Archives of Mechanics • Archiwum Mechaniki Stosowanej • 31, 2, pp. 151-163, Warszawa 1979

Limit analysis of thick and thin circular plates subjected to transverse pressure

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Two LOWER bound analyses for the calculation of the limit transverse pressure on simply supported and clamped plates are presented: a thick plate analysis which takes account of all stresses and which is based on the von Mises yield criterion, and a thin plate with shear analysis which ignores the through thickness direct stresses and which uses a close approximation to the true thin plate interaction yield surface. The lower bound limit pressures are optimised using a non-linear programming method. Results for the different ranges of plate radius to thickness are compared between the two analyses and with results by others.

Przedstawiono dwie dolne oceny nośności granicznej wolno podpartych i utwierdzonych płyt obciążonych poprzecznie; dla płyt grubych ocenę znaleziono uwzględniając pełny stan naprężenia i przyjmując warunek plastyczności Misesa; dla płyt cienkich pominięto naprężenia normalne w kierunku prostopadłym do płyty, a za warunek plastyczności przyjęto pewną aproksymację ścisłego warunku obowiązującego dla tych płyt. Najlepszą dolną ocenę nośności określono metodą programowania nieliniowego. Obliczone dla różnych stosunków promienia płyty do jej grubości wartości obu ocen porównano z wynikami uzyskanymi przez innych autorów.

Представлены две нижние оценки предельной несущей способности свободноподпертых и закрепленных плит, нагруженных поперечным образом; для толстых плит оценка найдена, учитывая полное напряженное состояние и принимая условие пластичности Мизеса, для тонких плит пренебрегается нормальными напряжениями в перпендикулярном направлении к плите, а условие пластичности принято в виде некоторой аппроксимации точного условия, обязывающего для этих плит. Наилучшая нижняя оценка несущей способности определена методом нелинейного программирования. Рассчитанные для разных отношений радиуса плиты к ее толщине значения обоих оценок сравнены результатами полученными другими авторами.

Notation

r_0 ratio of plate radius to thickness,
r ratio of radial coordinate to plate thickness,
z ratio of through thickness coordinate to plate thickness,
p pressure,
α ratio of radius of loaded circle to plate radius,
σ_0 yield stress in simple tension,
p^* dimensionless pressure $= \alpha^2 r_0^2 p/\sigma_0$,
σ_r ratio of radial stress to yield stress in simple tension,
σ_θ ratio of circumferential stress to yield stress in simple tension,
σ_z ratio of transverse stress to yield stress in simple tension,
$\tau_{zr} = \tau_{rz}$ ratio of shear stress to yield stress in simple tension.

Q_r transverse shear force,
M_r radial bending moment,
M_θ circumferential bending moment,
$q_r = Q_r/\sigma_0$ dimensionless shear force,
$m_r = 4M_r/\sigma_0$ dimensionless radial bending moment,
$m_\theta = 4M_\theta/\sigma_0$ dimensionless circumferential bending moment (N.B. thickness of plate = = unity),
$\mathbf{x}$ vector of stress parameters,
A_{nm}, B_{qv}, C_{qv} independent variables of stress vector,
F_y yield function.

Other symbols are defined as they appear in the text.

1. Introduction

THE THEORETICAL limit behaviour of rigid plastic thin isotropic circular plates has been studied by several authors [1-4]. HOPKINS and PRAGER [1] evaluated the limit load for simply supported and fixed circular plates subjected to circular and annular loads for a Tresca yield condition by ignoring the effect of shear on yielding. HOPKINS and WANG [2] also solved the same problem for circular loading, using the von Mises yield criterion, again neglecting the effect of shear force. The resulting differential equilibrium equation is nonlinear and is integrated numerically. DRUCKER and HOPKINS [3] discussed the case of a circular plate with overhangs for uniform and concentrated central loads using the Tresca yield criterion, again ignoring shear effect. SAWCZUK and JAEGER [4] solved the problem of circular plates subjected to line loads and annular plates subjected to uniform load using a yield condition that ignores the interaction between shear and bending.

The shear-bending interaction in circular plates was studied by BROTCHIE [5]. The Tresca yield criterion was used, but as a result of the assumed shear stress distribution across the plate thickness the analysis was only approximate. SHAPIRO [6] proposed a parametric form for the von Mises yield criterion for an arbitrary thin shell and derived general relations in an integrated form. SAWCZUK and DUSZEK [7] examined the effect of shear force on the load carrying capacity of plates by considering a simply supported circular plate under uniform circular loading. Two alternative yield conditions were used, a limited interaction yield surface between moment and shear force for a Tresca material and a linearised interaction surface for a von Mises material. The method presented, in spite of taking shear into account, ignored the effects of the direct transverse stress (like all the previous analyses). As such, it is not strictly applicable to thick plates.

In this paper two separate analyses will be presented. The first is an analysis which, in principle, applies to plates of any thickness but which in the form given is particularly suited to thick plates. The second is a thin plate analysis which takes into account the effect of shear on yielding. Both analyses are lower bounds for a rigid plastic material and use a nonlinear optimisation method to arrive at the optimum stress fields and their associated limit loads. The results presented are compared with the thin plate — without shear results by HOPKINS and WANG [2] and with the thin plate with shear results by SAWCZUK and DUSZEK [7].

2. Thick plate analysis

2.1. General

The method of analysis adopted here is that proposed by DINNO and GILL in a previous paper [8] in which it was formulated as a general method for shells of revolution and was utilised in determining the lower bound limit pressure for a thick cylindrical vessel with thick torispherical ends. This method considers a three-dimensional stress formulation in terms of stresses rather than stress resultants and uses a basic yield criterion of the material, assumed to be von Mises. The stress field takes full account of the transverse and shear stresses. Some of the stresses are expressed in terms of an independent set of variables and the remaining stresses are found from equilibrium and boundary conditions. Using a nonlinear optimisation process, namely the Sequential Unconstrained Minimisation Technique due to CARROLL [9], the variables are chosen to maximise the lower bound load with the constraint that the material yield criterion must not be violated anywhere in the plate.

For an element within the thickness of an axisymmetric plate loaded axisymmetrically, the only non-vanishing stresses are σ_r, σ_z, σ_θ and $\tau_{rz} = \tau_{zr}$. The equations of equilibrium are

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{rz}}{\partial z} = 0, \tag{2.1}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0. \tag{2.2}$$

The yield condition takes the form

$$F_y \leqslant 1,$$

where F_y is the yield function. Using the von Mises yield criterion the above inequality becomes

$$3\tau_{rz}^2 + \sigma_r^2 + \sigma_\theta^2 + \sigma_z^2 - \sigma_r\sigma_\theta - \sigma_r\sigma_z - \sigma_\theta\sigma_z \leqslant 1. \tag{2.3}$$

Of the four stresses it turns out to be most convenient to postulate τ_{rz} and σ_r and to determine σ_z and σ_θ in terms of them. The stresses τ_{rz} and σ_r are expressed in terms of a finite set of parameters $\mathbf{x}$ (which includes the applied load intensity p) and are expanded as polynomials in the radial and the through thickness coordinates.

2.2. Simply supported circular plate

The following expression for τ_{rz} is assumed to apply over the whole plate:

$$\tau_{rz} = \sum_{n=1}^{ns} \sum_{m=1}^{ms} A_{nm} z^n (1-z) r^m + C(r) z (1-z), \tag{2.4}$$

where A_{nm} is a group of variables which form part of the $\mathbf{x}$ vector in the optimisation and $C(r)$ is a function to be determined later. This expression for τ_{rz} satisfies the boundary conditions at $z = 0$ and at $z = 1$. Substituting for τ_{rz} in Eq. (2.2) and integrating,

$$\sigma_z = -\sum_{n=1}^{ns}\sum_{m=1}^{ms} A_{nm} r^{m-1}(m+1)\left(\frac{1}{n+1} - \frac{z}{n+2}\right) z^{n+1} - \left[C'(r) + \frac{C(r)}{r}\right]\left(\frac{z^2}{2} - \frac{z^3}{3}\right) + D(r). \tag{2.5}$$

Noting that at $z = 0$, $\sigma_z = -p(r)$ where $p(r)$ is the distributed load on the plate (counted positive along the positive direction of z), we get $D(r) = -p(r)$. Further, with $\sigma_z = 0$ at $z = 1$ (free boundary) and by integrating and noting that $\tau_{rz} = 0$ at $r = 0$, $C(r)$ can be determined. This finally leads to

$$\tau_{rz} = \sum_{n=1}^{ns}\sum_{m=1}^{ms} A_{nm}\left[r^m z^n(1-z) - \frac{6z(1-z)r^m}{(n+1)(n+2)}\right] - \frac{6z(1-z)}{r}\int_0^r p(r) r\,dr \tag{2.6}$$

and

$$\sigma_z = \sum_{n=1}^{ns}\sum_{m=1}^{ms} A_{nm}\left[-r^{m-1}(m+1)\left(\frac{1}{n+1} - \frac{z}{n+2}\right) z^{n+1} + \frac{r^{m-1}(m+1)}{(n+1)(n+2)}(3-2z)z^2\right] + [(3-2z)z^2 - 1]p(r). \tag{2.7}$$

For the radial stress σ_r a discontinuity is allowed within the thickness at $z = a$. Such a discontinuity is statically admissible as it affects only σ_r and σ_θ and its inclusion permits a better utilisation of the cross section to support radial and circumferential bending moments.

For $a \geqslant z \geqslant 0$, σ_r is postulated as

$$\sigma_r = \sum_{q=1}^{qs}\sum_{v=1}^{vs} B_{qv} z^q (r_0 - r)^v + \sum_{v=1}^{vs} B_{0v}(r_0 - r)^v. \tag{2.8}$$

For $1.0 \geqslant z > a$

$$\sigma_r = \sum_{q=1}^{qc}\sum_{v=1}^{vc} C_{qv} z^q (r_0 - r)^v + \sum_{v=1}^{vc} C_{0v}(r_0 - r)^v. \tag{2.9}$$

These distributions for σ_r satisfy the boundary conditions at the simply supported edge. At such an edge ($r = r_0$) it is assumed that $\sigma_r = 0$ and that the vertical external forces are supported by the shear reaction stress $\tau_{rz}(r = r_0)$, (see Fig. 1). From equilibrium it follows that for $a \geqslant z \geqslant 0$

$$\sigma_\theta = \sum_{n=1}^{ns}\sum_{m=1}^{ms} A_{nm} r^{m+1}\left[nz^{n-1} - (n+1)z^n - \frac{6(1-2z)}{(n+1)(n+2)}\right] - 6(1-2z)\int_0^r p(r) r\,dr + \sum_{q=1}^{qs}\sum_{v=1}^{vs} B_{qv} z^q[-vr(r_0 - r)^{v-1} + (r_0 - r)^v] + \sum_{v=1}^{vs} B_{0v}\left[-vr(r_0 - r)^{v-1} + (r_0 - r)^v\right] \tag{2.10}$$

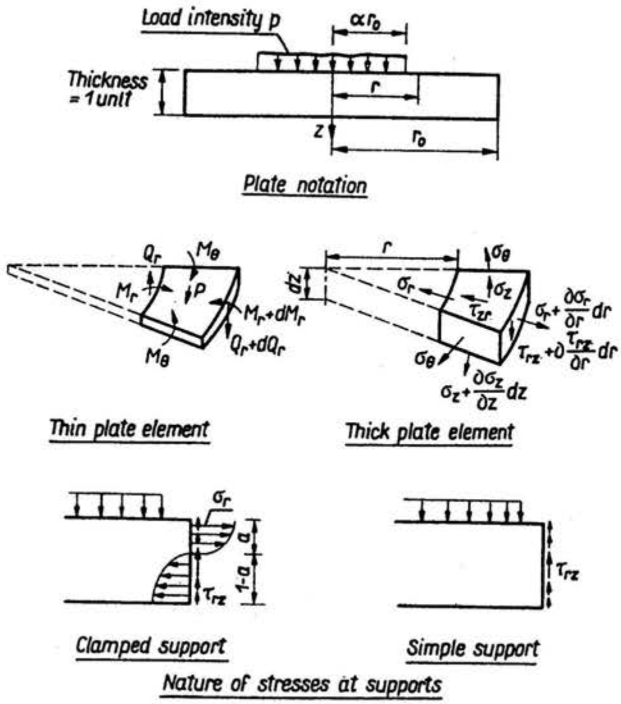

FIG. 1. Notation, elements and support conditions.

and for $1.0 \geqslant z > a$

$$\sigma_\theta = \sum_{n=1}^{ns}\sum_{m=1}^{ms} A_{nm} r^{m+1}\left[nz^{n-1}-(n+1)z^n-\frac{6(1-2z)}{(n+1)(n+2)}\right] \tag{2.11}$$

$$-6(1-2z)\int_0^r p(r)rdr+\sum_{q=1}^{qc}\sum_{v=1}^{vc} C_{qv}z^q[-vr(r_0-r)^{v-1}+(r_0-r)^v]$$

$$+\sum_{v=1}^{vc} C_{0v}[-vr(r_0-r)^{v-1}+(r_0-r)^v].$$

2.3. Clamped circular plate

The stresses τ_{rz} and σ_z follow the same boundary conditions as those for the simply supported case. Hence the expressions (2.6) and (2.7) for τ_{rz} and σ_z are valid for the present case. The clamped end is assumed to provide a bending constraint but the inplane resultant force is assumed to be zero. (The nature of the stresses at the support are shown in Fig. 1). As in the case of the simply supported plate a discontinuity in σ_r is allowed at $z = a$. σ_r is postulated as follows, which satisfies

$$\int_0^a \sigma_r dz+\int_a^1 \sigma_r dz = 0 \quad \text{at} \quad r = r_0; \tag{2.12}$$

for $a \geqslant z \geqslant 0$

$$\sigma_r = \sum_{q=1}^{qs}\sum_{v=1}^{vs} B_{qv} z^q r^v + \sum_{v=1}^{vs} B_{0v} r^v + \sum_{q=1}^{qs} B_{q0} z^q + B_{00}. \tag{2.13}$$

Consequently, from equilibrium

$$\sigma_\theta = \sum_{n=1}^{ns}\sum_{m=1}^{ms} A_{nm} r^{m+1}\left[nz^{n-1} + (n+1)z^n - 6\,\frac{(1-2z)}{(n+1)(n+2)}\right] - 6(1-2z)\int_0^r p(r)rdr + \sum_{q=1}^{qs}\sum_{v=1}^{vs} B_{qv}(v+1)z^q r^v + \sum_{v=1}^{vs} B_{0v}(v+1)r^v + \sum_{q=1}^{qs} B_{q0} z^q + B_{00}; \tag{2.14}$$

for $1.0 \geqslant z > a$

$$\sigma_r = \sum_{q=1}^{qc}\sum_{v=1}^{vc} C_{qv} z^q r^v + \sum_{v=1}^{vc} C_{0v} r^v + \sum_{q=1}^{qc} C_{q0} z^q + C_{00}. \tag{2.15}$$

Substituting in the boundary condition (2.12) defines one of the C variables in terms of the others and the B variables. If, in particular, C_{00} is chosen to be eliminated, its value is given by

$$C_{00} = -\frac{1}{1-a}\left\{\sum_{q=1}^{qs}\sum_{v=1}^{vs} B_{qv}\frac{a^{q+1}}{q+1} r_0^v + \sum_{v=1}^{vs} B_{0v} a r_0^v + \sum_{q=1}^{qs} B_{q0}\frac{a^{q+1}}{q+1} + B_{00} a + \sum_{q=1}^{qc}\sum_{v=1}^{vc} C_{qv}\left[\frac{1-a^{q+1}}{q+1}\right] r_0^v + \sum_{v=1}^{vc} C_{0v}(1-a) r_0^v + \sum_{q=1}^{qc} C_{q0}\left[\frac{1-a^{q+1}}{q+1}\right]\right\}. \tag{2.16}$$

Finally,

$$\sigma_\theta = \sum_{n=1}^{ns}\sum_{m=1}^{ms} A_{nm} r^{m+1}\left[nz^{n-1} - (n+1)z^n - 6\,\frac{(1-2z)}{(n+1)(n+2)}\right] - 6(1-2z)\int_0^r p(r)rdr + \sum_{q=1}^{qc}\sum_{v=1}^{vc} C_{qv}(v+1)z^q r^v + \sum_{v=1}^{vc} C_{0v}(v+1)r^v + \sum_{q=1}^{qc} C_{q0} z^q + C_{00}. \tag{2.17}$$

2.4. Computation

As stated earlier the Sequential Unconstrained Minimisation Technique due to CARROLL [9] was used for the optimisation. The variables in the **x** vector were optimised in order to determine the highest value of the lower bound limit load subject to the yield constraint

(2.3) applied at pre-chosen points in the domains of the plate. For the purpose of a typical computation the load $p(r)$ was taken as uniform $= p$. Figures 2 and 3 show results for the simply supported and the clamped plates, respectively.

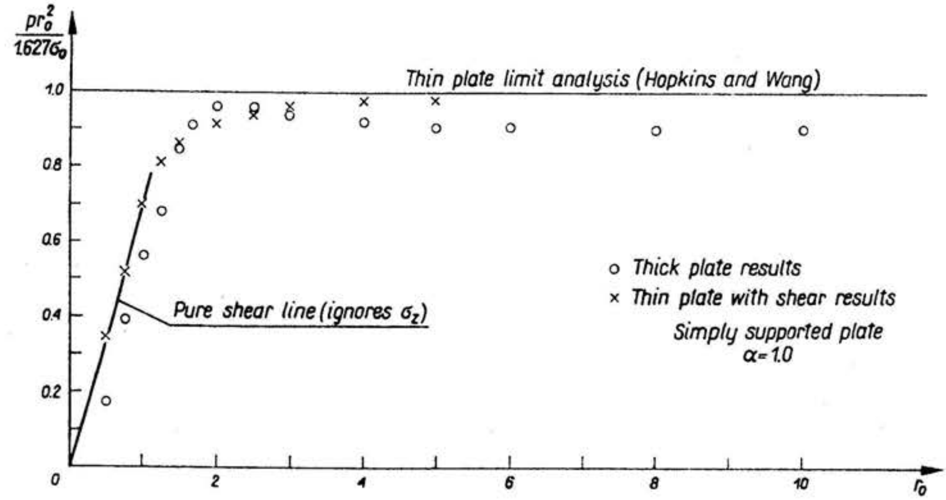

FIG. 2.

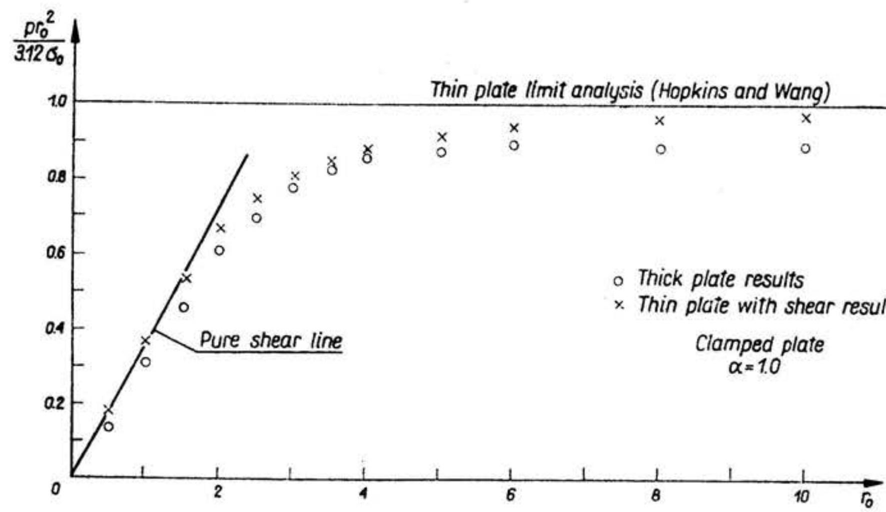

FIG. 3.

In each figure the dimensionless limit load (as a ratio of the calculated limit load to the thin plate —without shear limit load based on a von Mises yield criterion) is plotted against r_0, the radius to thickness ratio of the plate.

For the parameters investigated here the number of variables were: $ns = 6$, $ms = 6$, $qs = 4$, $vs = 7$, $qc = 4$ and $vc = 7$. The number of constraint points across the thickness was 12 (6 for zone $a \geqslant z \geqslant 0$ and 6 for zone $1.0 \geqslant z \geqslant a$) and the total number of constrained points for the plate was 120. The optimum value of a for the cases considered lay between 0.45 and 0.55.

At the end of the optimisation process the yield function was calculated at a larger number of points (usually 400) in order to survey the extent of yield violation. Yield violation was no more than about 3% for all values of r_0. A suitable reduction factor was imposed on p to produce a safe value of p for which no yield violation occurred.

3. Thin plate with shear analysis

3.1. General

The lower bound analysis is formulated in terms of stress resultants. The analysis is made for a simply supported and clamped circular plate of unit thickness and radius r_0. For simplicity the loading is assumed to be a uniform pressure p acting over a central area of radius αr_0 (see Fig. 1) where $0 < \alpha \leqslant 1$.

For an axisymmetric stress state the non-vanishing dimensionless stress resultants are m_r, m_θ and q_r. These must satisfy the following thin-plate equations of equilibrium

$$\frac{d}{ds}(sq_r) = \frac{p^* s}{\alpha^2 r_0}, \tag{3.1}$$

$$s\frac{dm_r}{ds} + m_r - m_\theta = 4r_0 sq_r, \tag{3.2}$$

where $s = r/r_0$ and p^* is the dimensionless pressure given by $p^* = \alpha r_0^2 p/\sigma_0$.

3.2. Formulation of stress resultants

Integration of Eq. (3.1) gives

$$q_r = \frac{1}{2}p^* \frac{\beta}{r_0}, \quad \text{where} \quad \beta = \frac{s}{\alpha^2} \quad \text{if } s < \alpha \quad \text{and} \quad \frac{1}{s} \quad \text{if } s \geqslant \alpha. \tag{3.3}$$

Equation (3.2) gives

$$m_\theta = s\frac{dm_r}{ds} + m_r - 2\beta sp^*. \tag{3.4}$$

The bending moment m_r is approximated by parabolas in a series of zones. The dimensionless pressure p^* is taken as the first component of the $\mathbf{x}$ vector, the rest of which describes m_r.

Let the ith zone be defined between $S = S_i$ and $S = S_{i+1}$ and let $m_r = X_{2i}$ at $S = S_i$ and X_{2i+2} at $S = S_{i+1}$. Thus in this zone m_r may be written as

$$m_r = X_{2i}\frac{(S-S_{i+1})}{S_i - S_{i+1}} + X_{2i+1}(S-S_i)(S-S_{i+1}) + X_{2i+2}\frac{(S-S_i)}{S_{i+1}-S_i}. \tag{3.5}$$

From Eq. (3.4)

$$m_\theta = X_{2i}\frac{(2S-S_{i+1})}{S_i-S_{i+1}} + X_{2i+1}[3S^2 - 2S(S_i+S_{i+1}) + S_i S_{i+1}] + X_{2i+2}\frac{2S-S_i}{S_{i+1}-S_i}. \tag{3.6}$$

If the number of zones of equal length used in the range $0 \leqslant S \leqslant \alpha$ is n_1 and in the range $\alpha \leqslant S \leqslant 1$ is n_2, then for a simply supported plate with $m_r = 0$ at $S = 1$, the variable $X_{2n_1+2n_2+2} = 0$.

3.3. Yield condition

For a thin plate theory it is necessary to make the kinematic assumption that plane normals to the plate middle surface remain plane (though not necessarily normal). It then becomes possible to obtain a yield condition in terms of m_r, m_θ and q_r. Defining $Q_m = m_r^2+m_\theta^2-m_r m_\theta$ and $Q_q = 3q_r^2$, it can be shown that for the exact thin plate yield condition $Y_0 = 1$ (with associated limit load P_0), Y_0 is a function of Q_m and Q_q only [6]. If we define $Y_1 = Q_m+Q_q$ and use $Y_1 = 1$ as an approximation to Y_0, and if P_1 is the associated limit load, it can be shown [10] that $0.955\, P_0 \leqslant P_1 \leqslant P_0$. A very good approximation for Y_0, accurate to 1/2%, was suggested by Ivanov (see [10]) and is given by

$$Q_q+Q_m-\frac{0.25Q_qQ_m}{Q_q+0.48Q_m} = 1.$$

Y_1 is used during the optimisation and the Ivanov yield condition is used afterwards for a final check on the stress field. If required a reduction factor is imposed so that the Ivanov yield condition is nowhere violated.

3.4. Computation

The principle of computation used here is exactly that used for the thick plate analysis. The yield function is constrained at, say, nc points in each zone. At the end of the

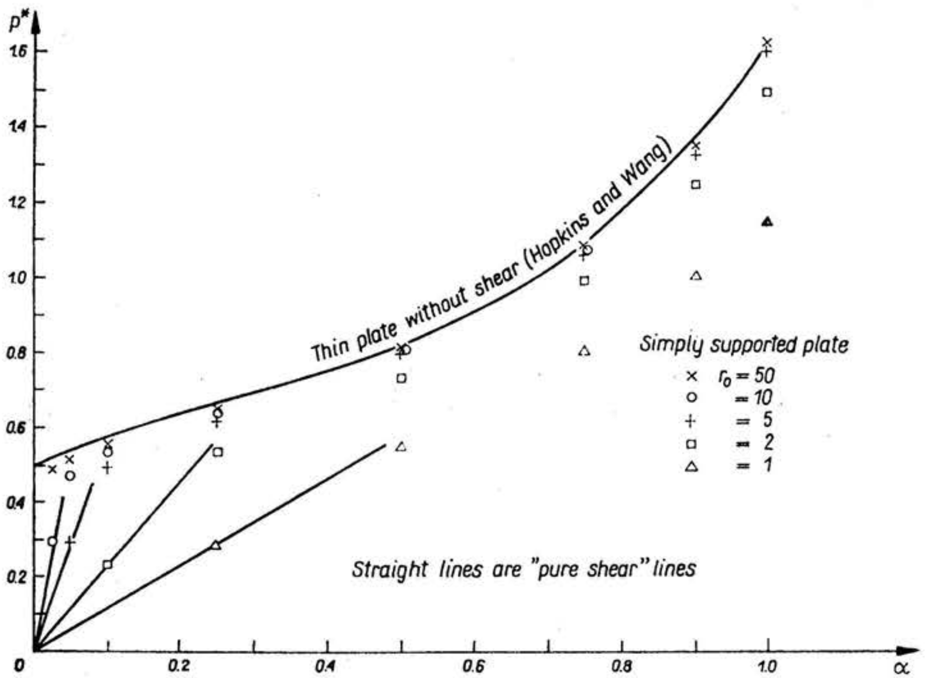

FIG. 4.

optimisation process the approximate yield surface Y_1 and the almost exact Ivanov yield function were evaluated at a large number of points (about twenty for each zone) and a suitable reduction factor for p imposed. This is $1/\sqrt{Y_{max}}$ where Y_{max} is the largest value of the Ivanov yield function (since this is quadratic in the stress resultants).

The results for this analysis are presented in the form of four graphs. The first two are for fully loaded plates ($\alpha = 1$) for both the simply supported and clamped cases. These are shown in Figs. 2 and 3 which enable direct comparison to be made with the thick plate results. The second two are for the dimensionless pressure p^* against α for

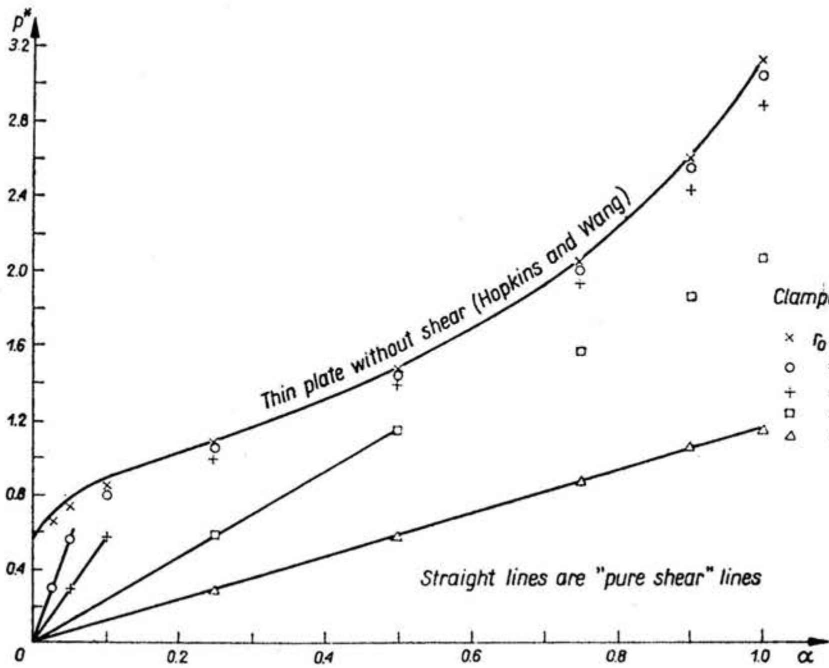

FIG. 5.

both simply supported (Fig. 4) and clamped (Fig. 5) cases. These are for values of r_0 (ratio of plate radius to thickness) of 1, 2, 5, 10, 50 and for α from 0.025 to 1.0.

The number of zones used in each of the regions $0 \leqslant S \leqslant \alpha$ and $\alpha \leqslant S \leqslant 1$ was 10. This was found to be optimum and very little difference was detected in the value of p^* when the number of zones was increased from 10 to 30. The number of constraint points nc in each zone was 5. This was found to be adequate and led to minimal yield violation elsewhere.

4. Discussion

Considering the lower range of r_0, say r_0 less than 2, the thick plate analysis is more relevant than the thin plate with shear analysis since it takes account of the through thickness stress σ_z which has a significant value at this range of r_0. Notwithstanding this, it is still worthwhile to examine the shear effect at this range of r_0 for the thin plate formulation. When r_0 is very small, failure may be visualised to occur by pure shear at the edge with bending moments being very small. For such a situation we have

$$2\pi\alpha r_0 \sigma_0 q_{max} = \pi p \alpha^2 r_0^2 .$$

With q_{max} being limited to $1/\sqrt{3}$ for a von Mises material the pure shear failure line is given by

$$p^* = 2\alpha r_0/\sqrt{3}. \tag{4.1}$$

This line is indicated on Figs. 2 and 3 where it is seen to pass through the origin and represent, as is to be expected, an upper estimate of the collapse load. The pure shear line is also plotted in Figs. 4 and 5.

Of course, it can easily be seen that this thin plate "pure shear" line represents an upper bound on the more accurate thick plate result, considering τ_{rz} alone. In order to obtain the q_r required by the thin shell analysis, one would have to have τ_{rz} = maximum value from top to bottom of the shell, whereas it must vanish on both surfaces. If $\partial\tau_{rz}/\partial z$ is large near the surfaces, this induces rapid changes in σ_r as can be seen from Eq. (2.1), and hence σ_r contributes to the yield function.

Secondly, for small r_0 the σ_z contribution becomes dominant. For, since $p/\sigma_0 = p^*/\alpha^2 r_0^2$ must remain finite as $r_0 \to 0$, it follows that p^* as a function of r_0 is of the power 2 in the neighbourhood of $r_0 = 0$. Therefore, a graph of p^* against r_0 has a zero slope at the origin for the thick plate analysis. This differs strikingly from the "thin plate with shear" results.

Comparing the behaviour of the simply supported and clamped plates, it is seen from Figs. 2 and 3 that both the thick plate and thin plate with shear collapse pressures rise more slowly towards the Hopkins and Wang thin plate without shear values for the clamped plates than for the simply supported plates. This is to be expected since taking the shear stress τ_{rz} into account uses up a considerable portion of the allowable limiting value of the yield function, particularly at the clamped end, and thus reduces the section's capacity for supporting the radial stresses σ_r. This argument which applies to the thin plate with shear case as well as to the thick plate case means a reduction in the ability of the plate to support a clamping moment relative to the no-shear case. The simply supported plate for which the radial stresses at the end, and hence the clamping moment, are zero is naturally less affected by allowing for shear.

Although the method for the thick plate analysis is perfectly general and can, in principle, be used for any plate radius to thickness ratio, it is observed from Figs. 2 and 3 that the thick plate results are nearly all lower than the thin plate with shear results. The difference at high r_0 is mostly due to the fact that in order to obtain a stress distribution where the whole of the plate is nearly at yield, very high order polynomials would have to be used (or an alternative formulation of the stresses). This in turn would require a far larger number of constraint points and create problems with computer storage. In spite of this difficulty, dimensionless pressures of 0.9 have been achieved at high r_0 which is considered fairly satisfactory.

As for the thin plate with shear results, it is seen from Figs. 2 and 3 that these tend towards the thin shell without shear lines for increasing values of r_0. The agreement between the two at values of $r_0 > 25$ has been found to be almost exact.

The plots of p^* against α for the thin plate with shear case are shown in Figs. 4 and 5. For comparison purposes the no-shear curves due to Hopkins and Wang have been superimposed. The pure shear lines have also been plotted for most values of r_0. From these plots it is seen that if r_0 is not very small, a straight line along the pure shear curve up to the no-shear curve and then a continuation along the no-shear curve appears to be a good upper bound approximation to the true behaviour of both simply supported and clamped plates.

Comparing the results obtained here with those by Sawczuk and Duszek [7], one must be careful to note the differences between the yield conditions employed in the various analyses. Sawczuk and Duszek used a separated shear and bending action based on the Tresca yield criterion. They also used linearised approximations to an interaction surface based on the von Mises yield criterion. As such, quantitative comparison of results is rather difficult. In particular, the pure shear case for $\alpha = 1$ and $r_0 = 0.5$ according to the present analysis gives $p^* = 0.577$ which, in Sawczuk and Duszek's notation, corresponds to $S/2\pi M_0$ of 1.155. The corresponding values obtained by Sawczuk and Duszek, as shown in Fig. 6 of their paper, are about 1.5 and 1.95 depending on the yield condition employed. Notwithstanding this, the qualitative similarity between Fig. 2 in their paper which shows a plot of $S/2\pi M_0$ against α for a simply supported plate and Fig. 4 in this paper is evident. Once again, however, as Sawczuk and Duszek remark in their paper their results do not strictly apply to thick plates as a result of ignoring the influence of σ_z.

5. Conclusions

Two lower bound analyses for the calculation of the limit transverse load on simply supported and clamped circular plates have been presented: a thick plate analysis which takes full account of all stresses including the through thickness direct stress, and a thin plate with shear analysis which ignores the through thickness direct stress. The thick plate method of analysis, although perfectly general and in principle applicable to plates of any thickness, is found to be more useful in the lower range of r_0 (i.e. $r_0 < 4$) than in the higher range of r_0 for which range further investigation of the stress formulation is needed. The thin plate-with shear results which are particularly applicable in the middle and higher ranges of r_0 have been found to converge to the thin plate without shear results by Hopkins and Wang at large values of r_0. From the analyses presented it is seen that the load carrying capacity of plates having a small radius to thickness ratio is drastically smaller than estimates based on ignoring shear and direct transverse stresses.

6. Acknowledgements

This investigation was carried out largely in the Structural Engineering Division of the Department of Civil and Structural Engineering, University of Manchester Institute of Science and Technology. The computations were carried out on the CDC 7600 computer at the University of Manchester Regional Computing Centre. Most of the results were presented at the WCOSE Conference, Montreal, July 1976.

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Received November 6, 1976.