^T(\tau)\mathbf{F}(\tau), \tag{2.4}$$

where the superposed dot denotes the material time derivative and superscript $T$ — the transpose. On applying the definition of the Rivlin-Ericksen kinematic tensors in the form (cf. e.g. [12])

$$\mathbf{A}_n(t) = \frac{d^n}{d\tau^n} \mathbf{C}(\tau)|_{\tau=t}, \quad n = 1, 2, 3, \ldots, \tag{2.5}$$

we arrive at the formula (cf. [10, 15, 16])

$$A^{(n+1)}_{\alpha\beta}(t) = \frac{\partial^n}{\partial t^n} A^{(1)}_{\alpha\beta}(t)|_{X^\alpha = \mathrm{const}}, \quad A^{(1)}_{\alpha\beta}(t) = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha, \tag{2.6}$$

if the convected coordinates $X^\alpha (\alpha = 1, 2, 3)$ are used, and $A^{(n)}_{\alpha\beta}$ denote the corresponding convected components of the tensor $\mathbf{A}_n$.

Since for complex axially symmetric squeeze-film flows the equation of motion (2.1) is not known a priori, we shall assume that the velocity field at present time $t$ may be written in the form

$$\mathbf{v}(t) = \boldsymbol{\varphi}(X^\beta)\,\dot{\lambda}(t) + \varepsilon\boldsymbol{\psi}(X^\beta)\,\frac{\dot{\lambda}^2(t)}{\lambda(t)}, \tag{2.7}$$

where $X^\beta(\beta = 1, 2, 3)$ denote the convected coordinates defined in the reference configuration at time $t$, and $\varepsilon$ is a small parameter proportional to the characteristic time $\theta$ of a fluid (cf. (2.19)). There exist infinitely many motions leading to velocity fields in the form (2.7), and the complexity of motion is rather connected with a way in which the system of convected coordinates moves and deforms together with a fluid. The material planes $X^\alpha$ = const at time $t$ may not be plane at any past time $\tau < t$. In particular, for Newtonian fluids $\varepsilon \equiv 0$ and the first term on the right hand side of Eq. (2.7) is fully sufficient (cf. [10]).

It results from the assumption (2.7) that

$$\mathbf{L}(t) = \mathbf{L}' + \varepsilon\mathbf{L}'' = \mathbf{N}_1\dot{k}_1(t) + \varepsilon\mathbf{N}_2\dot{k}_2(t), \tag{2.8}$$

where $\mathbf{N}_1, \mathbf{N}_2$ do not depend on time $t$, and

$$k_1(t) = \lambda(t), \quad k_2(t) = \int \frac{\dot{\lambda}^2(t)}{\lambda(t)}\,dt. \tag{2.9}$$

Since also,

$$\frac{d}{dt}\mathbf{F}_{t'}(t) = \mathbf{L}(t)\mathbf{F}_{t'}(t), \quad \mathbf{F}_{t'}(t') = \mathbf{1}, \tag{2.10}$$

where $t'$ denotes any other instant of time $(t' \neq t)$, and the subscript $t'$ means that the gradient is taken with respect to the reference configuration at time $t'$, we obtain

$$\mathbf{F}_{t'}(t) = \exp\left[\mathbf{N}_1\big(k_1(t) - k_1(t')\big) + \varepsilon\mathbf{N}_2\big(k_2(t) - k_2(t')\big)\right]. \tag{2.11}$$

After replacing $t'$ by $t$ and $t$ by $t-s$, where $s \in [0, \infty)$, we arrive at

$$\mathbf{F}(s) \equiv \mathbf{F}_t(t-s) = \exp[\mathbf{N}_1 l_1(s) + \varepsilon\mathbf{N}_2 l_2(s)], \tag{2.12}$$

where

$$l_i(s) = k_i(t-s) - k_i(t), \quad i = 1, 2. \tag{2.13}$$

It is known from our previous considerations on motions with proportional stretch history [11] that flows described by a history of the relative deformation gradient in the form (2.12) belong to the class of motions with superposed proportional stretch histories (MSPSH)([1]). Thus, flows for which the velocity field at present time $t$ is described by Eq. (2.7) may be treated as an instantaneous MSPSH. The right relative Cauchy-Green deformation tensor is of the form

$$\begin{aligned} \mathbf{C}(s) = \mathbf{F}^T(s)\mathbf{F}(s) &= \exp[\mathbf{N}_1^T l_1(s) + \varepsilon\mathbf{N}_2^T l_2(s)]\exp[\mathbf{N}_1 l_1(s) + \varepsilon\mathbf{N}_2 l_2(s)] \\ &= \exp\left[\mathbf{L}'^T\frac{l_1(s)}{\dot{k}_1(t)} + \varepsilon\mathbf{L}''^T\frac{l_2(s)}{\dot{k}_2(t)}\right]\exp\left[\mathbf{L}'\frac{l_1(s)}{\dot{k}_1(t)} + \varepsilon\mathbf{L}''\frac{l_2(s)}{\dot{k}_2(t)}\right], \end{aligned} \tag{2.14}$$

([1]) These motions may also be called unsteady homothermal motions [17].

where, according to Eqs. (2.8), we have denoted

(2.15) $$\mathbf{L}'(t) = \mathbf{N}_1 \dot{k}_1(t), \quad \mathbf{L}''(t) = \mathbf{N}_2 \dot{k}_2(t).$$

Thus, the constitutive equation of an incompressible simple fluid (cf. [12]) can be presented in the following form:

(2.16) $$\mathbf{T}(t) = -p\mathbf{1} + \overset{\infty}{\underset{s=0}{\mathscr{G}}} \big(\mathbf{C}(s)\big) = -p\mathbf{1} + \overset{\infty}{\underset{s=0}{\mathscr{H}}} \big(l_1(s), l_2(s); \mathbf{L}'(t), \varepsilon\mathbf{L}''(t)\big),$$

where $\mathbf{T}(t)$ is the stress tensor at time $t$, $p$ — the hydrostatic pressure, and $\mathscr{H}$ denotes a functional with respect to $l_1(s), l_2(s)$, being an isotropic tensor function of tensors $\mathbf{L}'$ and $\varepsilon\mathbf{L}''$, and allowing for interchange of arguments. If $\mathbf{L}'^T+\mathbf{L}'$ and $\varepsilon(\mathbf{L}''^T+\mathbf{L}'')$ have three eigen-values distinct, it can be proved (see Appendix) that also

(2.17) $$\mathbf{T}(t) = -p\mathbf{1} + \overset{\infty}{\underset{s=1}{\mathscr{K}}} \big(l_1(s), l_2(s); \mathbf{A}_1', \varepsilon\mathbf{A}_2'', \mathbf{A}_2', \varepsilon\mathbf{A}_2''\big),$$

where $\mathbf{A}_i', \mathbf{A}_i''$ $(i = 1, 2)$ are the partial Rivlin-Ericksen kinematic tensors defined by Eqs. (A.1) and (A.2).

If we take an expansion of the function (2.17) with respect to $\mathbf{A}_i'$, $\varepsilon\mathbf{A}_i''$ $(i = 1, 2)$, the corresponding coefficients will be functionals in $l_1(s), l_2(s)$ and scalar functions of all joint invariants composed of $\mathbf{A}_i'$ and $\varepsilon\mathbf{A}_i''$. To obtain more explicit results, allowing for an effective solution of the problem, we shall seek the simplest representation of Eq. (2.17), valid for sufficiently small Deborah numbers and linear with respect to parameter $\varepsilon$.

Bearing in mind the well-known form of an isotropic tensor function of two arguments (cf. [12]) and the fact that $\mathscr{K}$ must be symmetric with respect to the pairs of arguments $\mathbf{A}_i'$ and $\varepsilon\mathbf{A}_i''$ $(i = 1, 2)$, we have

(2.18) $$\mathbf{T}(t) = -p\mathbf{1} + \alpha_1\mathbf{A}_1' + \alpha_1\varepsilon\mathbf{A}_1'' + \alpha_2\mathbf{A}_2' + \alpha_3\mathbf{A}_1'^2 + \mathbf{O}(\varepsilon^2),$$

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are assumed as material parameters, constant in such short periods of time in which the flows are realized. The parameter $\varepsilon$, proportional to the characteristic time of a fluid $\theta$, may be expressed as

(2.19) $$\varepsilon = \left|\frac{\alpha_2}{\alpha_1}\right| = \theta \quad \text{or} \quad \varepsilon = \left|\frac{\alpha_3}{\alpha_1}\right| = c\theta,$$

what justifies the rejection of terms proportional to $\alpha_2\varepsilon$, $\alpha_3\varepsilon$ etc.

Equation (2.18) differs from that for a second-order fluid (cf. [12]) if the second term in the velocity equation (2.7) is not identically equal to zero. This difference becomes much greater when all tensor quantities at time $t$ are related to a system of convected coordinates $X^\alpha$ $(\alpha = 1, 2, 3)$, moving and deforming together with a fluid.

## 3. Axially symmetric squeeze-film flows

We consider the test fluid contained between two horizontal discs of radii $a$ which are at rest for $t < 0$ (Fig. 1). At some instant $t = 0$ the top disc is released and falls down under the constant load $F$. The distance $h$ ($h_0$ means the initial value) between the discs is measured as a function of current time $t$ (cf. [6]). We use the spatial system

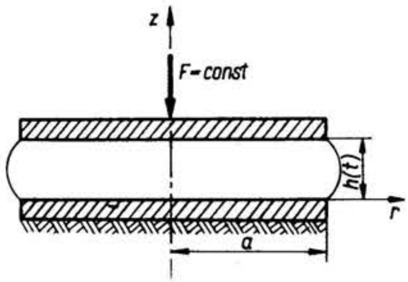


FIG. 1.

of the cylindrical polar coordinates $r, \vartheta, z$ fixed in the centre of the lower disc, with the $z$-axis directed upwards, and the system of convected coordinates $R, \theta, Z$ (cf. Eqs. (3.2)).

Bearing in mind the formalism developed in Sect. 2, we assume an instantaneous velocity field in the following form:

$$(3.1)\qquad \begin{aligned} v^{\langle R\rangle} &= -\frac{1}{2}Rf'(Z)\dot{h}(t)-\varepsilon\frac{1}{2}Rg'(Z)\frac{\dot{h}^2(t)}{h(t)},\\ v^{\langle\theta\rangle} &= 0,\\ v^{\langle Z\rangle} &= f(Z)\dot{h}(t)+\varepsilon g(Z)\frac{\dot{h}^2(t)}{h(t)}, \end{aligned}$$

where primes denote derivatives with respect to $Z$, and $v^{\langle\alpha\rangle}$ are the velocity physical components referred to the system of convected coordinates $R, \theta, Z$, introduced as follows:

$$(3.2)\qquad R = r/h(t),\quad \theta = \vartheta,\quad Z = z/h(t),$$

where $h(t)$ denotes a function of $t$ only.

The velocity field (3.1) must satisfy the continuity equations as well as the boundary conditions:

$$(3.3)\qquad \begin{aligned} v^{\langle R\rangle} = v^{\langle Z\rangle} &= 0 \qquad \text{for}\quad Z = 0,\\ v^{\langle R\rangle} = 0,\quad v^{\langle Z\rangle} &= \dot{h}(t) \quad \text{for}\quad Z = 1, \end{aligned}$$

and this requires that

$$(3.4)\qquad \begin{aligned} &f'(0) = f(0) = g'(0) = g(0) = 0,\\ &f'(1) = g'(1) = g(1) = 0,\quad f(1) = 1. \end{aligned}$$

Taking into account the definitions (2.6), we arrive at the following partial Rivlin-Ericksen kinematic tensors (cf. Appendix) expressed in convected coordinates at time $t$:

$$(3.5)\qquad [A'^{(1)}_{\alpha\beta}] = \begin{bmatrix} -f' & 0 & -\frac{1}{2}f''R\\ 0 & -f' & 0\\ -\frac{1}{2}f''R & 0 & 2f' \end{bmatrix}\dot{h}h,$$

$$[A''^{(1)}_{\alpha\beta}] = \begin{bmatrix} -g' & 0 & -\frac{1}{2}g''R\\ 0 & -g' & 0\\ -\frac{1}{2}g''R & 0 & 2g' \end{bmatrix}\dot{h}^2,$$

(3.6) $$[A'^{(2)}_{\alpha\beta}] = [A'^{(1)}_{\alpha\beta}]\left(\frac{\ddot{h}}{\dot{h}} + \frac{\dot{h}}{h}\right), \quad [A''^{(2)}_{\alpha\beta}] = [A''^{(1)}_{\alpha\beta}]2\frac{\ddot{h}}{\dot{h}},$$

(3.7) $$[A'^{(1)}_{\alpha\beta}]^2 = \begin{bmatrix} f'^2 + \frac{1}{4}f''^2R^2 & 0 & -\frac{1}{2}f'f''R \\ 0 & f'^2 & 0 \\ -\frac{1}{2}f'f''R & 0 & 4f'^2 + \frac{1}{4}f''^2R^2 \end{bmatrix} (\dot{h}h)^2.$$

Physical components of the above tensors result from the relations:

(3.8) $$[A'^{(n)}_{\langle\alpha\beta\rangle}] = [A'^{(n)}_{\alpha\beta}]\frac{1}{h^2}, \quad [A''^{(n)}_{\langle\alpha\beta\rangle}] = [A''^{(n)}_{\alpha\beta}]\frac{1}{h^2}.$$

It can easily be shown that all eigen-values of $\mathbf{A}'_1$ as well as $\mathbf{A}''_1$ are distinct, what justifies the application of the constitutive equation (2.18). The corresponding Deborah numbers may be defined as follows:

(3.9) $$\mathrm{De} = \theta\frac{\dot{h}}{h_0} \quad \text{or} \quad \mathrm{De} = c\theta\frac{\dot{h}}{h_0},$$

where $\theta$ is the characteristic time of a fluid (cf. (2.19)).

Physical components of the stress tensor resulting from Eq. (2.18) take the form

(3.10)
$$T^{\langle RZ\rangle} = \frac{1}{2}f''R\left(\alpha_1\dot{h} + \alpha_2\left(\frac{\ddot{h}}{h} + \frac{\dot{h}^2}{h^2}\right)\right) - \frac{1}{2}f'f''R\alpha_3\frac{\dot{h}^2}{h^2} - \varepsilon\frac{1}{2}g''R\alpha_1\frac{\dot{h}^2}{h^2},$$
$$T^{\langle RR\rangle} = -p - f'\left(\alpha_1\dot{h} + \alpha_2\left(\frac{\ddot{h}}{h} + \frac{\dot{h}^2}{h^2}\right)\right) + \left(f'^2 + \frac{1}{4}f''^2R^2\right)\alpha_3\frac{\dot{h}^2}{h^2} - \varepsilon g'\alpha_1\frac{\dot{h}^2}{h^2},$$
$$T^{\langle\theta\theta\rangle} = -p - f'\left(\alpha_1\dot{h} + \alpha_2\left(\frac{\ddot{h}}{h} + \frac{\dot{h}^2}{h^2}\right)\right) + f'^2\alpha_3\frac{\dot{h}^2}{h^2} - \varepsilon g'\alpha_1\frac{\dot{h}^2}{h^2},$$
$$T^{\langle ZZ\rangle} = -p + 2f'\left(\alpha_1\dot{h} + \alpha_2\left(\frac{\ddot{h}}{h} + \frac{\dot{h}^2}{h^2}\right)\right) + \left(4f'^2 + \frac{1}{4}f''^2R^2\right)\alpha_3\frac{\dot{h}^2}{h^2} - \varepsilon 2g'\alpha_1\frac{\dot{h}^2}{h^2},$$

where the terms of orders $\varepsilon^2$, $\alpha_2\varepsilon$, $\alpha_3\varepsilon$ (cf. (2.19)) have been disregarded. After substituting the above stress components into the inertialess equations of equilibrium (cf. Appendix)

(3.11) $$\frac{\partial T^{\langle RR\rangle}}{\partial R} + \frac{1}{R}(T^{\langle RR\rangle} - T^{\langle\theta\theta\rangle}) + \frac{\partial T^{\langle RZ\rangle}}{\partial Z} = 0,$$
$$\frac{\partial T^{\langle RZ\rangle}}{\partial R} + \frac{1}{R}T^{\langle RZ\rangle} + \frac{\partial T^{\langle ZZ\rangle}}{\partial Z} = 0,$$

and eliminating the pressure $p$, we arrive at the differential equation

(3.12) $$f^{\mathrm{IV}}\left[\alpha_1\dot{h} + \alpha_2\left(\frac{\ddot{h}}{h} + \frac{\dot{h}^2}{h^2}\right)\right](f^{\mathrm{IV}}f' + 2f''f''')\alpha_3\frac{\dot{h}^2}{h^2} + \varepsilon g^{\mathrm{IV}}\alpha_1\frac{\dot{h}^2}{h^2} = 0.$$

Since the Newtonian solution ($\varepsilon = \alpha_2 = \alpha_3 = 0$) is of the form

$$f^{IV} = 0, \quad f(Z) = Z^2(3-2Z), \tag{3.13}$$

we also have the solution

$$\varepsilon g(Z) = -\frac{6}{5}\frac{\alpha_3}{\alpha_1} Z^2(2Z^3-5Z^2+4Z-1). \tag{3.14}$$

It may be verified that $f(Z)$ as well as $g(Z)$ satisfy the boundary conditions (3.4).

The total force $P(t)$ acting on the top disc due to the fluid, balancing the applied constant load $F$ (if the inertia force $M\ddot{h}$ may be disregarded) can be calculated as follows:

$$F = P(t) = -\int\limits_0^a T^{\langle ZZ\rangle}\Big|_{Z=1} 2\pi r dr = -h^2 \int\limits_0^{a/h} T^{\langle ZZ\rangle}\Big|_{Z=1} 2\pi R dR = h^2 \int\limits_0^{a/h} \pi R^2 \frac{\partial T^{\langle ZZ\rangle}}{\partial R}\Big|_{Z=1} dR, \tag{3.15}$$

where integration by parts and the boundary condition

$$T^{\langle ZZ\rangle} = 0 \quad \text{for} \quad R = a/h, \quad Z = 1, \tag{3.16}$$

have been used. The above condition discussed in [6] means that the normal stress vanishes at the rim if the upper disc is not immersed in a fluid.

On the basis of Eqs. (3.15) and (3.10) we arrive at the equation

$$F = -\frac{3\pi a^4 \alpha_1 \dot{h}}{2h^3}\left[1+\frac{\alpha_2}{\alpha_1}\left(\frac{\ddot{h}}{\dot{h}}+\frac{\dot{h}}{h}\right)+\frac{9}{10}\frac{\alpha_3}{\alpha_1}\frac{\dot{h}}{h}\right]. \tag{3.17}$$

This ordinary second-order differential equation is highly nonlinear in $h(t)$, being a particular type of the Emden-Fowler equation (cf. [18]), the solution of which can be obtained in a finite form only in certain exceptional cases.

For Newtonian fluids ($\alpha_2 = \alpha_3 = 0$) the order of Eq. (3.17) is reduced by one and we obtain at once

$$h_N(t) = \left(\frac{1}{h_0^2}+\frac{4Ft}{3\pi a^4 \alpha_1}\right)^{-\frac{1}{2}}, \quad \dot{h}_N(0) = -\frac{2Fh_0^3}{3\pi a^4 \alpha_1}, \tag{3.18}$$

where the subscript $N$ denotes Newtonian quantities and $h_0 \equiv h(0)$.

To get other approximate solutions of Eq. (3.17) we can either seek the solution as an expansion about $h_N(t)$ for very small values of $\alpha_2, \alpha_3$ ($\alpha_2^2 = \alpha_3^2 = \alpha_2\alpha_3 = 0$) (cf. [6]), or, for finite values of $\alpha_2, \alpha_3$, solve Eq. (3.17) in the vicinity of an arbitrarily chosen instant $t'$ for which $h' = h(t')$ can be treated as a constant parameter (cf. [10]).

For the first type of solution, we take into account the fact that for a Newtonian fluid loaded with the same constant $F$

$$F = -\frac{3\pi a^4 \alpha_1 \dot{h}_N}{2h_N^3}, \quad \frac{\ddot{h}_N}{\dot{h}_N} = 3\frac{\dot{h}_N}{h_N}, \tag{3.19}$$

and then Eq. (3.17) leads to

$$\dot{h}(t) = \dot{h}_N\left[1-\frac{\dot{h}_N}{\alpha_1 h_N}\left(4\alpha_2+\frac{9}{10}\alpha_3\right)\right], \tag{3.20}$$

or to

$$h(t) = h_N\left[1-\frac{\dot{h}_N}{\alpha_1 h_N}\left(4\alpha_2+\frac{9}{10}\alpha_3\right)\ln\left(\frac{h_N}{h_0}\right)\right], \tag{3.21}$$

where $h_N$ is determined by Eq. $(3.18)_1$.

It can easily be deduced from the above results that the fluid considered will behave as a better lubricant ($\dot{h} > \dot{h}_N, h > h_N$) than a Newtonian fluid of the same viscosity $\alpha_1$ only for negative $4\alpha_2+\frac{9}{10}\alpha_3$. Thus, in terms of the normal stress differences defined for steady shearing flows (cf. [19]), we obtain the following conditions of improved lubrication:

$$4\alpha_2+\frac{9}{10}\alpha_3 < 0 \quad \text{or} \quad \nu_1-\frac{9}{11}\nu_2 > 0, \tag{3.22}$$

where $\nu_1$ denotes the first normal stress difference, while $\nu_2$ — the second one. These conditions are quite realistic, at least for the majority of viscoelastic fluids (polymer solutions and melts) for which $\nu_1$ is positive and $\nu_2$ — negative, with absolute values much smaller than $\nu_1$ (cf. e.g. [6, 19, 20]). The reason why our conclusion differs from certain previously reported theoretical results (cf. [1, 4, 5, 6]) is mainly due to the constitutive equation applied. Our simplified constitutive equation (2.18) differs from that for a second-order fluid.

To get the second type of solution we consider the following nonlinear differential equation:

$$\ddot{h}+\frac{9\alpha_3+10\alpha_2}{10\alpha_2}\frac{\dot{h}^2}{h'}+\frac{\alpha_1}{\alpha_2}\dot{h}+\frac{2Fh'^3}{3\pi a^4\alpha_2} = 0, \tag{3.23}$$

where $h' \equiv h(t')$ is a constant parameter. In the case of a Newtonian fluid, Eq. (3.17) leads to

$$h_N(t) = h'_N-\frac{2Fh'^3}{3\pi a^4\alpha_1}(t-t'), \qquad \dot{h}_N \equiv \dot{h}'_N = -\frac{2Fh'^3}{3\pi a^4\alpha_1}, \tag{3.24}$$

where $h'_N \equiv h_N(t')$.

Equation (3.23) can be integrated in quadratures and its general solution satisfying the initial conditions $h(t') = h'$, $\dot{h}(t') = \dot{h}'$ depends on sign of the expression:

$$\Delta = \left(\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}+2\dot{h}_N\right)\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}. \tag{3.25}$$

Since in our case it is reasonable to assume that (cf. (3.22))

$$10\alpha_2+9\alpha_3 > 0 \quad \text{or} \quad 4\nu_1+9\nu_2 > 0, \tag{3.26}$$

where $\nu_1, \nu_2$ are the corresponding normal stress differences, we obtain either

$$\text{A:} \quad \Delta > 0 \quad \text{if} \quad \frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3} > -2\dot{h}_N, \tag{3.27}$$

or

$$\text{B:}\quad \Delta < 0 \quad \text{if} \quad \frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3} < -2\dot{h}_N. \tag{3.28}$$

For $\Delta > 0$ (the case A) we arrive at

$$\dot{h}(t) = \sqrt{\Delta}\,\mathrm{th}\left[\sqrt{\Delta}\left(C+\frac{10\alpha_2+9\alpha_3}{10\alpha_2}\frac{t}{h'}\right)\right]-\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}, \tag{3.29}$$

$$h(t) = \frac{10\alpha_2}{10\alpha_2+9\alpha_3}h'\ln\mathrm{ch}\left[\sqrt{\Delta}\left(C+\frac{10\alpha_2+9\alpha_3}{10\alpha_2}\frac{t}{h'}\right)\right]-\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}t+D, \tag{3.30}$$

where

$$\begin{aligned} C &= \frac{1}{\sqrt{\Delta}}\,\mathrm{Ar\,th}\left[\frac{1}{\sqrt{\Delta}}\left(\dot{h}'+\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}\right)\right]-\frac{10\alpha_2+9\alpha_3}{10\alpha_2}\frac{t'}{h'},\\ D &= h'-\frac{10\alpha_2}{10\alpha_2+9\alpha_3}h'\ln\mathrm{ch}\left[\sqrt{\Delta}\left(C+\frac{10\alpha_2+9\alpha_3}{10\alpha_2}\frac{t'}{h'}\right)\right]+\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}t'. \end{aligned} \tag{3.31}$$

Similarly, for $\Delta < 0$ (the case B) we have

$$\dot{h}(t) = -\sqrt{-\Delta}\,\mathrm{tg}\left[\sqrt{-\Delta}\left(C+\frac{10\alpha_2+9\alpha_3}{10\alpha_2}\frac{t}{h'}\right)\right]-\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}, \tag{3.32}$$

$$h(t) = \frac{10\alpha_2}{10\alpha_2+9\alpha_3}h'\ln\left|\cos\left[\sqrt{-\Delta}\left(C+\frac{10\alpha_2+9\alpha_3}{10\alpha_2}\frac{t}{h'}\right)\right]\right| - \frac{5\alpha_1 h'}{10\alpha_2++9\alpha_3}t+D, \tag{3.33}$$

where

$$\begin{aligned} C &= \frac{1}{\sqrt{-\Delta}}\,\mathrm{arctg}\left[\frac{1}{\sqrt{-\Delta}}\left(-\dot{h}'-\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}\right)\right]-\frac{10\alpha_2+9\alpha_3}{10\alpha_2}\frac{t'}{h'},\\ D &= h'-\frac{10\alpha_2}{10\alpha_2+9\alpha_3}h'\ln\left|\cos\left[\sqrt{-\Delta}\left(C+\frac{10\alpha_2+9\alpha_3}{10\alpha_2}\frac{t'}{h'}\right)\right]\right|+\frac{5\alpha_1 h'}{10\alpha_2+9\alpha_3}t'. \end{aligned} \tag{3.34}$$

The above solutions may be useful to derive conditions under which the fluid considered will behave as a better lubricant than a Newtonian fluid of the same viscosity $\alpha_1$. To this end we shall analyse the conditions under which either $h(t)$ or $\dot{h}(t)$ are greater than the corresponding Newtonian quantities $h_N(t)$ or $\dot{h}_N(t)$.([2])

Taking into account the solutions (3.30), (3.29) and the inequality (3.27), we can deduce that

$$h(t) \underset{t\to t'}{>} h_N(t) \quad \text{if} \quad \{\alpha_2 < 0,\ 10\alpha_2+9\alpha_3 > 0,\ h' > h'_N\}, \tag{3.35}$$

$$\dot{h}(t) \underset{t\to t'}{>} \dot{h}_N(t) \quad \text{if} \quad \{10\alpha_2+9\alpha_3 > 0,\ \dot{h}' > \dot{h}'_N\}. \tag{3.36}$$

The above conditions show that if the distance between the discs $h'$ and the rate $\dot{h}'$ at any instant $t'$ are greater than the corresponding Newtonian quantities, a similar tendency

([2]) Let us note that if $h(t) > h_N(t)$ for any $t$, the fluid is really a better lubricant. This is not true if $\dot{h}(t) > \dot{h}_N(t)$ and, simultaneously, $h_N(t) < h_N(t)$.

will be observed after a very short time lapse $t-t'$, only for negative $\alpha_2$ and positive $10\alpha_2+9\alpha_3$. These inequalities are again quite realistic for the majority of viscoelastic fluids for which the first normal stress difference is positive and much greater than the absolute value of the negative second normal stress difference (cf. (3.26)).

Taking into account the solutions (3.33), (3.32) and the inequality (3.28), we can also deduce that

$$h(t) \underset{t\to t'}{>} h_N(t) \quad \text{if} \quad \{\alpha_2 < 0,\ 10\alpha_2+9\alpha_3 > 0,\ h' > h'_N\}, \tag{3.37}$$

$$\dot{h}(t) \underset{t\to t'}{>} \dot{h}_N(t) \quad \text{if} \quad \{10\alpha_2+9\alpha_3 > 0,\ \dot{h}'_N > \dot{h}'\}. \tag{3.38}$$

The above results are very similar to those described by the inequalities (3.35) and (3.36). The only difference is connected with the rate $\dot{h}'$. Bearing in mind the inequality (3.28), it is easy to note that such cases are possible for slightly higher loads.

## 4. Other viscoelastic properties of squeezing flows

BRINDLEY *et al.* [6] reported that for certain viscoelastic fluids under more severe loading conditions (or higher Deborah numbers) the distance $h(t)$ did not always decrease monotonically with time as one would expect from inelastic analysis. Sometimes, they

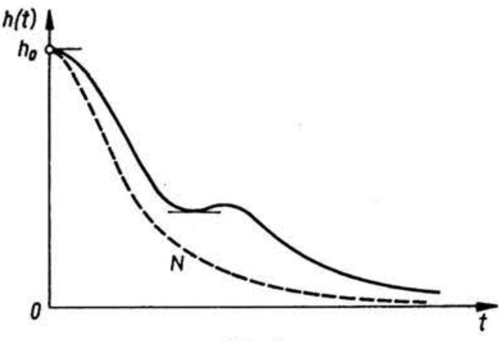


FIG. 2.

observed a "bounce" appearing on the experimental $h(t)$ curves (Fig. 2) being a manifestation of solid-like or elastic behaviour.

Although our solutions are valid only for light loading conditions (or low Deborah numbers), there exists a possibility of similar elastic or solid-like behaviour. To show this we shall seek conditions under which either the negative rate $\dot{h}(t)$ may change its sign or the distance $h(t)$ may become greater than the distance $h'$ at slightly earlier instant $t'$. The case of $\Delta > 0$ as well as the case of $\Delta < 0$ must be considered separately.

For $\Delta > 0$ (cf. inequality (3.27)) we can deduce on the basis of Eqs. (3.29) and (3.30) that

$$\text{neither} \quad \dot{h}(t) \underset{t\to t}{\geqslant} 0 \quad \text{nor} \quad h(t) \underset{t\to t'}{\geqslant} h' \quad \text{if} \quad \{\alpha_2 < 0\}, \tag{4.1}$$

what means that a "bounce" is not possible for negative $\alpha_2$.

On the other hand, for $\Delta < 0$ (cf. inequality (3.28)) we can deduce on the basis of Eqs. (3.32) and (3.33) that

$$\dot{h}(t) \underset{t \to t'}{\geq} 0 \quad \text{if} \quad \{\alpha_2 < 0,\ 10\alpha_2 + 9\alpha_3 > 0\}, \tag{4.2}$$

$$h(t) \underset{t \to t'}{\geq} h' \quad \text{if} \quad \{\alpha_2 < 0,\ 10\alpha_2 + 9\alpha_3 > 0\}. \tag{4.3}$$

The above results show that a "bounce" on the $h(t)$ curve may appear for negative $\alpha_2$ and positive $10\alpha_2+9\alpha_3$, if simultaneously the inequality (3.28) holds. This inequality corresponds to the case of loading conditions slightly heavier than those described by the inequality (3.27). A small but finite time lapse $t-t'$ after which $\dot{h}(t)$ may change its sign can be estimated from the following relation:

$$\dot{h}(t) \geq 0 \quad \text{if} \quad \left\{ -\frac{10\alpha_2+9\alpha_3}{10\alpha_2} \frac{t-t'}{h'} \leq \frac{\dot{h}'}{\Delta} \right\}, \tag{4.4}$$

where $\Delta < 0$, $\alpha_2 < 0$ and $10\alpha_2+9\alpha_3 > 0$.

Since our present considerations can be repeated for many various instants of time denoted by $t'$, we do not exclude the possibility of several "bounces" existing on the $h(t)$ curves. We want to emphasize, moreover, that from the fact that "bounces" are possible on the $h(t)$ curves it does not result that they should appear or could be observable for lighter loading conditions.

## Appendix A

To prove the corresponding representation theorem we use the following definitions of the partial Rivlin-Ericksen kinematic tensors:

$$\mathbf{A}_1' = \mathbf{L}'^T + \mathbf{L}', \quad \mathbf{A}_1'' = \mathbf{L}''^T + \mathbf{L}'', \tag{A.1}$$

$$\mathbf{A}_2' = \dot{\mathbf{A}}_1' + \mathbf{A}_1'\mathbf{L}' + \mathbf{L}'^T\mathbf{A}_1', \quad \mathbf{A}_2'' = \dot{\mathbf{A}}_1'' + \mathbf{A}_1''\mathbf{L}'' + \mathbf{L}''^T\mathbf{A}_1'', \tag{A.2}$$

where $\mathbf{L}'$ and $\mathbf{L}''$ are defined by Eqs. (2.15) and superposed dots denote the material time derivatives.

Moreover, the following lemma is very useful (cf. [11]):

LEMMA. Let [S] be a $3\times3$ diagonal matrix and [W] $3\times3$ skew-symmetric matrix:

$$[\mathbf{S}] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad [\mathbf{W}] = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ y & -z & 0 \end{bmatrix}, \quad a \neq b \neq c \neq a, \tag{A.3}$$

then:

$$[\mathbf{SW}] = [\mathbf{WS}] \quad \text{if, and only if,} \quad x = y = z = 0. \tag{A.4}$$

This lemma can be proved in a straightforward manner by direct multiplication of matrices.

THEOREM. *If the partial Rivlin-Ericksen kinematic tensors* $\mathbf{A}_1'$ *and* $\mathbf{A}_2''$ *have three eigen-values distinct, the partial velocity gradients* $\mathbf{L}'$ *and* $\mathbf{L}''$ *are uniquely determined by* $\mathbf{A}_1', \mathbf{A}_2'$ *and* $\mathbf{A}_1'', \mathbf{A}_2''$, *respectively.*

The proof of the above theorem will be presented only for the primed quantities $\mathbf{L}', \mathbf{A}_1', \mathbf{A}_2'$; for $\mathbf{L}'', \mathbf{A}_1'', \mathbf{A}_2''$ it can be repeated in an analogous way.

Let $\mathbf{L}'$ not be uniquely determined by $\mathbf{A}_1', \mathbf{A}_2'$. Then, taking an $\overline{\mathbf{L}}'$ such that

$$\mathbf{A}_1' = \mathbf{L}'^T + \mathbf{L}' = \overline{\mathbf{L}}'^T + \overline{\mathbf{L}}', \tag{A.5}$$

$$\mathbf{A}_2' = \dot{\mathbf{A}}_1' + \mathbf{A}_1'\mathbf{L}' + \mathbf{L}'^T\mathbf{A}_1' = \dot{\mathbf{A}}_1' + \mathbf{A}_1'\overline{\mathbf{L}}' + \overline{\mathbf{L}}'^T\mathbf{A}_1', \tag{A.6}$$

we obtain from Eq. (A.5)

$$(\mathbf{L}' - \overline{\mathbf{L}}')^T = -(\mathbf{L}' - \overline{\mathbf{L}}'); \tag{A.7}$$

thus the difference $\mathbf{L}' - \overline{\mathbf{L}}'$ is skew-symmetric. From Eq. (A.6) we also see that $\mathbf{L}' - \overline{\mathbf{L}}'$ commutes with $\mathbf{A}_1'$, namely

$$(\mathbf{L}' - \overline{\mathbf{L}}')\mathbf{A}_1' = \mathbf{A}_1'(\mathbf{L} - \overline{\mathbf{L}}'). \tag{A.8}$$

Since, according to the lemma, $\mathbf{A}_1'$ does not commute with any non-zero skew-symmetric tensor, we have $\mathbf{L}' - \overline{\mathbf{L}}' \equiv 0$. Thus, $\mathbf{L}'$ is uniquely determined by $\mathbf{A}_1'$ and $\mathbf{A}_2'$.

## Appendix B

The proof that the inertialess equations of equilibrium in the system of convected coordinates $R, \theta, Z$ defined at time $t$ are of the form (3.11) can be outlined as follows.

For any system of curvilinear coordinates $X^\alpha$ ($\alpha = 1, 2, 3$) we have in general (cf. e.g. [14])

$$\partial_\alpha T^{\alpha\beta} + \begin{Bmatrix} \beta \\ \mu\varrho \end{Bmatrix} T^{\mu\varrho} + \begin{Bmatrix} \varrho \\ \mu\varrho \end{Bmatrix} T^{\mu\beta} = 0, \tag{B.1}$$

where $T^{\alpha\beta}$ are contravariant components of the stress tensor and $\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}$ are Christoffel symbols of the second kind. From the definition (3.2) it results that

$$[g_{\alpha\beta}(t)] = \begin{bmatrix} h^2 & 0 & 0 \\ 0 & R^2h^2 & 0 \\ 0 & 0 & h^2 \end{bmatrix} \quad \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -R, \quad \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{R}, \tag{B.2}$$

where $g_{\alpha\beta}$ denote covariant components of the variable metric tensor. Thus, Eqs. (B.1) lead to

$$\begin{gathered} \partial_R T^{RR} + \partial_Z T^{ZR} + \frac{1}{R} T^{RR} - RT^{\theta\theta} = 0, \\ \partial_R T^{RZ} + \partial_Z T^{ZZ} + \frac{1}{R} T^{RZ} = 0. \end{gathered} \tag{B.3}$$

Bearing in mind the fact that physical components of the stress tensor are defined as follows (cf. [14]):

$$T^{\langle\alpha\beta\rangle} = \sqrt{\frac{g_{\beta\beta}}{g^{\alpha\alpha}}}\, T^{\alpha\beta} \quad \text{(no summation)}, \tag{B.4}$$

we immediately arrive at Eqs. (3.11).

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