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Locally periodic medium and homogenization of random media

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CLASSICAL homogenization methods include the homogenization of periodic media and statistical theory. The paper presents a mixed method whose aim is to cumulate the advantages of these two methods. The application is presented in the case of filtration through porous media.

Klasyczne metody homogenizacji obejmują homogenizację ośrodków periodycznych i teorie statystyczne. W pracy zaproponowano metodę mieszaną, łączącą w sobie zalety obu metod składowych. Przedstawiono zastosowanie uzyskanej teorii do opisu filtracji w ośrodkach porowatych.

Классические методы гомогенизации охватывают гомогенизацию периодических сред и статистические теории. В работе предложен смешанный метод, соединяющий в себе достоинства обоих составных методов. Представлено применение полученной теории для описания фильтрации в пористых средах.

1. Introduction

THE MACROSCOPIC mechanics of media with a high density of strong heterogeneities necessitates a simplification which consists in introducing equivalent continuous media. If such a homogenization process is available, it leads to balance equations and to constitutive equations, when they exist. The quantities entering these equations are macroscopic quantities such as the mean stresses or the mean deformations and macroscopic or effective coefficients which are to be defined and calculated. Among all the homogenization processes two families can be pointed out:

the methods of homogenization for periodic structures,

the statistical methods for random media.

The first one is by far the strongest but it comes into play only when the structure of the medium is periodic. The aim of this paper is to show some results concerning the extension of the method of homogenization of periodic structure to the case of random media. We have in mind a mixed process — homogenization by local periodization — with a view to transfer some advantages from the first method to the second one.

The process is as follows:

From a given random medium to be homogenized, a periodic medium is built up by extending by periodicity a local pattern. This local pattern is, at this stage, arbitrarily chosen. In fact its dimension is of the order of a few heterogeneities, with a view to have the method tractable. Then the process of homogenization of periodic structures is applied and the corresponding macroscopic description is obtained.

The procedure is then repeated with a large number of randomly selected patterns, with different dimensions and positions in the medium.

The effective coefficients of the macroscopic equivalent medium are then obtained by a "properly" selected mean process over the above obtained statistical series.

In the first part we present generalities concerning the two methods and the simple example — a one-dimensional problem — to be studied in the sequel.

The second part is devoted to the analysis of the selected medium. Its simplicity is necessary since it is our aim to be capable of reckoning complete and exact solutions for the macroscopic description for each stage of the process. But it is clear that the validity of the method would have to be checked on more sophisticated media.

2. Scope

All the homogenization processes introduce at least two scales. The micro-scale is that of the heterogeneities, with a characteristic length l. The macro-scale is that of the bulk medium, with a characteristic length L. A necessary condition governing these processes is: $\varepsilon = l/L \ll 1$. In other words, a separation between two scales is assumed. Let us recall some features concerning the two homogenization processes mentioned above.

The homogenization of periodic structures [1, 2] makes possible a complete determination of the macroscopic description from the microscopic one, i.e.

the structure of the macroscopic laws and the macroscopic quantities describing the macroscopic state,

the numerical calculation of the effective parameters entering the relations,

the local fields of the microscopic quantities derived from the macroscopic ones (localization process).

The very strong hypothesis introduced, i.e. the periodicity, enables a clean separation between the two scales and the full use of the small parameter ε. Two important features must be pointed out. The first one is that, as opposed to the second method, no prerequisite is needed concerning the structure of the macroscopic description. This is a definite advantage when the structure of the macro-scale description is quite different from that of the micro-scale, particularly when the macroscopic behavior does not enter the family of classical laws. An example of quite different structures is the filtration of a liquid through a rigid porous medium (see [2] for more details). The micro-scale is described by the Navier–Stokes equations, i.e. the momentum balance and a constitutive law for the liquid, say a Newtonian incompressible liquid. At the macro-scale, the description consists of a volume balance and a momentum balance — the Darcy law — with no typical constitutive equation: the viscosity is introduced at the macro-scale through the homogenization process in a viscous force. Another example is the filtration of a liquid through a deformable porous medium. In that case different macroscopic descriptions are available, mono or diphasic, depending on the relative values of the viscous stress and the skeleton stress. "Relative values" are understood here as measured with the powers of the parameter ε. The second point is that the physical significance of the macroscopic quantities is not so obvious as for the second method. Special care must be taken at this point.

Statistical modelling for random media is based on the ergodic hypothesis, i.e. volume averages equal ensemble averages. For this method see KRÖNER [3]. The ergodic hypothesis permits 1) the use of statistics, 2) the permutation of the derivation and the mean values.

The method needs something more than the micro-scale description, i.e some hypotheses concerning the macro-scale. For example, in the case of an elastic composite, it is necessary to assume (for example) that the macroscopic behavior is also an elastic one. And this is not the only possibility [4].

Nevertheless, notwithstanding the strong apparent dissimilarities between the two methods, it is interesting to notice that a unification exists, at least for elastic composites with a macroscopic elastic behaviour [5]. This fact encourages to explore the mixed method presented in the sequel, which transfers some advantages of the homogenization of the periodic structure to the study of effective properties of random media.

The mixed method — homogenization by local periodization — consists in substituting to the random medium to be studied a random ensemble of periodic media. Each periodic medium is built up from a local pattern considered as a period. The different patterns are randomly selected by their dimensions and positions. We have in view a practical purpose and at this stage the method is only a formal procedure. We thus hope to be able:

To say something about the macroscopic description of random media when the structures of the descriptions at the micro and macro scales are different. We adopt here the hypothesis according to which each associated periodic medium behaves like a macroscopic equivalent one. This is quite admissible if the dimensions of the periods are of the order of a few heterogeneities so that the parameter ε for the periodic medium is of the same order as the parameter ε of the random initial medium.

To have information on quantities at the micro scale, using for this purpose the localization process.

We focus on the first step of the method, i.e on the periodization process in view to study its validity. So the sample to be analysed must be such that the second step, i.e the statistical treatment, is obvious for both the ensemble of periodic media and the initial random medium. This leads us, in particular, to select a case where both micro and macro scales have the same structure. In that case statistical modelling gives lower and upper bounds for the effective coefficients when theorems of the same type as those of the minima of the potential energy and the complementary energy of an elastic medium stand for the medium under consideration. And the lower bound, given by the theorem of the minimum of the potential energy, is effectively reached for a one-dimensional medium. So we select a one-dimensional rigid porous medium where the Darcy law is assumed to be valid at the local scale, $v = k\nabla p$, so that at the macro scale $\langle v\rangle = K\langle\nabla p\rangle$, k and K are the permeabilities at the micro and macro scales, respectively, and $\langle v\rangle$ and $\langle\nabla p\rangle$ are the mean values of the velocity v and the gradient of pressure p, respectively. And we consider the filtration of an incompressible liquid. The theorem of the minimum of the dissipated energy (gives we assume the media to be described by the same structure at the two scales)

$$\langle v\rangle K^{-1}\langle v\rangle = \langle vk^{-1}v\rangle \leqslant \langle v^*k^{-1}v^*\rangle,$$

where v^* is an arbitrary admissible velocity. In that problem the only possibility is to take v^* as a constant, then equal to the real velocity v. In fact, the minimum theorem is not needed here and the result, quite classical, follows the left-hand equality, i.e conservation of the dissipation density through the homogenization process:

$$K^{-1} = \langle k^{-1} \rangle .$$

And that harmonic mean value will also be used for statistical modelling.

The medium will be composed of two alternating porous media, with constant permeabilities k_1 and k_2, respectively, and with randomly distributed thicknesses. The extreme simplicity of the medium under consideration which is determined by the conditions above prescribed does not ensure the complete validity of the method. Particularly, difficulties appear in two- or three-dimensional situations when the studied phenomena exhibit a strong relation with the connexity of the components of the medium.

3. Homogenization processes by local periodization

3.1. The processes of local periodization

As described above, the processes which are studied below all consist in the following procedure:

a) Focus on a small volume of heterogeneous medium.

b) Consider this small volume as a motif to be periodically repeated.

c) Use the technique of homogenization of periodic media to obtain a value of the effective coefficients for such a medium.

d) Reproduce the steps a) to c) for a great number of small volumes with an equal thickness.

e) Perform a statistical treatment of the result obtained in d).

3.2. Elements for the description of a particular process

The description of a particular process necessitates the following choices:

A) Choice of the extension of the selected small volumes (denoted below by T).

B) Choice of the locations of the selected small volumes. This choice is represented by the locations of the centers Ci of the small volumes.

C) Choice of the number of the selected small volumes. In the following, the small volume will be called a "period" taking into account Step 2 of the procedure.

D) Choice of the statistical treatment of the results.

It is clear that a great number of possibilities may be used in the choices A) to D) but the four processes described below seem to represent examples where the main possibilities of choice are shown.

3.3. Criteria for the comparison of the processes

The *criteria* for the comparison of the processes are the following:

CRITERION C1: dispersion of the results. The process will be obviously considered as better if the dispersion of results is smaller.

CRITERION C2: complexity of the "period". If it does not constitute a problem for the 1D simulation below, the complexity of the geometry in a "period" is crucial for 2D or 3D problems. If a numerical resolution of the micro-scale problem is in view, only a small number of heterogeneities can be tractable. A great number of samples is, on the contrary, not a practical difficulty. This central part of the choice of a convenient process will be considered by studying the number of heterogeneities in a period.

The process will be considered as better if the number of heterogeneities is smaller.

3.4. Description of the medium

The artificial porous medium to be studied is described in Fig. 1. It consists of 205 alternating layers of permeability coefficients k_1 and k_2 equal to 10 and 20 darcy, respectively.

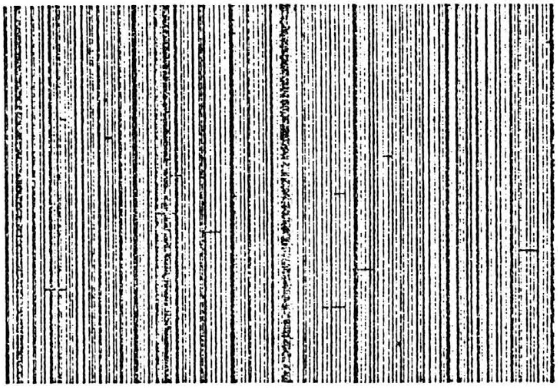

FIG. 1. Sketch of the studied medium.

3.5. Process number 1: slipping periods

a) Description of the process (Fig. 2)

A) The *extension* T of the period is considered as variable and related to the number N of periods by the expression $T = 10/(N+9)$.

B) The *locations* of the periods are shown in Fig. 2. The centers Ci are chosen to be periodocally-located within the medium. The medium is entirely covered by the "periods".

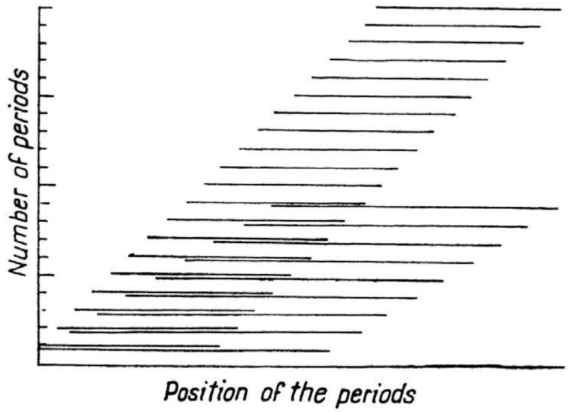

FIG. 2. Principle of the process by slipping periods

C) *The number* of the periods is chosen between 1 and 500.

D) The *statistical treatment* consists only in computing the mean value K^{-1} of the inverse value of the permeability, the standard deviation and the coefficient of variation of k^{-1}.

b) *Results*

The results are shown in Figs. 3 and 4. The value of the coefficient of variation may be seen as small (a few percent) but the inconveniences of the process are the following:

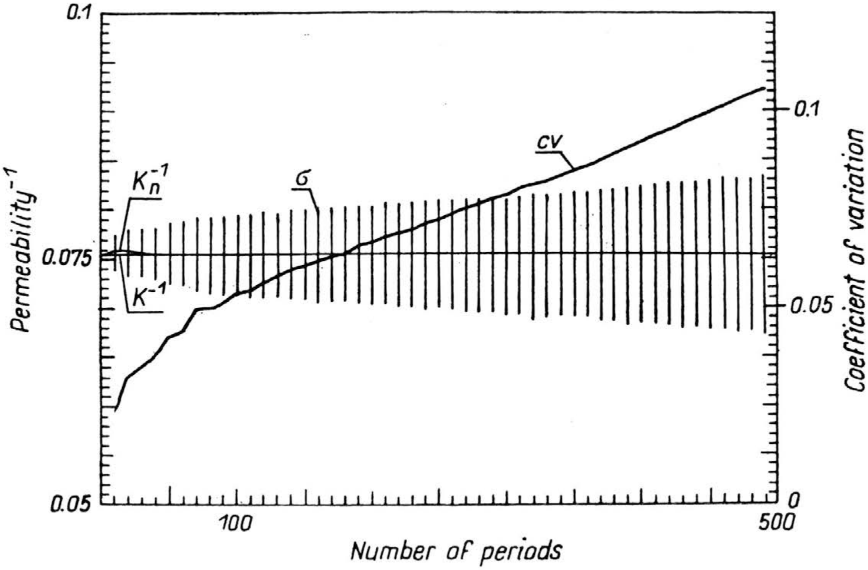

FIG. 3. Results of the process 1. Number of periods between 1 and 500.

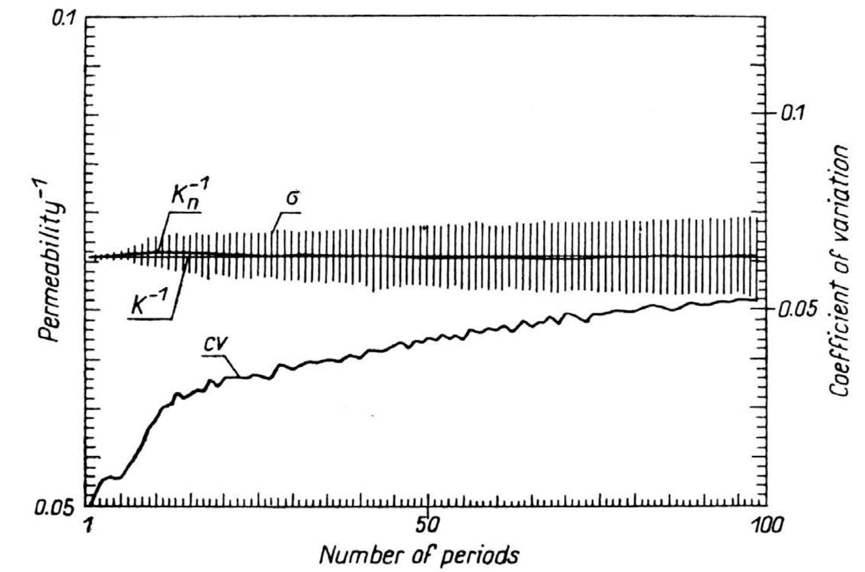

FIG. 4. Results of the process 1. Number of periods between 1 and 100.

for a small number of periods the number of heterogeneities is very important;

for a great number of periods, the coefficient of variation is more and more important.

It seems to be better fo *fix the number of heterogeneities* and to vary the other parameters as shown in the following processes.

3.6. Process number 2: fixed extension and independent series

The main idea in this process is to fix the number of heterogeneities in order to deal with periods as simple as possible (a few heterogeneities).

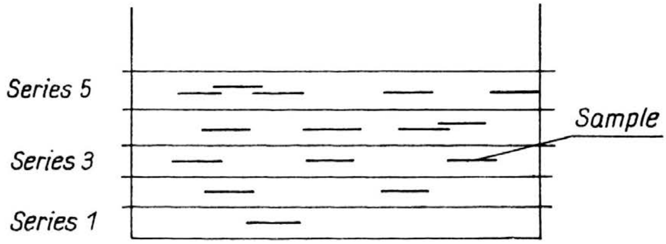

FIG. 5. Principle of the process 2. Fixed extension and independent series.

a) Description of the process (Fig. 5)

A) The *extension* of the period is fixed by the maximal number of heterogeneities by period, M, taken into account in the period. This number of heterogeneities M is taken between 2 and 10.

B) The *locations* of the periods are randomly taken within the medium. The medium is therefore not entirely covered by the "periods".

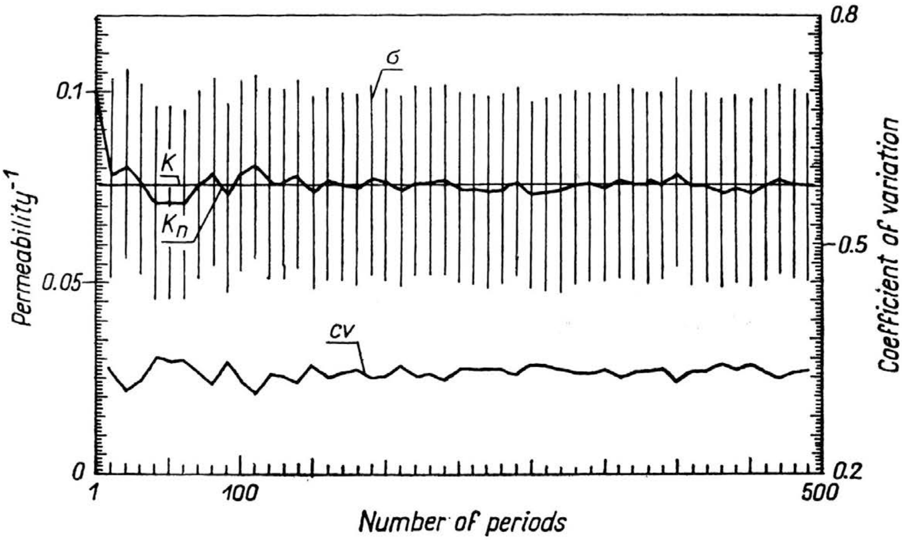

FIG. 6. Results of the process 2. Number of periods between 1 and 500.
Maximum number of heterogeneities of 2 by period.

C) The *number* of periods N is taken between "1 and 100" or "1 and 500".

D) *Statistical treatment.*

As before, the mean value and the standard deviation are computed for a fixed value of N and M.

b) *Results*

The results are shown in Figs. 6, 7, 8 and 9. The *coefficient of variation* is important for a given process (greater than 0.1) and *depends essentially on the number of heterogeneities.*

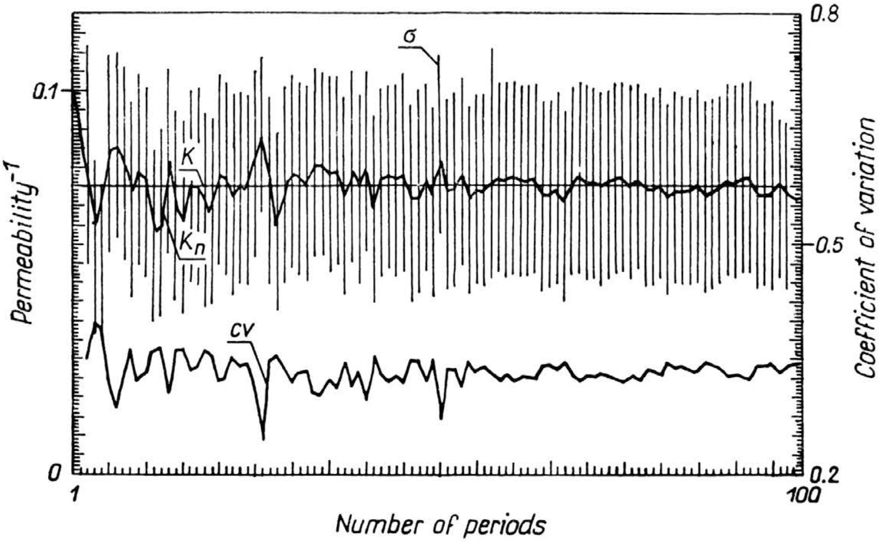

FIG. 7. Results of the process 2. Number of periods between 1 and 100.
Maximum number of heterogeneities of 2 by period.

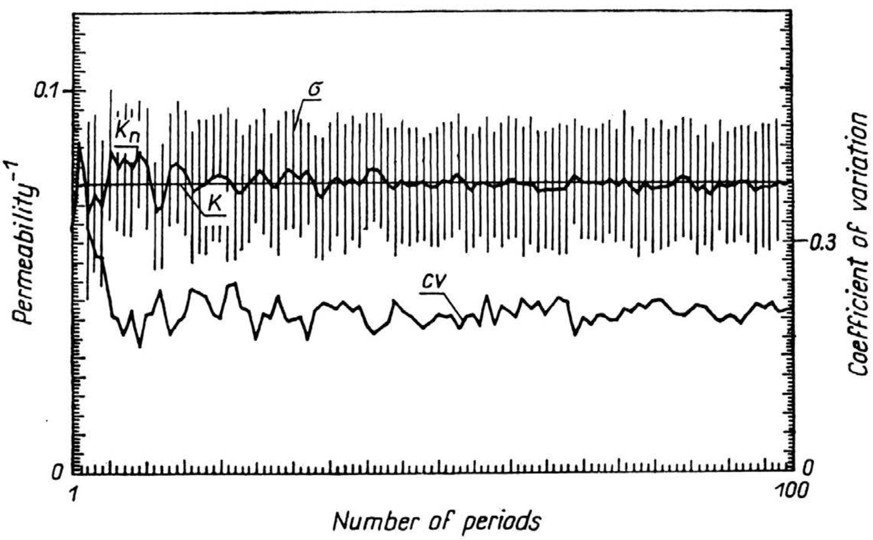

FIG. 8. Results of the process 2. Number of periods between 1 and 100.
Maximum number of heterogeneities of 5 by period.

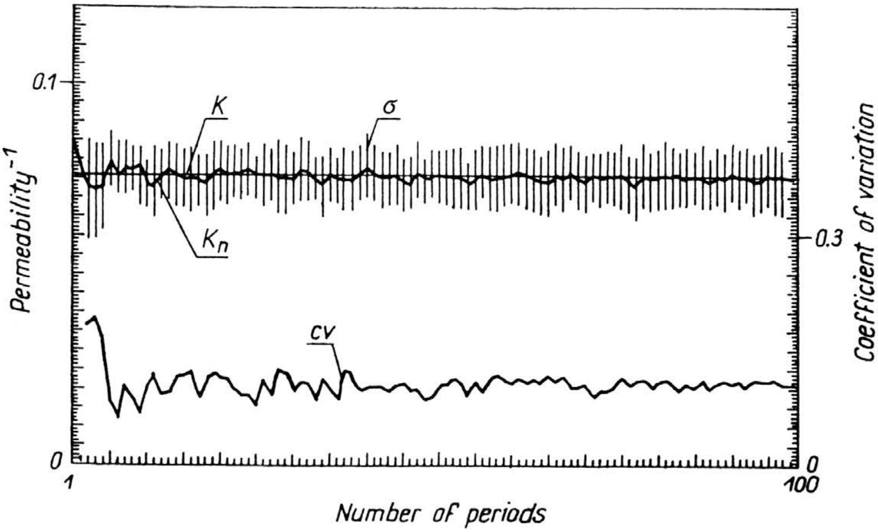

FIG. 9. Results of the process 2. Number of periods between 1 and 100.
Maximum number of heterogeneities of 10 by period.

The minimal value of 0.1 is obtained for $M = 10$; this value is not negligible. One inconvenience of this process is that it is very long in time, because for each value of N, the mean value and standard deviation have to be computed. The following process was designed to improve this point.

3.7. Process number 3: fixed extension and additive series

a) Description of the process (Fig. 10)

The process is similar to the previous one, except that when the number of periods increases, the periods already chosen are conserved (see Fig. 10 for the building of successive periods).

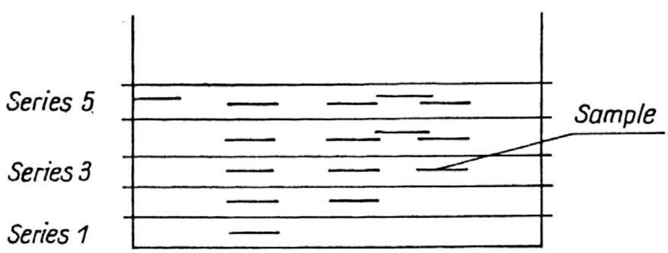

FIG. 10. Description of the process 3. Fixed extension and additive series.

b) Results

The results are shown in Figs. 11 to 15. It is noteworthy that the obtained coefficients of variation are very similar to those obtained in the process 2. A supplementary advantage of this process is to show smaller oscillations than in the previous process. It can then be remarked that the standard deviation for fixed N and M is very important as compared

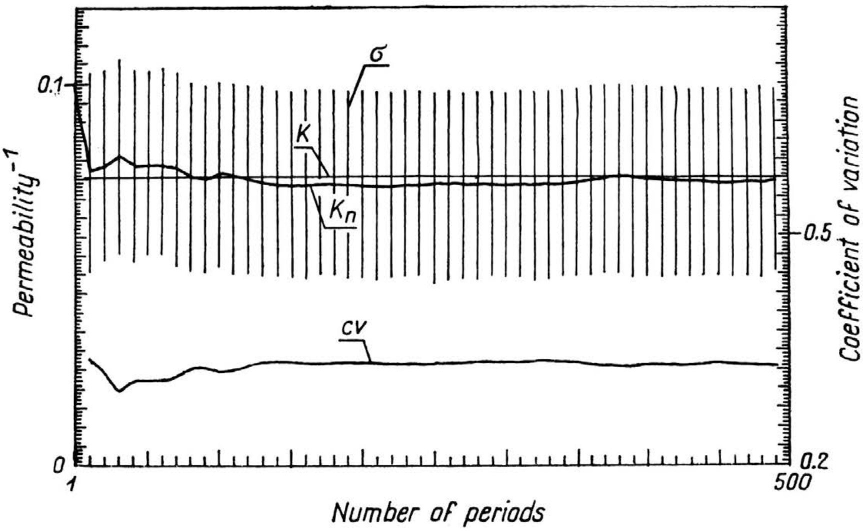

FIG. 11. Results of the process 3. Number of periods between 1 and 500.
Maximum number of heterogeneities of 2 by period.

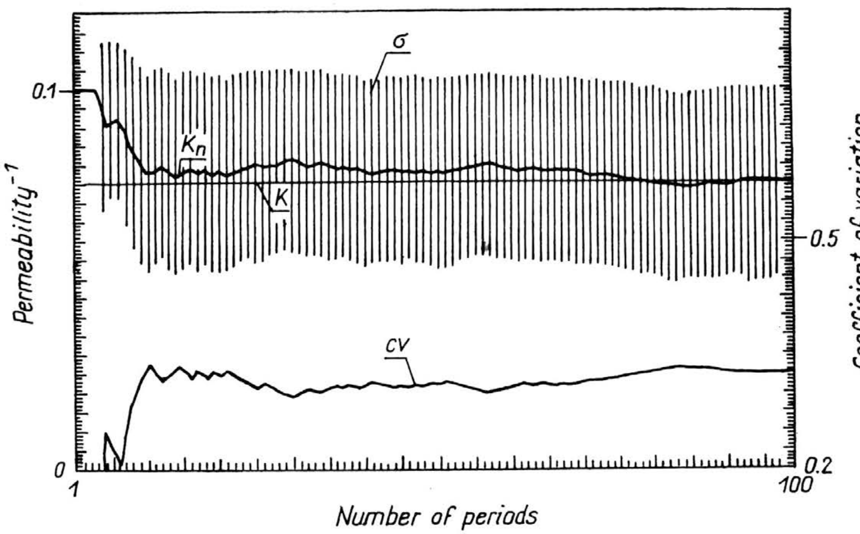

FIG. 12. Results of the process 3. Number of periods between 1 and 100.
Maximum number of heterogeneities of 2 by period.

to the variation of the mean values for a fixed value of N. The following process takes advantage of this property.

3.8. Process number 4: additive series and improved statistical treatment

a) Description of the process

As already obtained in 3.7, the mean values for a fixed value of N seem to be a better approximation than the one given by the standard deviation previously obtained.

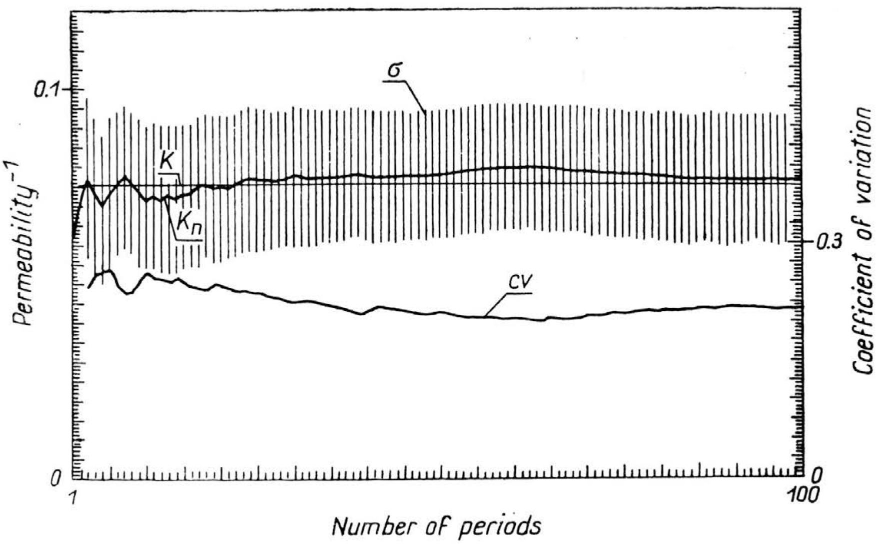

FIG. 13. Results of the process 3. Number of periods between 1 and 100.
Maximum number of heterogeneities of 5 by period.

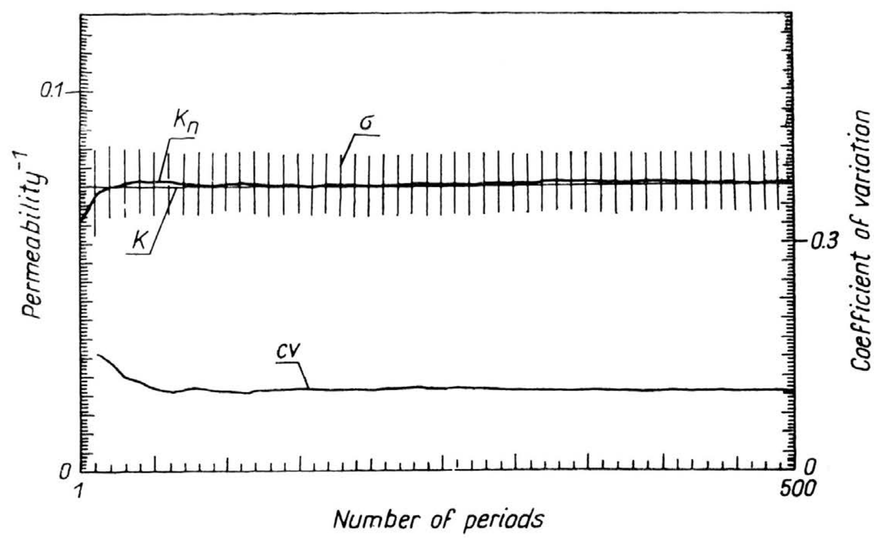

FIG. 14. Results of the process 3. Number of periods between 1 and 500.
Maximum number of homogeneities of 10 by period.

The process 4 is therefore similar to the previous one, except that the statistical treatment is improved in the following way:

The values taken into account are only the mean values for fixed values of N and M. A sampling is next obtained by varying the values of N, for a fixed value of M.

b) Results

The results are reported in Figs. 16 to 18, where the values of N were taken between N_1 to N_2 (1 to 10, 10 to 20 and 10 to 100), with the following notations:

$$K^{-1} = \frac{\sum_{n=N_1}^{N_2} K_n^{-1}}{N_2 - N_1 + 1},$$

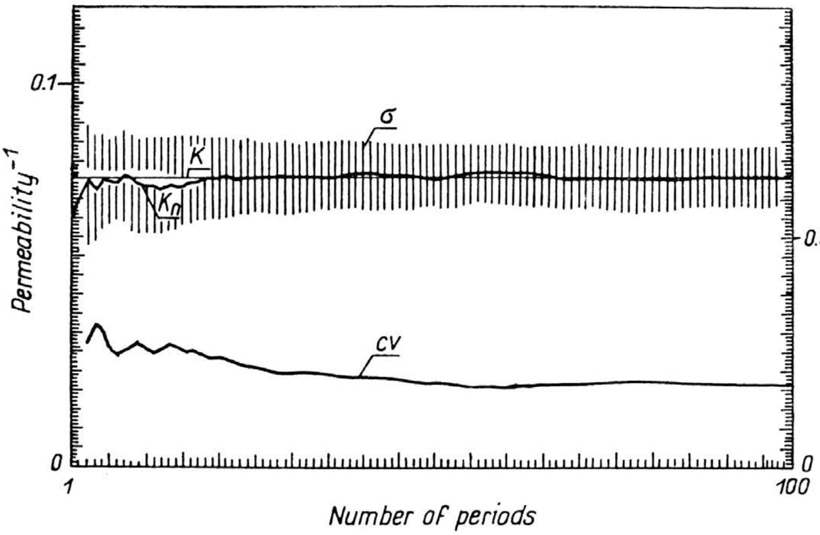

FIG. 15. Results of the process 3. Number of periods between 1 and 100.
Maximum number of heterogeneities of 10 by period.

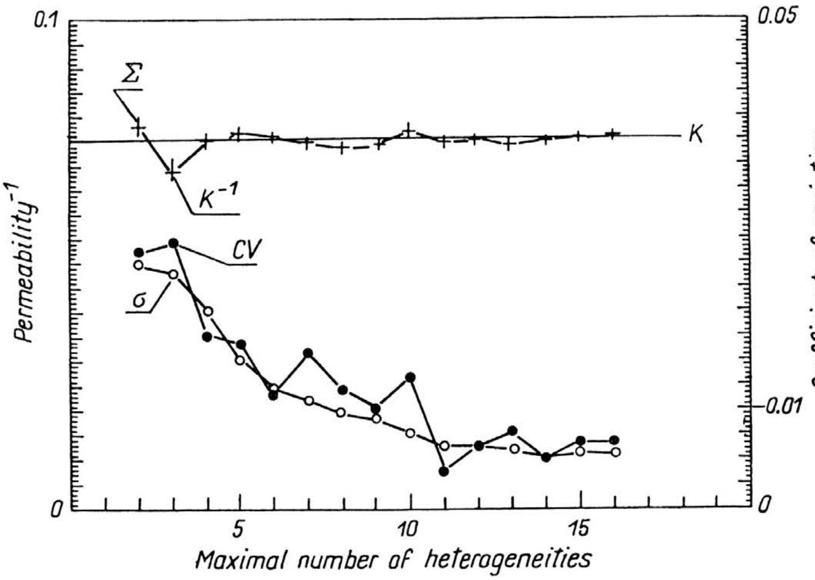

FIG. 16. Results of the process 4. Number of periods between 10 and 100.

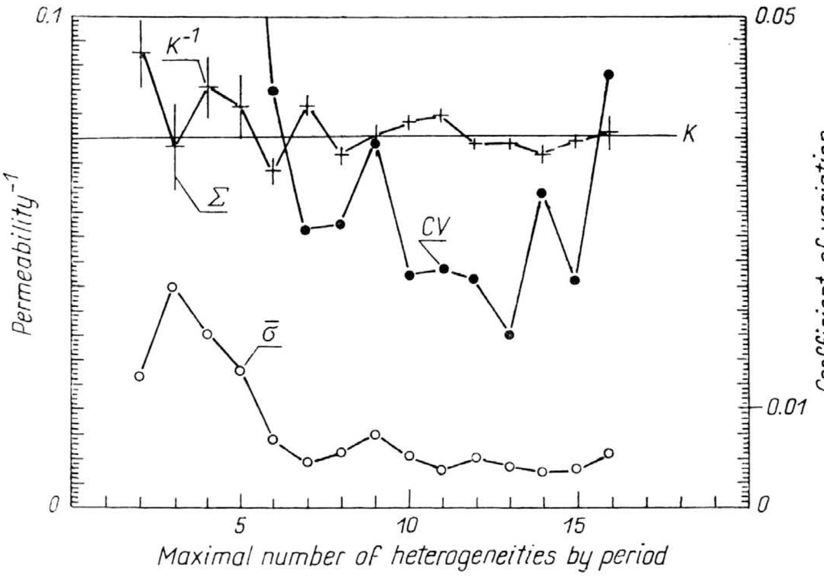

FIG. 17. Results of the process 4. Number of periods between 1 and 10.

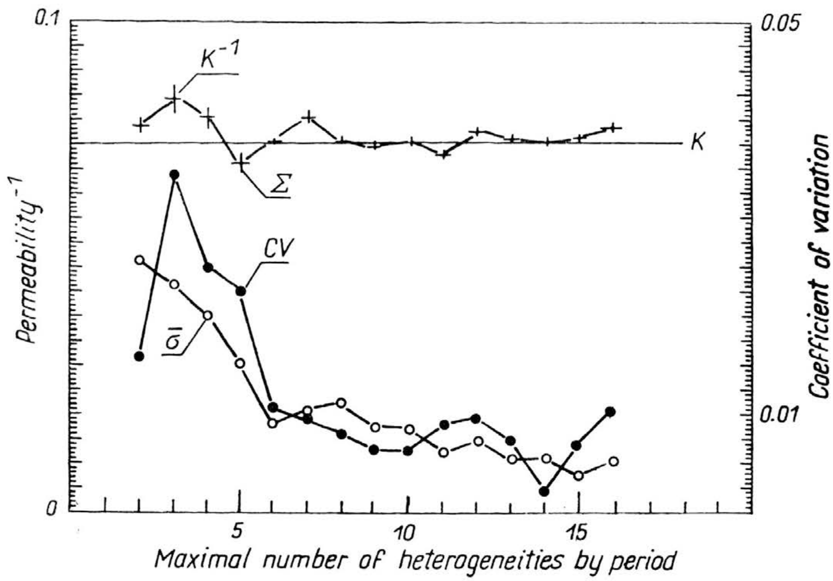

FIG. 18. Results of the process 4. Number of periods between 10 and 20.

where K_n is the mean value of permeability of the series n

$$\Sigma = \sqrt{\frac{\sum_{n=N_1}^{N_2} (K^{-1} - K_n^{-1})^2}{N_2 - N_1 + 1}}, \quad \sigma = \frac{\sum_{n=N_1}^{N_2} \sigma_n}{N_2 - N_1 + 1},$$

where the values of σ_n are computed for different numbers of periods varying from N_1 to N_2

$$CV = \Sigma / K^{-1}.$$

It is noteworthy that very good results are obtained for N varying between 10 to 100, with a coefficient of variation below 3×10^{-2}.

In addition, the mean value K^{-1} seems to be stable for 4 heterogeneities and more. Even for such a small number of heterogeneities, a coefficient of variation below 2 percent and a correct value of the mean coefficient of permeability are obtained.

4. Conclusion

After a review and discussion of the two main methods for dealing with a finely heterogeneous medium (homogenization of periodic media and statistical theory), a new method was proposed, which is a synthesis of these two methods.

The four processes used showed that a very complex medium may be well represented by taking into account a small number of heterogeneities and by using a convenient homogenization process.

It is obvious that the results obtained may be extended without difficulty to the computation of the elastic coefficient of a stratified elastic medium or to other physical problems.

The generalization of this method to 2D or 3D problems will constitute a new way to study finely heterogeneous media.

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