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A new solution of the Navier-Stokes equation for the motion of a fluid contained between two parallel plates rotating about the same axis(*)

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THE EXISTENCE of an infinite set of nontrivial solutions has been proved for the problem of incompressible viscous fluid contained between the two parallel infinite plates rotating with constant angular velocity around the fixed normal axis. Stability of the solutions has been investigated and the conditions enabling to point out the subset of the solutions which are stable with regard to arbitrary disturbance.

Wykazano istnienie nieskończonego zbioru nietrywialnych rozwiązań zadania o przepływie nieściśliwej cieczy lepkiej zawartej między dwoma równoległymi nieskończonymi płytami, obracającymi się ze stałą prędkością kątową wokół ustalonej prostopadłej osi. Rozpatrzono stateczność rozwiązań oraz zbadano warunki pozwalające na wskazanie podzbioru rozwiązań statecznych względem dowolnych zaburzeń.

Найдено, что существует бесконечное множество нетривиальных решений задачи о течении вязкой несжимаемой жидкости содержащейся между двумя плоскопараллельными бесконечными пластинами вращающимися с постоянной угловой скоростью вокруг фиксированной нормальной оси. Исследована устойчивость решений и найдены условия позволяющие выделить из полученного класса решения устойчивые к любым возмущениям.

1. Introduction

LET Π_1 and Π_2 be two infinite parallel plates rotating about a fixed normal axis D with the same constant angular velocity ω. A classical incompressible fluid fills the infinite domain limited by the plates Π_1 and Π_2. These plates are solid walls. The fluid is assumed to be in steady motion.

The problem of finding the motion of the fluid filling the domain described above has a trivial solution in which the fluid is rotating as a rigid body about the axis D with the angular velocity ω. It will be shown in the present paper that the problem admits an infinite number of other solutions which are exact solutions of the Navier-Stokes equation and which satisfy the boundary conditions. For each of these new solutions the velocity field, the pressure field, and the stress exerted by the fluid on the plates Π_1 and Π_2 will be given. The stability of the new solutions will be studied using the energy method, and it will be proved that among the new solutions those satisfying the following condition are stable for arbitrary disturbances:

$$(1.1) \qquad \frac{l}{h} < \frac{\pi^2}{2} \frac{\tanh\left(\frac{R}{8}\right)^{1/2}}{R^{3/2}}.$$

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In this inequality h is half of the distance between the two plates Π_1 and Π_2, and l is the distance to the axis of rotation D of a point P which is uniquely determined for each new solution. (P is the point situated at the same distance from the plates Π_1 and Π_2 and where the velocity of the fluid is zero.) Moreover, in the inequality (1.1) R is the Reynolds number defined by the equation

$$R = \frac{\omega h^2}{\nu},$$

where ν is the coefficient of kinematic viscosity of the fluid.

I owe my thanks to Mr. Göktürk Üçoluk who made the calculations necessary for the drawing of Figs. 3, 4, 5, 6, 7 and 8.

2. Statement of the problem. Theorem giving all solutions

Let Π_1 and Π_2 be two infinite parallel plates rotating about a fixed normal axis D with the same constant angular velocity ω. A classical viscous incompressible fluid fills the infinite domain limited by the plates Π_1 and Π_2. These plates are solid walls. The fluid is assumed to be in steady motion.

Let $Oxyz$ be a fixed system of axes chosen such that the axis Oz coincides with the axis of rotation D, and such that the equations of the planes Π_1 and Π_2 be $z = h$ and $z = -h$, respectively.

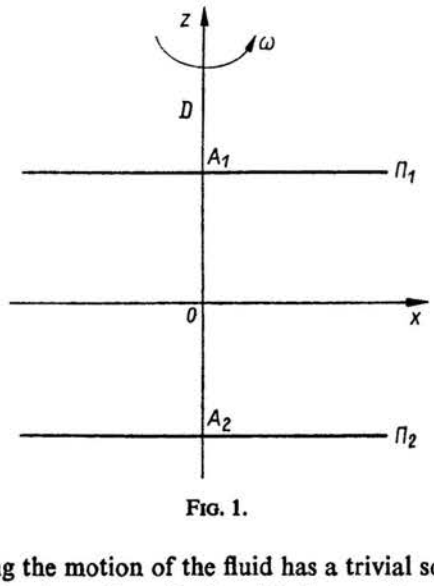

Fig. 1.

The problem of finding the motion of the fluid has a trivial solution in which the fluid is rotating as a rigid body about the axis Oz with the angular velocity ω. In this trivial solution the streamlines are evidently concentric circles contained in the planes Π parallel to the plane Oxy, the center of the circles being, for each plane Π, the intersection I of this plane and the axis Oz. Therefore, the locus of the point I when the plane Π shifts from Π_2 to Π_1 is the segment A_2A_1 of the axis Oz.

Now we ask the following question: is it possible to find solutions of the problem in

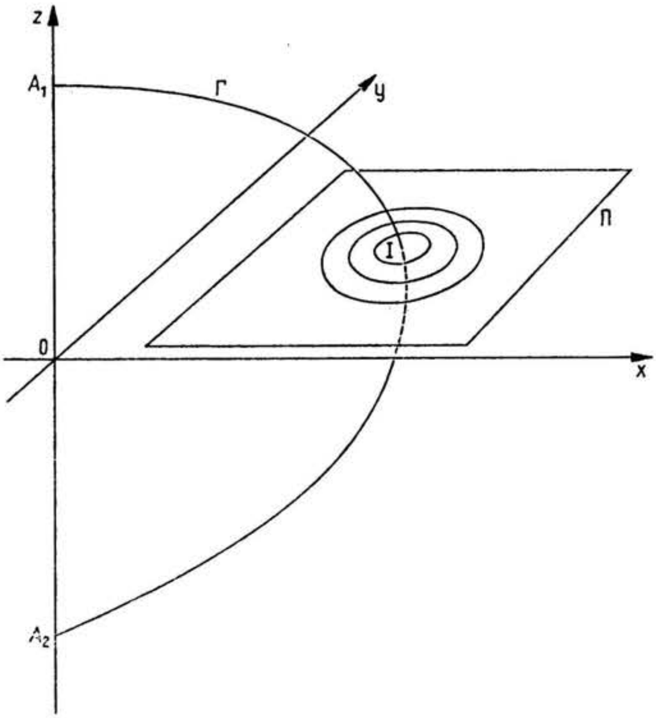

FIG. 2.

which the streamlines are again, in each plane Π parallel to the plane Oxy, concentric circles, the locus of the point I, center of the circles, being no more a segment of the axis Oz, but a curve Γ? The answer to this question is affirmative, and all the solutions which satisfy the condition of the question are given in the following theorem.

THEOREM. Consider the steady motion of a viscous incompressible fluid in which the streamlines are in each plane Π parallel to the plane Oxy concentric circles. All motions, other than the trivial motion mentioned above, having these streamlines and which are solutions of the Navier-Stokes equation and of the equation of continuity, which furhermore meet the boundary condition on the rotating plates Π_1 and Π_2, are given below.

Let P be an arbitrary point of the plane Oxy. For each position of the point P other than 0 one has a new solution. Let the axes Oxy be chosen such that the coordinates of the point P are $(l, 0)$, $l > 0$. The components with respect to the axes $Oxyz$ of the velocity field of the new solutions are given by the equations

$$u = -\omega[y-g(z)], \quad v = \omega[x-f(z)], \quad w = 0, \tag{2.1}$$

the function $f(z)$ and $g(z)$ appearing in these equations being given by the equations

$$\begin{aligned} \frac{f(z)}{l} &= \frac{1-\phi(h)}{\Delta}[\phi(z)-\phi(h)] - \frac{\chi(h)}{\Delta}[\chi(z)-\chi(h)], \\ \frac{g(z)}{l} &= \frac{\chi(h)}{\Delta}[\phi(z)-\phi(h)] + \frac{1-\phi(h)}{\Delta}[\chi(z)-\chi(h)]. \end{aligned} \tag{2.2}$$

The functions $\phi(z)$, $\chi(z)$ and the constant Δ which appear in Eqs. (2.2) are given by the equations

$$\phi(z) = \cosh mz \cdot \cos mz, \quad \chi(z) = \sinh mz \cdot \sin mz, \tag{2.3}$$
$$\Delta = [1-\phi(h)]^2 + [\chi(h)]^2 = (\cosh mh - \cos mh)^2,$$

the parameter m being given by

$$m = \left(\frac{\omega}{2\nu}\right)^{1/2},$$

where ν is the coefficient of kinematic viscosity of the fluid. The pressure field of the new solutions is given by the equation

$$\frac{p}{\varrho} + \Omega = \frac{\omega^2}{2}[(x-x_1)^2 + (y-y_1)^2] + C, \tag{2.4}$$

where ϱ is the density of the fluid, Ω is the potential of the body force, C is an arbitrary constant, and x_1 and y_1 are given by the equation

$$\frac{x_1}{l} = 1 - \frac{1-\phi(h)}{\Delta}, \quad \frac{y_1}{l} = -\frac{\chi(h)}{\Delta}. \tag{2.5}$$

The equations of the curve Γ locus of the points I are

$$x = f(z), \quad y = g(z).$$

The proof of this theorem will be given in Sects. 3 and 4.

Consider a motion which satisfies the condition mentioned in the first sentence of the statement of the theorem. This motion belongs to the family of pseudo-plane motions of the first kind(¹). A pseudo-plane motion of the first kind is a motion in which the components of the velocity are of the following form:

$$u = u(x, y, z, t), \quad v = v(x, y, z, t), \quad w = 0.$$

As a result of the equation of continuity, a motion of this family admits a stream function $\psi(x, y, z, t)$ such that the components of the velocity are given with the aid of ψ by the following equations:

$$u = \psi'_y, \quad v = -\psi'_x, \quad w = 0.$$

The motion reduces to a plane motion if ψ does not depend on z.

We suppose that the body force acting on the fluid depends on a potential Ω. Then the Navier-Stokes equation gives the following equations(²):

$$\left(\frac{p}{\varrho} + \Omega\right)'_x = -\psi''_{yt} + \frac{D(\psi, \psi'_y)}{D(x, y)} + \nu\nabla'^2\psi'_y, \tag{2.6}$$
$$\left(\frac{p}{\varrho} + \Omega\right)'_y = \psi''_{xt} - \frac{D(\psi, \psi'_x)}{D(x, y)} - \nu\nabla'^2\psi'_x,$$

where

(¹) For these motions see [3], pp. 71–80 and pp. 134-140. These motions were introduced for the first time in [3]. See also [4], pp. 84-89 and pp. 161-163.

(²) See [4], p. 162.

$$\nabla'^2\psi = \psi''_{x^2}+\psi''_{y^2}+\psi''_{z^2}.$$

The Navier-Stokes equation gives furthermore the equation

$$\left(\frac{p}{\varrho}+\Omega\right)'_z = 0.$$

This equation shows that the quantity $p/\varrho+\Omega$ does not depend on z, just as in the case of plane motion.

As it is well known, it is possible to eliminate the pressure p and the potential Ω of the body force from the Navier-Stokes equation. In order to do this elimination, it is sufficient to take the curl of the two sides of the Navier-Stokes equation. The equation thus obtained, which contains the velocity field as the only unknown, expresses the condition necessary and sufficient for a vector field $\mathbf{v}(x, y, z, t)$ to be the velocity field of the motion of a viscous incompressible fluid(3). The equation mentioned can be called the compatibility equation(4).

In the case of a pseudo-plane motion of the first kind, the compatibility equation gives the following three scalar equations:(5)

$$\text{(2.7)}\qquad \begin{aligned}
&\nu\nabla'^2\psi''_{xz}+\left[\frac{D(\psi, \psi'_x)}{D(x, y)}\right]'_z-\psi'''_{xzt} = 0,\\
&\nu\nabla'^2\psi''_{yz}+\left[\frac{D(\psi, \psi'_y)}{D(x, y)}\right]'_z-\psi'''_{yzt} = 0,\\
&\nu\nabla'^2(\nabla^2\psi)+\frac{D(\psi, \nabla^2\psi)}{D(x, y)}-\nabla^2\psi'_t = 0,
\end{aligned}$$

where

$$\nabla^2\psi = \psi''_{x^2}+\psi''_{y^2}.$$

3. The velocity field

Consider the steady motion of a viscous incompressible fluid such that in each plane Π parallel to the fixed plane Oxy the stream lines are concentric circles having a point I of the plane Π as center. Let

$$x = f(z), \qquad y = g(z)$$

be the equations of the curve Γ which is the locus of the point I when the plane Π shifts remaining parallel to the plane Oxy. The previous motion is evidently a pseudo-plane motion of the first kind. It is easy to see that the stream function ψ of this motion must be of the form

$$\text{(3.1)}\qquad \psi = H(q, z),$$

(3) The statement of this sentence is correct only if the domain V occupied by the fluid is simply connected. If the domain V is not simply connected, it is necessary to add to the equation mentioned in the statement supplementary conditions in order that the pressure field deduced from $\mathbf{v}$ has a unique value at each point of the domain V.

(4) See [4], p. 3.

(5) See [3], p. 72 and p. 135, [4], p. 85 and p. 162.

where q is defined by the equation

$$q = [x-f(z)]^2+[y-g(z)]^2,$$

and where H is a function of q and z which must be a solution of Eqs. (2.7).

I have proved elsewhere([6]) that all solutions of Eqs (2.7), which are of the form (3.1), are given by the equation

$$\psi = K.\{[x-f(z)]^2+[y-g(z)]^2\}, \tag{3.2}$$

where K is an arbitrary constant, and where the functions $f(z)$ and $g(z)$ must be solutions of the following differential system:

$$\nu f'''-2Kg' = 0, \quad \nu g'''+2Kf' = 0. \tag{3.3}$$

From Eq. (3.2) we deduce, for the components of the velocity, the equations

$$u = 2K[y-g(z)], \quad v = -2K[x-f(z)].$$

These two equations show that the totality of the particles of fluid contained in any plane Π parallel to the plane Oxy move as if the plane Π were a rigid plate rotating with the constant angular velocity $-2K$ about a fixed point I of this plane; if the equation of the plane Π is $z = \zeta$, the coordinates of the point I are $x_I = f(\zeta)$, $y_I = g(\zeta)$, $z_I = \zeta$.

Consider for a while not the problem of the present paper but the problem of the motion of a fluid filling all the space and whose stream function is of the form (3.2). For this case, I have determined in [5] all the solutions of the differential system (3.3). The motion so obtained has been called in [5] a vortex with a curvilinear axis. The curve Γ, which is the locus of the fixed points of the fluid, is in this motion a skew curve wrapped round a surface of a revolution which looks like a hyperboloid of revolution of one sheet.

Consider now the problem of the present paper. We must search, among the motions given by Eq. (3.2) which satisfy Eqs (3.3), those which meet the boundary condition of adherence on the rotating plates Π_1 and Π_2. In order to satisfy this condition, the angular velocity $-2K$ of the planes Π must first of all, be equal to the angular velocity of the plates Π_1 and Π_2, that is to say we must have

$$-2K = \omega.$$

Then Eq. (3.2) takes the form

$$\psi = -\frac{\omega}{2}\{[x-f(z)]^2+[y-g(z)]^2\}, \tag{3.4}$$

and the components of the velocity become

$$u = -\omega[y-g(z)], \quad v = \omega[x-f(z)], \quad w = 0.$$

These are Eqs. (2.1).

In order to satisfy the boundary condition, in addition to the previous condition, we must have the following one: the curve Γ, which is the locus of the fixed points, must pass through the points $A_1(0, 0, h)$ and $A_2(0, 0, -h)$ which are the fixed points of the plates Π_1 and Π_2, respectively. This condition gives the equations

$$f(h) = g(h) = 0, \quad f(-h) = g(-h) = 0. \tag{3.5}$$

([6]) See [5]. See also [4], pp. 87-88 where the main results of [5] are given without the proofs.

In order to integrate the differential system (3.3), we put

$$F(z) = f(z)+ig(z). \tag{3.6}$$

From Eqs (3.3) one deduces

$$F''' - \frac{i\omega}{\nu} F' = 0. \tag{3.7}$$

Suppose that $\omega > 0$ and put(7)

$$m = \left(\frac{\omega}{2\nu}\right)^{1/2}.$$

The differential equation (3.7) takes the form

$$F''' - 2im^2 F' = 0.$$

The general solution of this differential equation is given by

$$F = C_1 + C_2 e^{m(1+i)z} + C_3 e^{-(1+i)z}, \tag{3.8}$$

where C_1, C_2, C_3 are arbitrary constants. The conditions (3.5) give, for the function $F(z)$, the conditions

$$F(h) = 0, \quad F(-h) = 0. \tag{3.9}$$

Let P be the intersection of the curve Γ and the plane Oxy, that is to say the point of this plane where the velocity of the fluid is zero. We will choose the fixed axes Ox and Oy such that the coordinates of the point P with respect to these axes are $x_P = l$, $y_P = 0$ ($l \geqslant 0$). Then the condition that the curve Γ passes through the point P gives, for $f(z)$ and $g(z)$,

$$f(0) = l, \quad g(0) = 0,$$

and therefore for $F(z)$,

$$F(0) = l. \tag{3.10}$$

It is easy to determine the constants C_1, C_2, C_3 appearing in Eq. (3.8) so that the conditions (3.9) and (3.10) are satisfied. One obtains

$$C_3 = C_2, \quad C_1 = -2C_2 \cosh m(1+i)h,$$

$$C_2 = \frac{l}{2\Delta}[1-\phi(h)+i\chi(h)],$$

where the functions ϕ and χ and the constant Δ are given by Eqs. (2.3). From these values of C_1, C_2, C_3, one deduces the expression of $F(z)$, and by utilizing Eq. (3.6) the following expressions for $f(z)$ and $g(z)$ are obtained:

$$\begin{aligned} f(z) &= \frac{l[1-\phi(h)]}{\Delta}[\phi(z)-\phi(h)] - \frac{l\chi(h)}{\Delta_{\text{i}}}[\chi(z)-\chi(h)], \\ g(z) &= \frac{l\chi(h)}{\Delta}[\phi(z)-\phi(h)] + \frac{l[1-\phi(h)]}{\Delta}[\chi(z)-\chi(h)]. \end{aligned} \tag{3.11}$$

(7) If $\omega < 0$, the changes to be made in the solution are trivial and will be indicated in the note following.

These equations are, for $l \neq 0$, Eqs. (2.2) given in the statement of the theorem([8]).

We have obtained for the velocity field of the fluid an infinite number of solutions. Indeed, the velocity field depends on the position of the point P in the plane Oxy. The point P can be chosen arbitrarily in this plane. For each position of P in the plane Oxy, different from O, we have a solution which is different from the trivial solution mentioned in the statement of the theorem. If the point P is taken in O, $l = 0$ and Eqs. (3.11) give

$$f(z) \equiv 0, \quad g(z) \equiv 0.$$

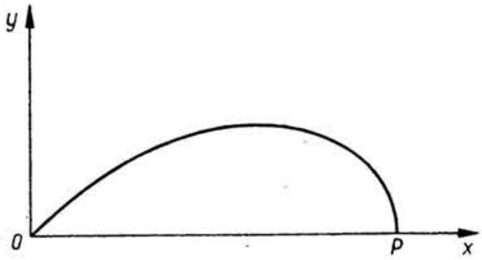

FIG. 3.

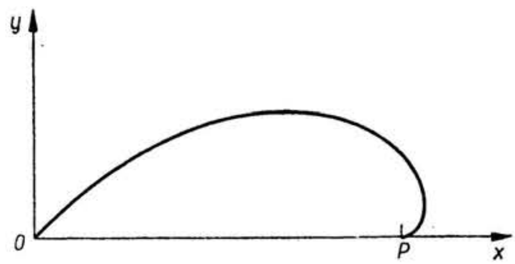

FIG. 4.

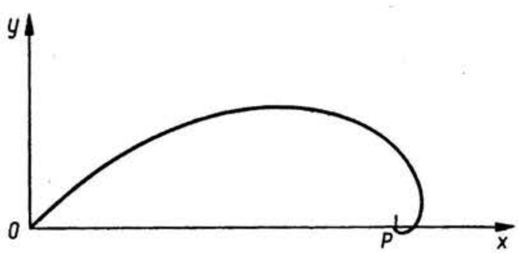

FIG. 5.

([8]) Equations (2.2) are thus obtained by supposing that $\omega > 0$. If $\omega < 0$, one must put $m = \left(-\frac{\omega}{2\nu}\right)^{1/2}$, and it is easy to see that Eqs. $(2.2)_1$ and (2.3) remain as they are, but the right-hand side of Eq. $(2.2)_2$ must be multiplied by -1. This means that the curve given in Figs. 3, 4 and 5 must be replaced by the curves which are symmetric of the previous ones with respect to the axis Ox. The curves given in Figs. 6, 7 and 8 remain as they are.

Then the curve Γ reduces to the segment $A_1 A_2$ of the axis Oz, and the motion of the fluid is a rigid rotation about the axis Oz, that is to say the motion reduces to the trivial solution.

The curve Γ which is the locus of the points of the fluid which are fixed has as equations

$$x = f(z), \quad y = g(z).$$

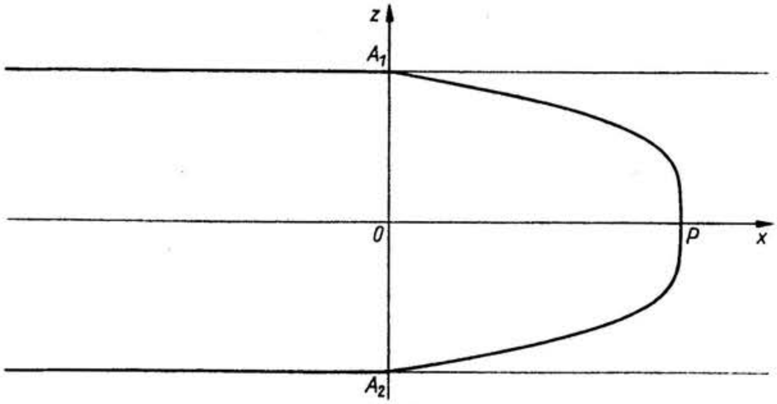

FIG. 6.

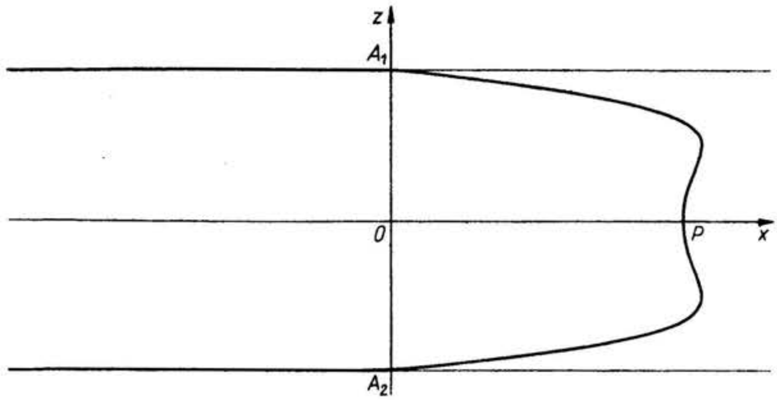

FIG. 7.

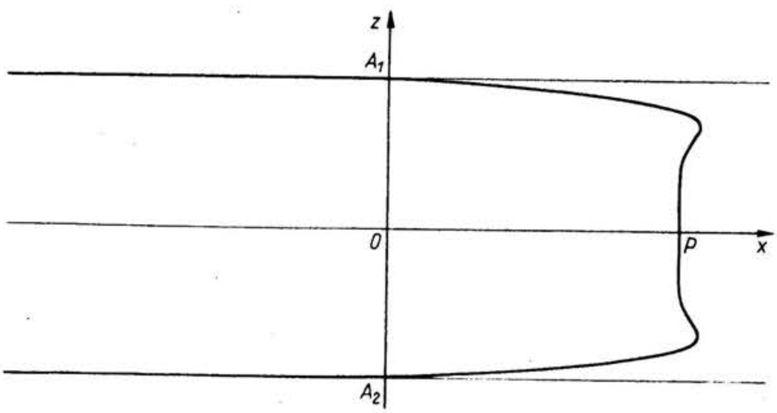

FIG. 8.

The projection of this curve on the plane Oxy has been drawn in Figs. 3, 4 and 5 for different values ($R = 20$, 44, 123, respectively) of the Reynolds number R defined by the equation

$$R = \frac{\omega h^2}{\nu}. \tag{3.12}$$

This Reynolds number is linked to the parameter m by the equation

$$R = 2m^2h^2. \tag{3.13}$$

The projection of the curve Γ on the plane Oxz for the same values of R has been drawn in Figs. 6, 7 and 8.

4. The pressure field

Let us calculate the pressure field of the motion, the velocity field of which is given by Eqs. (2.1), (2.2) and (2.3). If we replace in Eqs. (2.6) the stream function ψ by its value given by Eq. (3.4), we obtain:

$$\left(\frac{p}{\varrho}+\Omega\right)'_x = \omega^2 x+\omega(\nu g''-\omega f),$$

$$\left(\frac{p}{\varrho}+\Omega\right)'_y = \omega^2 y-\omega(\nu f''+\omega g).$$

Now if we replace in these equations f and g by their values given by Eqs. (2.2) and (2.3), we find

$$\left(\frac{p}{\varrho}+\Omega\right)'_x = \omega^2 x+\omega^2 l\left[\frac{1-\phi(h)}{\Delta}-1\right],$$

$$\left(\frac{p}{\varrho}+\Omega\right)'_y = \omega^2 y+\omega^2 l\,\frac{\chi(h)}{\Delta}.$$

From these equations we deduce

$$\frac{p}{\varrho}+\Omega = \frac{\omega^2}{2}[(x-x_1)^2+(y-y_1)^2]+C, \tag{4.1}$$

where x_1 and y_1 are given by Eqs. (2.5), and where C is an arbitrary constant(⁹)

Equation (4.1) shows how $p/\varrho+\Omega$ varies. This quantity, which depends only on x and y and not on z, is minimum for $x = x_1$ and $y = y_1$. Let M_1 be a point having x_1, y_1, and z_0 as coordinates, where z_0 is arbitrary ($-h \leqslant z_0 \leqslant h$). Let M be a point having x, y, and z_0 as coordinates; then Eq. (4.1) shows that we have

$$\left(\frac{p}{\varrho}+\Omega\right)_M = \left(\frac{p}{\varrho}+\Omega\right)_{M_1}+\frac{\omega^2}{2}\overline{MM}_1^2.$$

(⁹) The solutions obtained in the present paper and given by Eqs. (2.1)-(2.5) are entirely different from the solution given by ABBOTT and WALTERS [2] (see also [1]), where the fluid fills the domain between two parallel plates rotating about two parallel and distinct axes. None of the solutions of the present paper can be obtained as a particular case or limiting case of the solution of Abbott and Walters.

5. Stress applied by the fluid on the rotating plates

We will now determine the stress $\mathbf{t}_1$ applied by the fluid to a point $M(x, y, h)$ of the plate [10][11]Π_1. Let t_{1x}, t_{1y}, t_{1z} be the components of $\mathbf{t}_1$ with respect to the axes $Oxyz$. One has

$$t_{1x} = -\mu(u'_z + w'_x), \quad t_{1y} = -\mu(v'_z + w'_y),$$
$$t_{1z} = p - 2\mu w'_z.$$

In these equations μ is the coefficient of viscosity of the fluid. By replacing in these equations u, v, w by their values given by Eqs (2.1), we obtain

$$t_{1x} = -\mu\omega g'(h), \quad t_{1y} = \mu\omega f'(h), \quad t_{1z} = p. \tag{5.1}$$

If we suppose that there is no body force, or if the body force is the gravity, and if the plate Π_1 is horizontal, the third equation in the set (5.1) and Eq. (4.1) give

$$t_{1z} = \frac{\varrho\omega^2}{2}[(x-x_1)^2 + (y-y_1)^2] + D, \tag{5.2}$$

where D is an arbitrary constant.

The normal stress t_{1z} is minimum at the point $M_1(x_1, y_1, h)$ of the plate Π_1, and its value at an arbitrary point M of this plate is given by the equation

$$(t_{1z})_M = (t_{1z})_{M_1} + \frac{\varrho\omega^2}{2}\overline{MM}_1^2.$$

When the Reynolds number R tends to $+\infty$, that is to say, according to Eq. (3.13), when mh tends to $+\infty$, the equations (2.3) and (2.5) show that

$$\lim_{R\to+\infty} x_1 = l, \quad \lim_{R\to+\infty} y_1 = 0.$$

Therefore, when R tends to $+\infty$, the point M_1 approaches the point $P_1(l, 0, h)$ of the plate Π_1. This approach is very fast: for $R > 60$ one finds that the percentage errors

$$\left|\frac{x_1 - l}{l}\right|, \quad \left|\frac{y_1}{l}\right|$$

are smaller than 5‰ and Eq. (5.2) can be replaced with a very good approximation by the equation

$$t_{1z} = \frac{\varrho\omega^2}{2}[(x-l)^2 + y^2] + D.$$

Let us now consider the tangential stress $\mathbf{t}_{1t}$ at the point $M(x, y. h)$, whose components are t_{1x} and t_{1y}. The first two equations in the set (5.1) show that t_{1x} and t_{1z} do not depend on x and y, and are constant when the position of M various on the plane Π_1. Define α and t_0 by the equations

$$\alpha = \left(\frac{R}{2}\right)^{1/2}, \quad t_0 = \frac{\omega\mu l}{h}.$$

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(10) It is easy to see that the stress $\mathbf{t}_2$ applied by the fluid to the point of the plate Π_2 which is symmetric of the point M with respect to the plane Oxy is symmetric of $\mathbf{t}_1$ with respect to the same plane.

(11) It is possible to measure the stress applied on the rotating plate by utilizing the orthogonal rheometer of MAXWELL and CHARTOFF (see [8]).

If we utilize the values of the functions $f(z)$ and $g(z)$ given by Eqs. (2.2), the first two equations in the set (5.1) give the following equations:

$$\text{(5.3)}\qquad \begin{aligned} \frac{t_{1x}}{t_0} &= \alpha\,\frac{\sinh\alpha-\sin\alpha}{\cosh\alpha-\cos\alpha},\\ \frac{t_{1y}}{t_0} &= -\alpha\,\frac{\sinh\alpha+\sin\alpha}{\cosh\alpha-\cos\alpha}. \end{aligned}$$

These equations show that the ratios t_{1x}/t_0 and t_{1y}/t_0 are functions of the Reynolds number R only.

Let the tangential stress $\mathbf{t}_{1t}$ be drawn in the plane Oxy with its origin at the point O. The locus of the end point of $\mathbf{t}_{1t}$ when the Reynolds number R varies is the curve represented in Fig. 9, in which the projection of Γ on the plane Oxy given in Fig. 5 is also

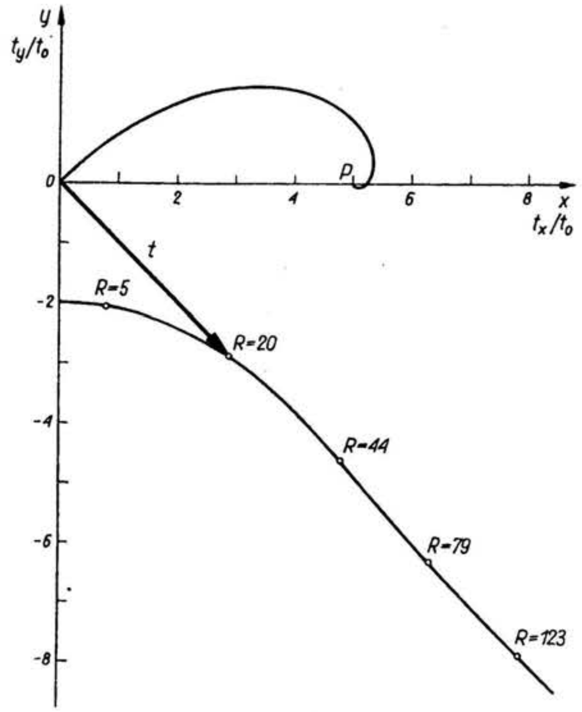

Fig. 9.

drawn[12]. When the Reynolds number R tends to $+\infty$, the direction of the tangential stress $\mathbf{t}_{1t}$ approaches very rapidly the direction making the angle $-\pi/4$ with the axis Ox, that is to say, with the straight line OP. If we call θ_1 the angle of $\mathbf{t}_{1t}$ with the axis Ox, it is easy to deduce from Eqs. (5.3) that for $R > 60$, one has $\theta_1 = -\pi/4$ with a percentage error which is smaller than 1%.

[12] Figure 9 has been obtained by supposing that $\omega > 0$. If $\omega < 0$, it is easy to see that the curve of Fig. 9 representing the locus of the end point of the tangential stress $\mathbf{t}_{1t}$ must be replaced by the curve which is symmetric of the previous one with respect to Ox.

Let $t_{1t} = (t_{1x}^2+t_{1y}^2)^{1/2}$ be the intensity of the tangential stress. Again Eqs. (5.3) show that for $R > 60$, one has

$$t_{1t} = t_0\left(\frac{R}{2}\right)^{1/2},$$

with a percentage error which is smaller than 3‰.

6. Stability of the motions obtained

In the linear stability method the disturbances are assumed to be infinitesimal and the equations are linearized. In the method of energy there are no such approximations: the disturbances are not supposed to be infinitesimal, they are arbitrary, and the method is mathematically rigorous. We will now study, using the method of energy ([13]), the stability of the motions obtained in the present paper.

Consider a viscous incompressible fluid filling a bounded domain $V(t)$ of space. Suppose that for the motion of this fluid there are two solutions $\mathbf{v}$ and $\dot{\mathbf{v}}$ of the Navier-Stokes equation, having the same velocity distribution on the boundary $S(t)$ of the domain $V(t)$. Put

$$\mathbf{u} = \dot{\mathbf{v}}-\mathbf{v}, \quad K = \frac{1}{2}\int\limits_{v} u^2 dv.$$

If $\mathbf{v}$ is the basic motion and $\dot{\mathbf{v}}$ the perturbed motion, then $\mathbf{u}$ is the disturbance and K is the kinetic energy of the disturbance.

The rate of change of K is given by the following equation which is the Reynolds-Orr energy equation:

$$\frac{dK}{dt} = -\int\limits_{V} [\nu(\nabla\mathbf{u}):(\nabla\mathbf{u})+\mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u}]dv, \tag{6.1}$$

where $\mathbf{D}$ is the rate of the deformation tensor of the flow $\mathbf{v}$.

From Eq. (6.1) it is possible to deduce the following theorem of stability:([14])

THEOREM OF STABILITY. *Let* $\mathbf{v}$ *be a solution of the Navier-Stokes equation for the motion in the bounded domain* $V(t)$. *Let* d *be the diameter of this domain. Let* $-c$ *be a lower bound for the characteristic values of the rate of the deformation tensor* D *of the flow* $\mathbf{v}$ *in the time interval* $(0, t)$. *If one has*

$$c < \frac{3\pi^2\nu}{d^2} \tag{6.2}$$

for all t, *then for any disturbance* $\mathbf{u}$ *one has*

$$\lim_{t\to+\infty} K = 0,$$

and the motion $\mathbf{v}$ *is stable*([15]).

([13]) See SERRIN [9], pp. 253-256 and [10], pp. 2-3.

([14]) This theorem is due to SERRIN [9], p. 254 and [10], p. 4. See GEORGESCU [6], pp. 52-53 and JOSEPH [7], vol. 1, p. 15 and p. 24.

([15]) It is possible to improve the inequality (6.2) by replacing the number 3 in the right-hand side of this inequality by greater numbers, see SERRIN [10], pp. 4-5 and VELTE [11], p. 14.

The previous theorem is stated for a domain $V(t)$ which is bounded. In the problem of the present paper the fluid fills the infinite domain V_1 which is limited by two parallel planes Π_1 and Π_2. In order to be able to apply the theorem of stability to the solutions given in the present paper, it is necessary to extend this theorem to the infinite domain V_1.

We will at first extend Eq. (6.1) to the infinite domain V_1. If we repeat the proof of this equation after having replaced the bounded domain $V(t)$ by the infinite domain V_1, we see that it is necessary to impose to the disturbance $\mathbf{u}$ and to the field $p_u = \dot{p}-p$ where $\dot{p}$ and p are the pressure fields associated respectively with the velocity fields $\dot{\mathbf{v}}$ and $\mathbf{v}$, the following conditions concerning their asymptotic behaviour as the distance r to the axis Oz tends to $+\infty$:

$$\mathbf{u} = O(r^{-k}), \quad \nabla\mathbf{u} = O(r^{-k}), \quad p = O(r^{-k_1}),$$

with $k > 1$, $k_1 > 0$. If these conditions are satisfied, Eq. (6.1) is valid for the infinite domain V_1.

Then it is possible to extend the theorem of stability to the domain[16] V_1. One obtains the following result: the theorem of stability is valid for the infinite domain V_1, the inequality (6.2) being replaced by the inequality

$$c < \frac{\pi^2 \nu}{4h^2}, \tag{6.3}$$

where $2h$ is the distance between the planes Π_1 and Π_2.

Now we will apply this last result to the new solutions given in the present paper. At first we must determine the number c. It is easy to see that the characteristic values of the rate of the deformation tensor $\mathbf{D}$ of the flow given by Eqs. (2.1), (2.2) and (2.3) are

$$\lambda = 0, \quad \lambda = \pm\frac{\omega}{2}(f'^2+g'^2)^{1/2}.$$

Therefore c is an upper bound of $\frac{\omega}{2}(f'^2+g'^2)^{1/2}$. From Eqs. (2.2) and (2.3) one deduces

$$f'^2+g'^2 = \frac{2m^2l^2}{\Delta}(\cosh^2 mz-\cos^2 mz).$$

For $-h \leqslant z \leqslant h$ we have

$$0 \leqslant \cosh^2 mz-\cos^2 mz \leqslant \cosh^2 mh-\cos^2 mh.$$

By utilizing the value of Δ given by the third equation in the set (2.3), we find that for $-h \leqslant z \leqslant h$ one has

$$(f'^2+g'^2)^{1/2} \leqslant ml2^{1/2}\left(\frac{\cosh mh+\cos mh}{\cosh mh-\cos mh}\right)^{1/2}.$$

From this result one deduces that we can take

$$c = m\omega l 2^{-1/2}\left(\frac{\cosh mh+\cos mh}{\cosh mh-\cos mh}\right)^{1/2}.$$

[16] See SERRIN [10], p. 6.

The inequality (6.3) then gives

$$\text{(6.4)} \qquad \frac{l}{h} < \frac{\pi^2}{2} \cdot R^{-3/2} \left[\frac{\cosh\left(\frac{R}{2}\right)^{1/2} - \cos\left(\frac{R}{2}\right)^{1/2}}{\cosh\left(\frac{R}{2}\right)^{1/2} + \cos\left(\frac{R}{2}\right)^{1/2}} \right]^{1/2}.$$

In this inequality R is the Reynolds number defined by Eq. (3.12).

Those solutions of the set given by Eqs. (2.1), (2.2), (2.3), (2.4), and (2.5) for which the inequality (6.4) is satisfied are stable for arbitrary disturbances. Call $\phi(R)$ the function which constitutes the right-hand side of the inequality (6.4); the graph of the function $\phi(R)$

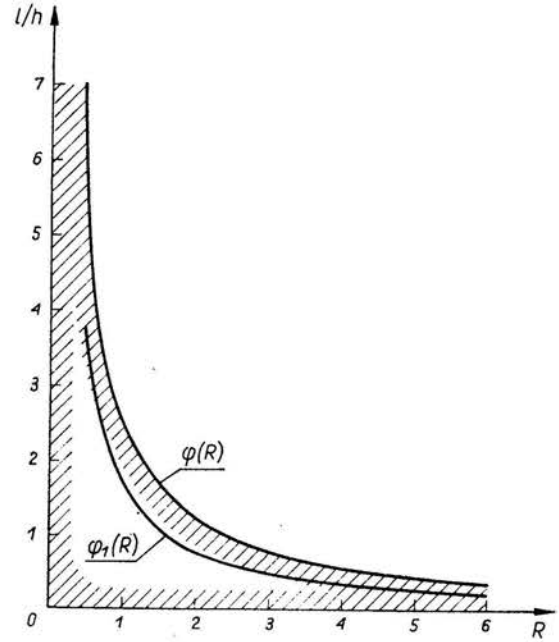

FIG. 10.

is given in Fig. 10. If, for a solution belonging to the mentioned set of solutions, the point having l/h and R as coordinates is situated in the streaked domain, this solution is stable.

It is possible to obtain for the stability condition an inequality which gives a smaller region of stability, but which is simpler than the inequality (6.4). Indeed, we have for all ξ

$$\frac{\cosh\xi - \cos\xi}{\cosh\xi + \cos\xi} \geqslant \frac{\cosh\xi - 1}{\cosh\xi + 1} = \tanh^2\frac{\xi}{2}.$$

If we utilize this inequality, we can obtain a lower bound for the right-hand of the inequality (6.4). By utilizing this result, we see that if the inequality

$$\text{(6.5)} \qquad \frac{l}{h} < \frac{\pi^2}{2} \cdot \frac{\tanh\left(\frac{R}{8}\right)^{1/2}}{R^{3/2}}$$

is satisfied, then the inequality (6.4) is also satisfied. Therefore, those solutions given by Eqs. (2.1), (2.2), (2.3), (2.4), and (2.5) for which the inequality (6.5) is satisfied are stable. Call $\phi_1(R)$ the function which constitutes the right-hand side of the inequality (6.5); the graph of the function $\phi_1(R)$ is given in Fig. 10.

References

1. T. N. G. ABBOTT, G. W. BOWEN and K. WALTERS, J. Phys. D. Appl. Phys., **4**, 190–203, 1971.
2. T. N. G. ABBOTT and K. WALTERS, J. Fluid Mech., **40**, 205–213, 1970.
3. R. BERKER, *Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible*, Paris, Lille 1936.
4. R. BERKER, *Integration des équations du mouvement d'un fluide visqueux incompressible*, Handbuch der Physik, Bd. VIII/2, 1–384, Berlin, Göttingen, Heidelberg 1963.
5. R. BERKER, *An exact solution of the Navier-Stokes equation: the vortex with curvilinear axis* [To be published soon].
6. A. GEORGESCU, *Teoria stabilitătii hidrodinamice*, Bucuresti 1976.
7. D. D. JOSEPH, *Stability of fluid motions*, 2 vol., Berlin, Heidelberg, New York 1976.
8. B. MAXWELL and R. P. CHARTOFF, Trans. Soc. Rheol., **9**, 41–52, 1965.
9. J. SERRIN, *Mathematical principles of classical fluid mechanics*, Handbuch der Physik, Bd. VIII/1, 125–263, Berlin, Göttingen, Heidelberg 1959.
10. J. SERRIN, Arch. for Rat. Mech. and Anal., **3**, 1–13, 1959.
11. W. VELTE, Arch. for Rat. Mech. and Anal., **9**, 9–20, 1962.

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