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Growth of cylindrical voids in nonlinear viscous material at arbitrary void volume fractions: a simple model

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THIS WORK is devoted to the growth of a cylindrical cavity in a finite shell of a nonlinear power law viscous material. The solution, given in a closed form, is used to evaluate the influence of void volume fraction and of stress triaxiality. A significant effect of void volume fraction is observed. We propose an interpolation formula, relating explicitly the void growth-rate, the stress triaxiality ratio, the void volume fraction and the material rate sensitivity parameter.

W pracy zbadano wzrost pustki cylindrycznej w ograniczonej otoczce tworzącej nieściśliwą matrycę lepkoplastyczną. Podano rozwiązania analityczne. Pozwalają one na obliczenie wpływu udziału objętościowego pustek, pomijanego w pracach klasycznych z tego zakresu, jak również trójosiowości naprężeń. Wreszcie zaproponowano prostą formułę interpolacji, całkowicie jawną, łączącą prędkość wzrostu, trójosiowość naprężeń, porowatość i wrażliwość materiału na prędkość deformacji.

В работе исследован рост цилиндрической пустоты в ограниченной оболочке, образующей несжимаемую вязкопластическую матрицу. Приведены аналитические решения. Позволяют они вычислить влияние объемного участия пустот, пренебрегаемого в классических работах из этой области, как тоже трехосности напряжений. Наконец, предложена простая формула интерполяции, полностью явная, соединяющая скорость роста, трехосность напряжений, пористость и чувствительность материала на скорость деформации.

1. Introduction

DUCTILE rupture mainly results from initiation, growth and coalescence of microscopic voids. Several theoretical analysis have considered the problem of a single cavity growing in a viscous or rigid-plastic incompressible matrix material. MCCLINTOCK'S [3] and RICE and TRACEY'S [16] pioneering works put forth the exponential dependence of voids growth-rate on the triaxiality ratio of remote stresses in a rigid-plastic material. More recently BUDIANSKY, HUTCHINSON, SLUTSKY [2] (B.H.S.) established that this dependence is polynomial for a power law matrix material (for recent reviews on theoretical aspects of cavity growth the reader is referred to B.H.S. [2], MCMEEKING [13] or GILORMINI, LICHT, SUQUET [7]).

These pioneering studies were mainly devoted to an isolated cavity growing in a matrix extending to infinity. However, basing on theoretical arguments, B.H.S. [2], DUVA, HUTCHINSON [5], DUVA [6] pointed out the importance on the void growth-rate of so-called "interactions"(1) at void volume fractions as small as one per cent. At the same time

(1) In authors' opinion, "interactions" between cavities should be distingued from effects of "existing porosity" which is a diffuse mode of interactions, and which is responsible for effects pointed out in [5, 6] and in the present work. Strong interactions between two neighbouring cavities seem to be essential in the stage of coalescence [21].

MARINI, MUDRY, PINEAU [12] reached a similar conclusion basing on experimental results. Probably motivated by such arguments several numerical works have been devoted to the growth of a spherical cavity in a cylindrical block of finite radius and height (ANDERSSON [1], STIGH [19] for simple tension, DUVA [6] for more general loadings). Growth of cylindrical cavities in a viscoplastic or rigid-plastic matrix material subjected to plane strains have also been studied (NEEDLEMAN [14], NEMAT-NASSER, TAYA [15], RICHARD [17], GUENNOUNI, FRANÇOIS [8]). The present work is devoted to the growth of a cylindrical cavity into a finite cylinder of matrix material. Although the cylindrical geometry is less general than the spherical one, its simplicity allows to derive solutions almost in closed forms, and therefore to state almost explicitly the influence of several parameters such as void volume fraction, stress triaxiality ratio, rate sensitivity of the matrix material.

Before proceeding in Sect. 2 to the detailed analysis, we recall a few notations and classical results of interest in such micromechanical approaches of cavities growth. Following techniques initiated by HILL [10] we consider a representative volume element Y containing traction-free voids denoted by V, and subjected to macroscopic stresses Σ or macroscopic strain-rates $\dot{\mathbf{E}}$. Throughout the following, upper case letters Σ, $\dot{\mathbf{E}}$ refer to macroscopic quantities while lower case letters $\boldsymbol{\sigma}$, $\dot{\boldsymbol{\varepsilon}}$ refer to local (also called microscopic) ones. Since voids are traction-free, macroscopic stresses are just the average of microscopic stresses

$$\boldsymbol{\Sigma} = \frac{1}{Y}\int\limits_{Y^*} \boldsymbol{\sigma}\, dx \overset{\text{def}}{=} \langle\sigma\rangle \qquad (\langle\cdot\rangle \text{ average symbol}),$$

where Y^* denotes the portion of Y occupied by the matrix material. The void volume change is simply related to the local displacement rate $\dot{u}$ by

$$\frac{\dot{V}}{V} = \frac{1}{V}\int\limits_{\partial V} \dot{u}_i n_i\, ds. \tag{1.1}$$(²)

The first way to determine the voids growth-rate $\dot{V}/V$ is to compute the local displacement rate $\dot{\mathbf{u}}$ and to perform the integration (1.1). A second approach is to note that, due to matrix incompressibility, the macroscopic dilatation is exclusively due to the volume change of voids:

$$\operatorname{Tr}\dot{E} = \frac{\dot{Y}}{Y} = f\frac{\dot{V}}{V} \qquad \left(f = \frac{V}{Y} \text{ is the void volume fraction}\right). \tag{1.2}$$

Therefore $\dot{V}/V$ can be derived from an approximation of the macroscopic constitutive law (or at least of the part relative to the macroscopic dilatation). This approach is used in [5, 6, 18] for dilutely voided materials and in [11] for arbitrary void volume fractions. The present work will use the first approach.

(²) When anisotropic damage is to be considered, an adequate theory basing on a tensorial relation generalizing (1.1) has been proposed by DRAGON [4].

2. Thick-walled cylinder under axisymmetric loading

2.1. Assumptions of the model

The thick-walled cylinder under axial and radial tension provides a simple guide to estimate the cavity growth law as a function of several parameters such as void volume fraction, macroscopic stress triaxiality, rate sensitivity of the matrix. First assumption of the model: cavities are assumed to be cylinders, aligned in the axial direction and with a circular cross-section. This assumption is obviously a crude one, but can be accepted for elongated cavities resulting from a sheet rolling process. It enables us to carry most calculations in closed forms, and we hope that it contains most desirable qualitative information on the dependence of the growth law on the above listed parameters (let us only recall that, in the same spirit, McCLINTOCK's model [3] basing on thick cylinders in a rigid plastic material, announced the exponential influence of stress triaxiality on cavities growth, later on established by RICE and TRACEY [15]). The inner and outer radii, the height of the cylinder are respectively denoted by a, b, H. Second assumption of the model: the matrix is assumed to be a power law viscous and incompressible material

$$\tag{2.1} \begin{gathered} \sigma_{ij}^D = \frac{2}{3}\mu \dot{\varepsilon}_{eq}^{s-2}\dot{\varepsilon}_{ij}, \quad \operatorname{div}\dot{\mathbf{u}} = 0, \\ \dot{\varepsilon}_{eq} = \left(\frac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}\right)^{1/2}, \qquad \sigma_{ij} = \sigma_m \delta_{ij} + \sigma_{ij}^D. \end{gathered}$$

Standard notations have been used for the mean stress σ_m, the deviatoric stress σ_{ij}^D, the equivalent strain rate $\dot{\varepsilon}_{eq} \cdot s$ is the rate sensitivity parameter of the material. Eq. (2.1), once inverted, takes the common form of a power law

$$\dot{\varepsilon}_{ij} = \frac{3}{2\mu^n}\sigma_{eq}^{n-1}\sigma_{ij}^D, \qquad \sigma_{eq} = \left(\frac{3}{2}\sigma_{ij}^D\sigma_{ij}^D\right)^{1/2}, \qquad n = 1/(s-1).$$

Third assumption of the model: attention is restricted to axisymmetric macroscopic stress states:

$$\Sigma_{11} = \Sigma_{22} = T, \qquad \Sigma_{33} = S \qquad \text{other} \qquad \Sigma_{ij} = 0.$$

Once more this assumption allows to carry calculations in a closed form, and does not pretend to be universal. However, it is of special interest in the important problem of striction of axisymmetric specimen under simple tension. The triaxiality of macroscopic stresses is measured through the triaxiality ratio

$$\tag{2.2} \frac{\Sigma_m}{\Sigma_{eq}} = \left(\frac{1}{3} + \frac{T}{S-T}\right)\operatorname{sgn}(S-T).$$

2.2. Details of the solution

Although the resolution is straightforward, we give here a few details for reader's convenience. Similar arguments can be found in BUDIANSKY *et al.* [2], TRACEY [20].

Equations of the problem consist of (2.1) and equilibrium equations

$$\operatorname{div}\boldsymbol{\sigma} = 0, \qquad a \leqslant r \leqslant b,$$

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \quad \text{for} \quad r = a,$$
$$\sigma_{rr} = T, \quad \sigma_{r\theta} = \sigma_{rz} = 0 \quad \text{for} \quad r = b;$$

$$\text{(2.3)} \qquad \frac{1}{\pi b^2}\int_0^{2\pi}\int_{a\leq r\leq b} \sigma_{zz} r\,dr\,d\theta = S(\text{resp. } -S) \quad \text{for} \quad z = H \text{ (resp. } z = 0),$$

$$\frac{1}{\pi b^2}\int_0^{2\pi}\int_{a\leq r\leq b} \sigma_{iz} r\,dr\,d\theta = 0, \quad i = \theta, r, \quad z = 0 \quad \text{and} \quad H.$$

Equations (2.3) do not constitute a sufficient set of boundary conditions and we have to prescribe kinematical end conditions. For this purpose we assume that generalized plane strains prevail in the cylinder:

$$\dot{u}_z = \dot{E}_{33} z,$$

where $\dot{E}_{33}$ is the axial macroscopic strain rate. Simple symmetry arguments show that the remaining components of the displacement rate $\dot{\mathbf{u}}$ are such that $\dot{u}_r$ is a function of r only, while $\dot{u}_\theta$ vanishes identically.

Incompressibility of the matrix restricts the number of candidates $\dot{u}_r$

$$\dot{u}_r = -\frac{\dot{E}_{33} r}{2} + \frac{A}{r}.$$

The equivalant strain $\dot{\varepsilon}_{eq}$ is

$$\dot{\varepsilon}_{eq} = \left(\dot{E}_{33}^2 + \frac{4A^2}{3r^4}\right)^{1/2},$$

and we deduce the following relations from the constitutive law (2.1)

$$\sigma_{rr} - \sigma_{\theta\theta} = \sigma_{rr}^D - \sigma_{\theta\theta}^D = -\frac{4}{3}\mu\dot{\varepsilon}_{eq}^{s-2}\frac{A}{r^2},$$

$$\sigma_{zz} - \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2} = \sigma_{zz}^D - \frac{\sigma_{rr}^D + \sigma_{\theta\theta}^D}{2} = \mu\dot{\varepsilon}_{eq}^{s-2}\dot{E}_{33}.$$

Equilibrium equations and boundary conditions are used to relate A, $\dot{E}_{33}$, T and S:

$$T = \sigma_{rr}(b) - \sigma_{rr}(a) = \int_a^b \frac{\partial \sigma_{rr}}{\partial r}\,dr = \int_a^b \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r}\,dr,$$

$$S - T = \Sigma_{33} - \frac{\Sigma_{11} + \Sigma_{22}}{2} = \left\langle \sigma_{zz} - \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2}\right\rangle.$$

After straightforward algebraic computations we obtain

$$\text{(2.4)} \qquad T = \frac{\mu}{\sqrt{3}}|\dot{E}_{33}|^{s-2}\dot{E}_{33}\int_{\omega f}^{\omega}(1+t^2)^{\frac{s-2}{2}}\,dt,$$

$$\text{(2.5)} \qquad S - T = \mu\omega f|\dot{E}_{33}|^{s-2}\dot{E}_{33}\int_{\omega f}^{\omega}(1+t^2)^{\frac{s-2}{2}}\,dt/t^2,$$

where $f = a^2/b^2$ is the void volume fraction, and ω is defined as:

$$\omega = \frac{2A}{\sqrt{3}\,\dot{E}_{33}a^2}.$$

The cavity volume V is $\pi a^2 H$ and its logarithmic rate is

$$\frac{\dot{V}}{V} = 2\frac{\dot{a}}{a} + \frac{\dot{H}}{H} = 2\frac{\dot{u}_r(a)}{a} + \dot{E}_{33} = \frac{A}{a^2}.$$

Therefore an alternate expression for ω evidencing the cavity growth rate is

$$\omega = \frac{1}{\sqrt{3}}\frac{\dot{V}}{\dot{E}_{33}V}, \tag{2.6}$$

ω is related to the macroscopic stress triaxiality ratio (2.2) by the following relation deduced from Eqs. (2.4), (2.5):

$$\frac{T}{S-T} = \frac{\int\limits_{\omega f}^{\omega}(1+t^2)^{\frac{s-2}{2}}dt}{\sqrt{3}\,\omega f\int\limits_{\vartheta f}^{\omega}(1+t^2)^{\frac{s-2}{2}}\frac{dt}{t^2}}. \tag{2.7}$$

2.3. Contact with other results

Contact is made with B.H.S. [2] result when f tends to 0 in (2.7). We obtain

$$\frac{T}{S-T} = \frac{1}{\sqrt{3}}\int\limits_0^{\omega}(1+t^2)^{\frac{s-2}{2}}dt. \tag{2.8}$$

If in addition we take $s = 1$ in (2.8) we obtain McClintock's result [3]

$$\frac{\dot{V}}{E_{33}\dot{V}} = \sqrt{3}\,\mathrm{sh}\left(\sqrt{3}\frac{T}{(S-T)}\right). \tag{2.9}$$

If we do not neglect the existing void volume fraction ($f \neq 0$), we can still eliminate ω from Eqs. (2.4) and (2.5) in the case of a rigid-plastic matrix material ($s = 1$). After algebraic computations we obtain Gurson's criterion for axisymmetric loadings [9]

$$\frac{\Sigma_{eq}^2}{\mu^2} + 2f\mathrm{ch}\left(\frac{\sqrt{3}}{2}\frac{\Sigma_{\alpha\alpha}}{\mu}\right) - 1 - f^2 = 0, \tag{2.10}$$

where μ is the yield stress of the matrix material, $\Sigma_{eq} = |S-T|$, $\Sigma_{\alpha\alpha} = 2T$.

3. Exact growth rate of cavities

3.1. Absolute and relative growth rates

In order to study the variations of the dilatation rate $\dot{V}/V$ with significant parameters such as stress triaxiality or porosity, we have to normalize it by a physically relevant strain rate of the problem. It is readily seen from homogeneity considerations that the macroscopic stress-strain rate law is a power law with the same exponents s and n as the microscopic one, and that

(3.1) $$\frac{\dot{V}}{V} = \frac{1}{\dfrac{1}{\mu^{s-1}}} |S-T|^{\frac{2-s}{s-1}} (S-T) F\left(f, s, \frac{T}{S-T}\right).$$

Therefore, a first choice for the normalizing strain rate is the equivalent strain rate which would take place in the unvoided material under the same stress conditions:

(3.2) $$\dot{\mathscr{E}}_{eq} = \frac{1}{\dfrac{1}{\mu^{s-1}}} |S-T|^{\frac{2-s}{s-1}} (S-T).$$

However Eqs. (2.6), (2.7) suggest that an alternative choice is to normalize $\dot{V}/V$ by the macroscopic axial strain rate $\dot{E}_{33}$. It should be noted that, while $\dot{\mathscr{E}}_{eq}$ does not depend on the void volume fraction and on f and stress triaxiality, $\dot{E}_{33}$ is a function of these parameters(³). Therefore $\dot{V}/\dot{\mathscr{E}}_{eq} V$ is the *absolute dilatation rate*, while $V/E_{33} V$ is the *relative dilatation rate* on which the relative effects of elongation and dilatation of the cavity may be seen.

3.2. Discussion of the results

Absolute and relative dilatation rates are plotted in solid lines on Figs. 1 to 4. Seven void volume fractions were considered:

$$f_1 = 0.012, \quad f_2 = 0.048, \quad f_3 = 0.11, \quad f_4 = 0.19, \quad f_5 = 0.30, \quad f_6 = 0.44, \quad f_7 = 0.59.$$

These porosities could seem fairly high. However, one should keep in mind that in the two-dimensional situation considered here, porosity varies with the square of the ratio between inner and outer radii of the cylindrical cell, while in the three-dimensional setting of spherical cavities it varies with the cube of this ratio. For instance, when the ratio diameter of cavities/distance between cavities is equal to 0.1, the void volume fraction amounts to 10^{-2} and 10^{-3}, respectively, in the two-dimensional or three-dimensional settings.

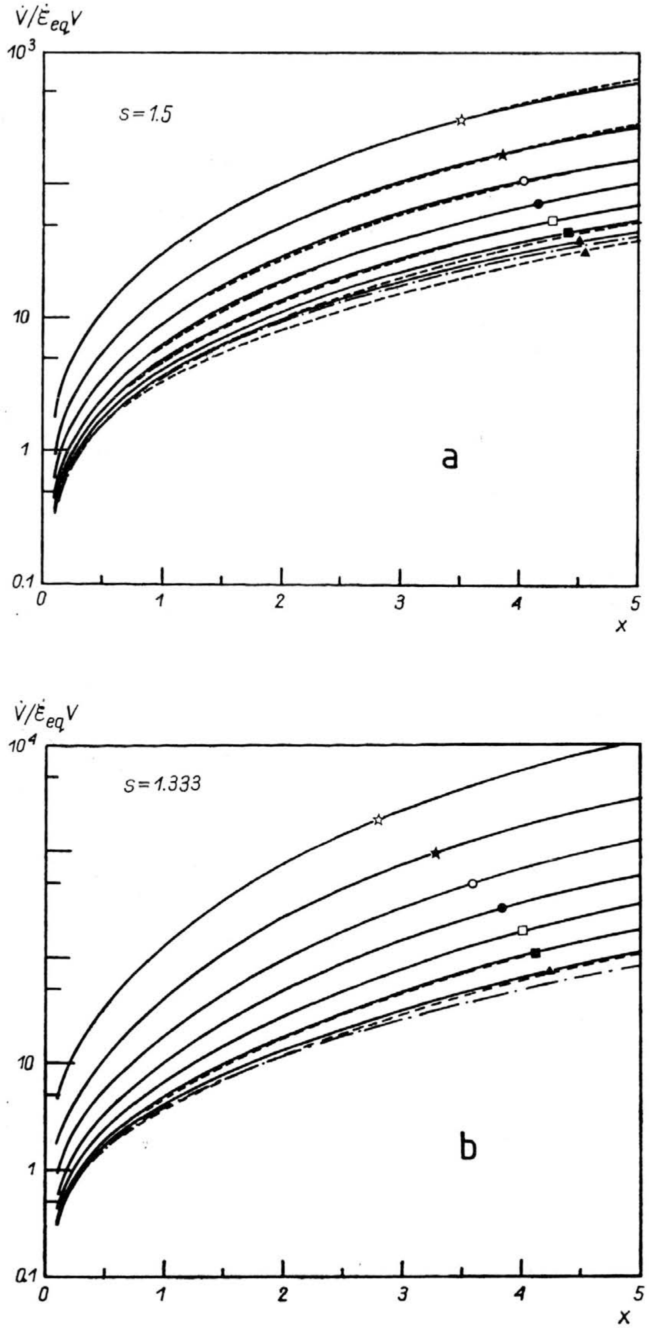
The dashed line reports the dilatation rate obtained by B.H.S. [2] in an infinite medium.

Observation of the *absolute void growth rate* $\dot{V}/\dot{\mathscr{E}} V$ clearly demonstrates its power law dependence on the stress ratio $T/(S-T)$. For nonlinear materials ($s \neq 2$) a strong influence of the void volume fraction can also be observed: when $s = 1.1$ and $T/S-T = 3.33$, a void volume fraction of 10^{-1} induces a dilatation rate 10^3 times greater than the value predicted for an infinite medium. However this ratio falls to 3 when $s = 1.3$, and is close to 1 for a Newtonian material ($s = 2$). The dependence of the dilatation rate on the void volume fraction is strongly enhanced by the material nonlinearity.

Observation of the *relative void growth rate* $\dot{V}/\dot{E}_{33} V$ shows that it increases with the stress triaxiality ratio (and is asymptotically a linear function of $T/(S-T)$ when this parameter approaches 0 and ∞, as will be shown in the next section), but is a decreasing

(³) Note that $\dot{E}_{33}$ and $\dot{\mathscr{E}}_{eq}$ are equal when $f = 0$, but are different in the general case.

function of void volume fraction. This latter result indicates a preponderance in the void growth of the axial elongation over the transverse dilatation. Sensitivity to void volume fraction is again enhanced by a strong material nonlinearity.



(FIG. 1a, b)

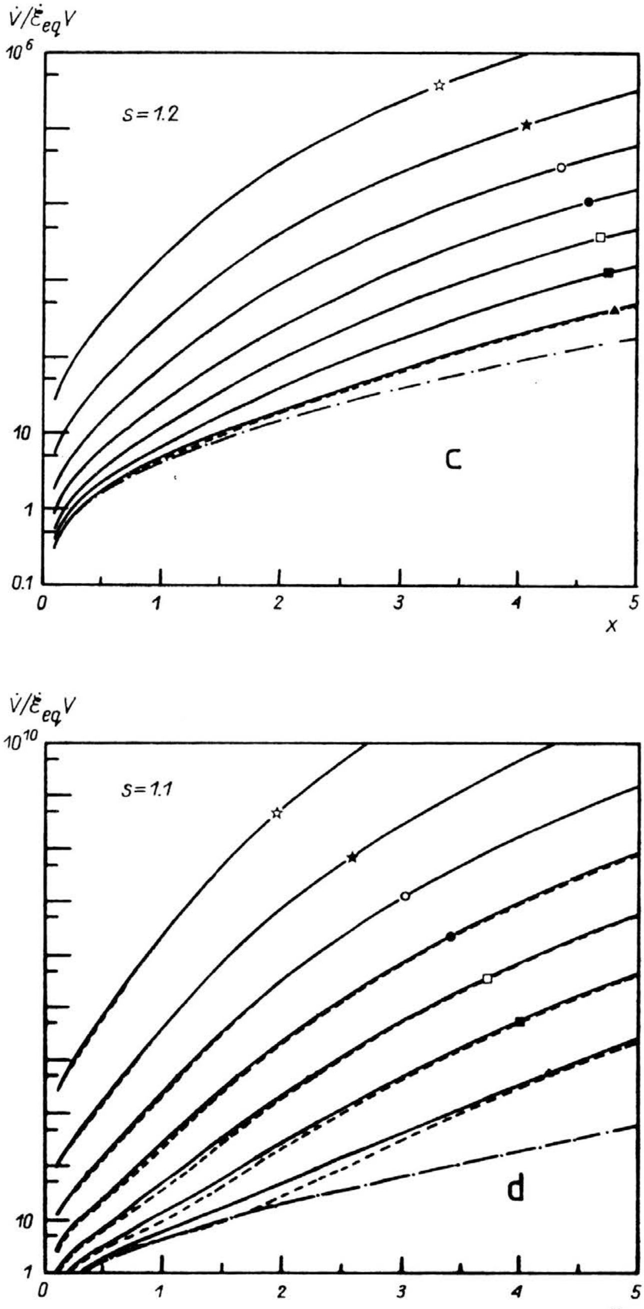
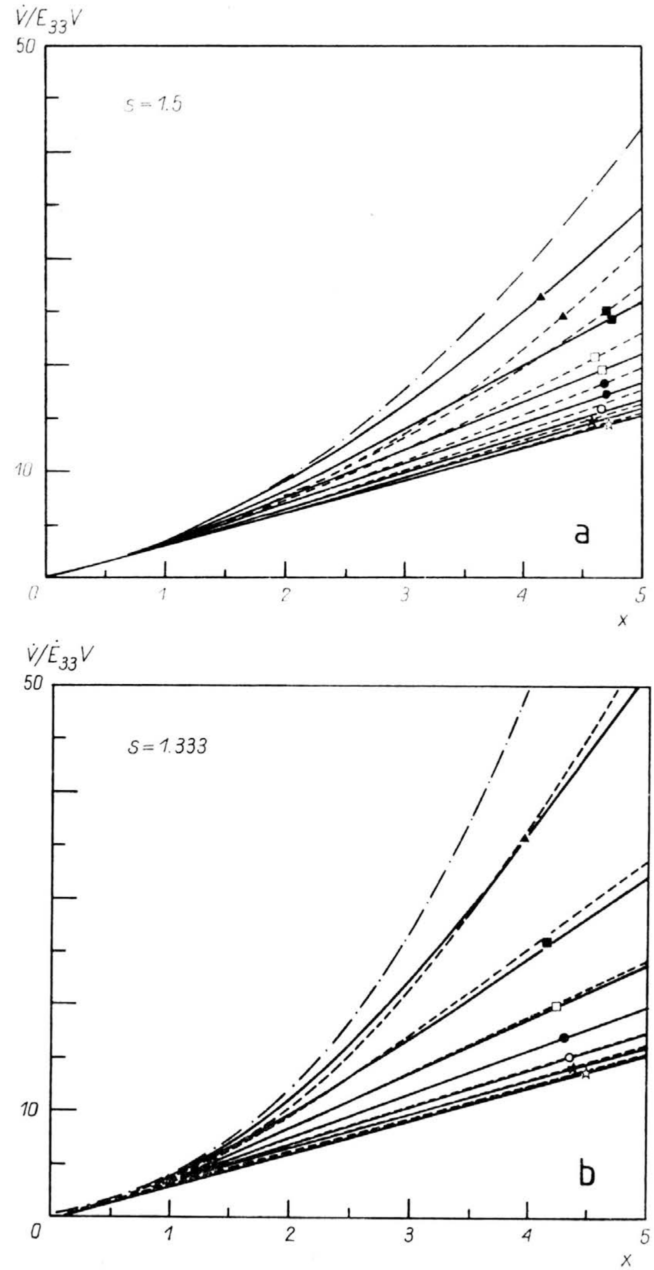


FIG. 1. Variations of the absolute growth rate with the stress ratio $x = T/(S-T)$;
—— exact, ----- interpolation formula (4.5), — . — infinite medium ([2]).
Influence of material nonlinearity a) $s = 1.5$, b) $s = 1.33$, c) $s = 1.2$, d) $s = 1.1$. Influence of void volume fraction f.
▲ — $f = 0.012$, ■ — $f = 0.048$, □ — $f = 0.11$, ● — $f = 0.19$, ○ — $f = 0.30$, ★ — $f = 0.44$, ✡ — $f = 0.59$.

4. Interpolated growth rates

4.1. Asymptotic expressions of the voids growth-rates

In order to quantify more precisely the variations of the voids growth-rates with the void volume fraction f and with the rate sensitivity parameter s, we derive asymptotic expressions for $\dot{V}/\dot{\mathscr{E}}_{eq}V$ and $\dot{V}/\dot{E}_{33}V$ for $T/(S-T)$ approaching 0 and $+\infty$.



(FIG. 2a, b)

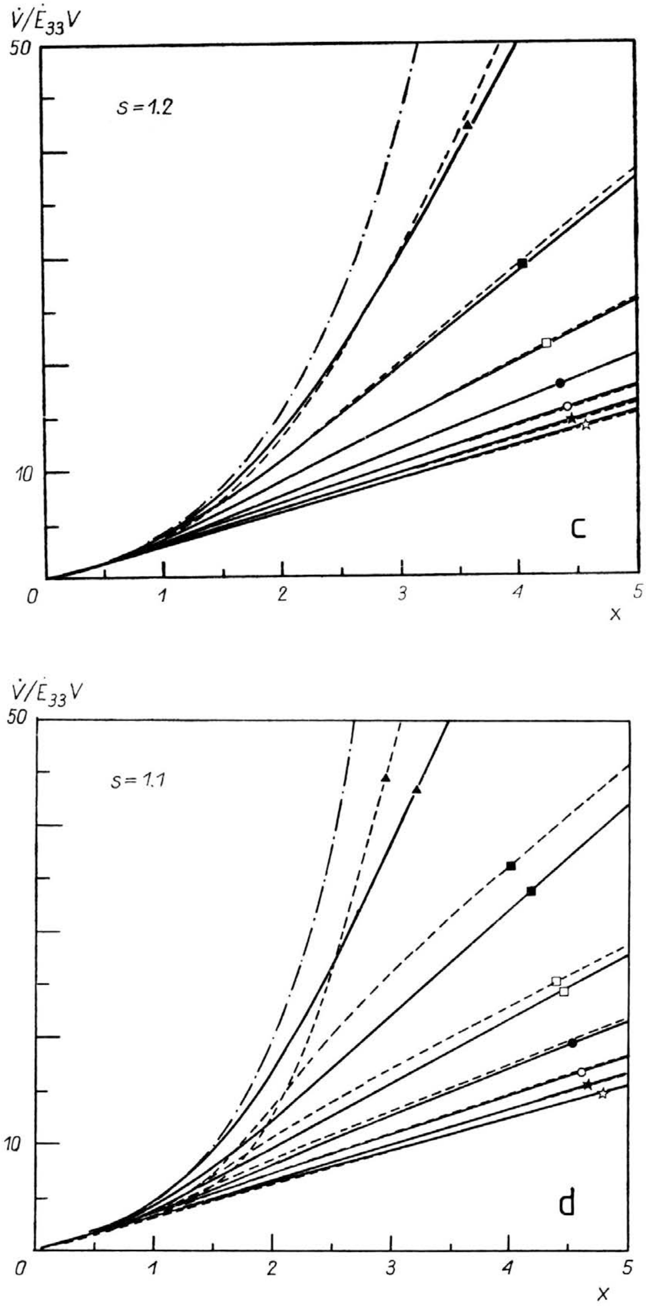


FIG. 2. Variations of the relative growth rate with the stress ratio $x = T/(S-T)$; —— exact, - - - - - interpolation formula (4.6), — . — infinite medium. Influence of material nonlinearity a) $s = 1.5$, b) $s = 1.33$, c) $s = 1.2$, d) $s = 1.1$. Influence of void volume fraction.

▲ — $f = 0.012$, ■ — $f = 0.048$, □ — $f = 0.11$, ● — $f = 0.19$, ○ — $f = 0.30$, ★ — $f = 0.44$, ✡ — $f = 0.59$.

These expressions are obtained from (2.4), (2.5) by means of the following simple expansions[4]:

$$(1+t^2)^{\frac{p-2}{2}} = 1+\frac{p-2}{2}t^2+0(t^2) \quad \text{in the neighborhood of } t=0,$$

$$(1+t^2)^{\frac{p-2}{2}} = |t|^{p-2}\left(1+\frac{p-2}{2t^2}+\ldots\right) \quad \text{in the neighborhood of } t=\infty.$$

We obtain for low values of $x = T/(S-T)$

$$\frac{\dot{V}}{\dot{\mathscr{E}}_{\text{eq}}V} = \operatorname{sgn}(S-T)3\left(\frac{1}{1-f}\right)^{\frac{1}{s-1}}x+0(x), \tag{4.1}$$

$$\frac{\dot{V}}{\dot{E}_{33}V} = 3x, \tag{4.2}$$

and for high values of $x = T/(S-T)$

$$\frac{\dot{V}}{\dot{\mathscr{E}}_{\text{eq}}V} \sim \operatorname{sgn}(T)(3)^{\frac{s}{2(s-1)}}|x|^{\frac{1}{s-1}}\left(\frac{s-1}{1-f^{s-1}}\right)^{\frac{1}{s-1}}, \tag{4.3}$$

$$\frac{\dot{V}}{\dot{E}_{33}V} \sim 3\,\frac{s-1}{s-3}\left(\frac{f-f^{s-2}}{1-f^{s-1}}\right)x, \tag{4.4}$$

where we recall that $s = (n+1)/n$.

These asymptotic expressions suggest a few brief comments. Concerning the absolute void growth-rate $\dot{V}/\dot{\mathscr{E}}_{\text{eq}}V$ we observe that it is governed for small values of x by $\left(\frac{1}{1-f}\right)^{1/(s-1)}$. When s varies from 2 to 1.1 this coefficient is raised to its tenth power. This operation has a spectacular effect as soon as the void volume fraction f is not equal to 0. When x is large the absolute void growth-rate varies as $|x|^{1/(s-1)}$ in agreement with B.H.S. [2] results for a spherical cavity in an infinite medium.

Concerning the relative void growth-rate we observe that, as soon as the void volume fraction f does not vanish, $\dot{V}/\dot{E}_{33}V$ has asymptotically a linear growth with x, for low or high values of x. The two slopes for low and high values of x are different, except in the Newtonian case ($s = 2$) for which $\dot{V}/\dot{E}_{33}V$ is identically equal to $3x$. However, when the effect of void volume fraction is neglected ($f = 0$) $\dot{V}/\dot{E}_{33}V$, which reduces to $\dot{V}/\dot{\mathscr{E}}_{\text{eq}}V$, is a power law function of $|x|$ with exponent $1/(s-1)$. Therefore the (relative) void growth-rate behaves differently in the two cases $f = 0$ and $f \neq 0$.

4.2. Interpolation of the void growth-rates for arbitrary stress triaxiality ratios

Interpolation formulas fitting the asymptotic expressions (4.1)–(4.4), can be proposed for the absolute and relative void growth-rates:

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[4] We use the fact that small (or large) values of $T/(S-T)$ corresponds to small (or large) values of ω in (2.7).

$$(4.5)\quad \frac{\dot{V}}{\dot{\mathscr{E}}_{eq} V} = \frac{\operatorname{sgn}(S-T)}{f}\left[2\left(B_L A_L - \frac{s'}{2} B_H A_H\right)x + s' B_H A_H x(1+A_H x^2)^{\frac{s'}{2}-1}\right],$$

$$(4.6)\quad \frac{\dot{V}}{\dot{E}_{33} V} = \frac{1}{f}\,\frac{2\left(B_L A_L - \frac{s'}{2} B_H A_H\right)x + s' B_H A_H x(1+A_H x^2)^{\frac{s'}{2}-1}}{s'(B_L - B_H) + (s'-2)\left(B_L A_L - \frac{s'}{2} B_H A_H\right) + s' B_H (1+A_H x^2)^{\frac{s'}{2}-1}}$$

where we have used the following notations

$$s' = \frac{s}{s-1}, \quad A_L = \frac{3f}{2} s', \quad B_L = \frac{1}{s'}\left(\frac{1}{1-f}\right)^{s'-1},$$

$$A_H = \frac{3(s-1)}{(s-3)} f^2 \left(\frac{1-f^{s-3}}{1-f^{s-1}}\right), \quad B_H = \frac{f}{s'}\left(\frac{3}{A_H}\right)^{s'/2}\left(\frac{s-1}{1-f^{s-1}}\right)^{\frac{1}{s-1}}.$$

Other interpolation formulas can obviously be proposed, but (4.5), (4.6) are not fortuitous. They derive from a single interpolation of the macroscopic stress potential. Additional information on the underlying theory can be found in [11].

4.3. Discussion of the results

The results obtained with the interpolation formulas are represented on Fiqs. 1 to 4 by dashed lines. In order to estimate properly the deviations between exact and interpolated results, readers' attention is called on the different scales of these figures, necessary to display the whole set of results. The choice of a logarithmic scale may hide a significant discrepancy: this is the case if Figs. 1 and 3 where the solid line and the dashed lines can hardly be distinguished while the deviation can reach 30% for $s = 1.1$. However, a detailed examination of the numerical results reveals the following general trends:

a. When $s = 1.2$ ($n = 5$), $s = 1.33$ ($n = 3$) the deviation of the interpolated formula from the exact result on both growth-rates is always lower than 10%, and the error is maximal at the lowest porosity $f = 1.1 \cdot 10^{-2}$ and for a stress ratio x in the range [1, 2].

b. When $s = 1.1$ ($n = 10$), the deviation can reach 40% for $f = 1.1 \cdot 10^{-2}$ and x larger than 1. The deviation decreases when f increases, and is smaller than 15% as soon as f is greater than 0.11.

c. Surprisingly when $s = 1.5$, and when $f = 1.1 \cdot 10^{-2}$, the deviation can reach 17% but decreases rapidly when f increases. It is smaller than 10% when f is larger than $4.8 \cdot 10^{-2}$.

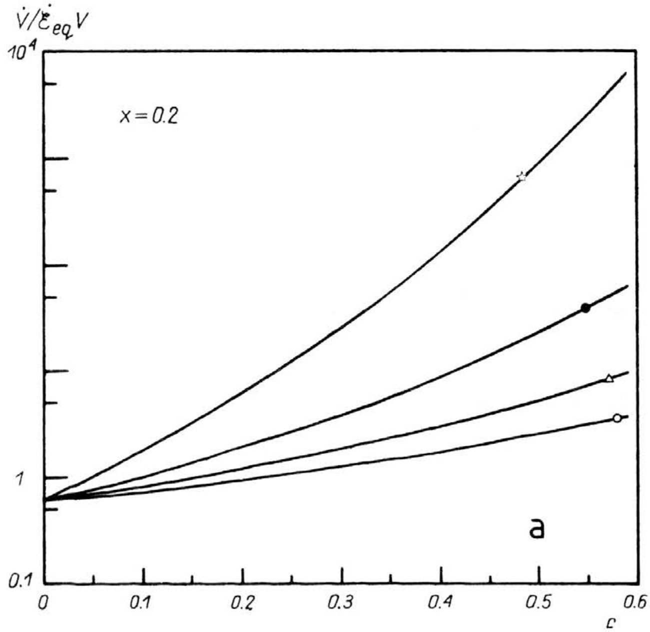
d. When $s = 2$ the interpolated formulas are exact.

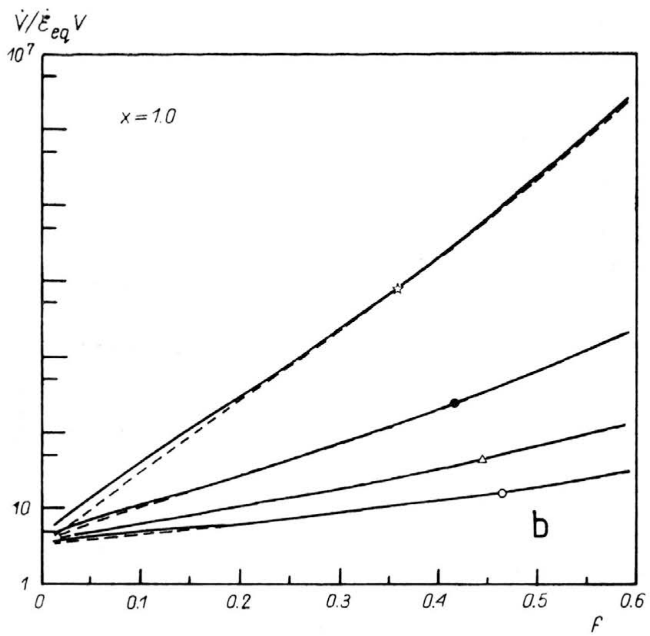
These observations suggest the following comments:

a. The interpolated formulas have an exact asymptotic behavior for small x or large x. We observe that the meaning of "large x" depends on the rate sensitivity s of the matrix material, and on the void volume fraction f. In most commonly encountered situations x is "large" when it is greater than 2. Consequently it is not surprising that greater deviations occur in a middle range of stress ratios x between 1 and 2.

b. The interpolation formulas assume in an essential manner that the void volume fraction f is not negligible. This assumption is used in the asymptotic expansions where f

is assumed to be of the same order as ω. Therefore the interpolation becomes inaccurate when f is very small. Dilutely voided materials have been studied, in a more complex geometry, by DUVA, HUTCHINSON [5] and we believe that their arguments should apply to the present situation, although we have not performed the complete analysis. Our interest in this paper was clearly to study the effect of a significant void volume fraction.





(FIG. 3a, b)

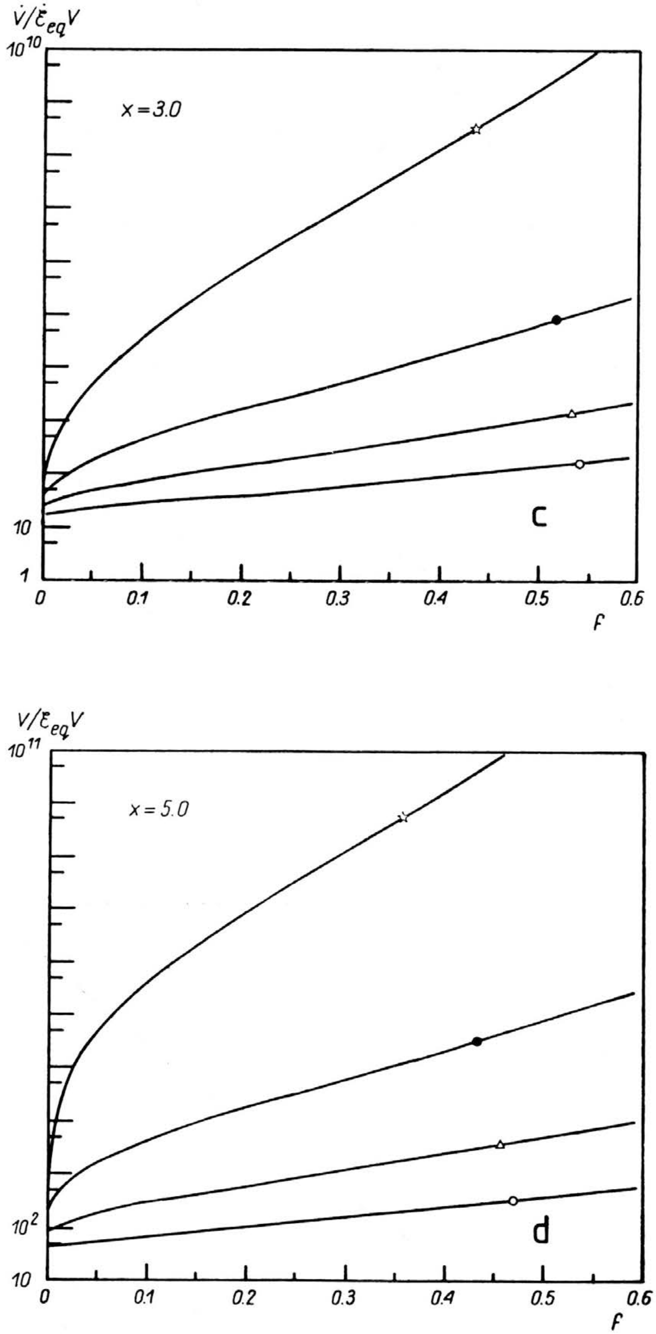
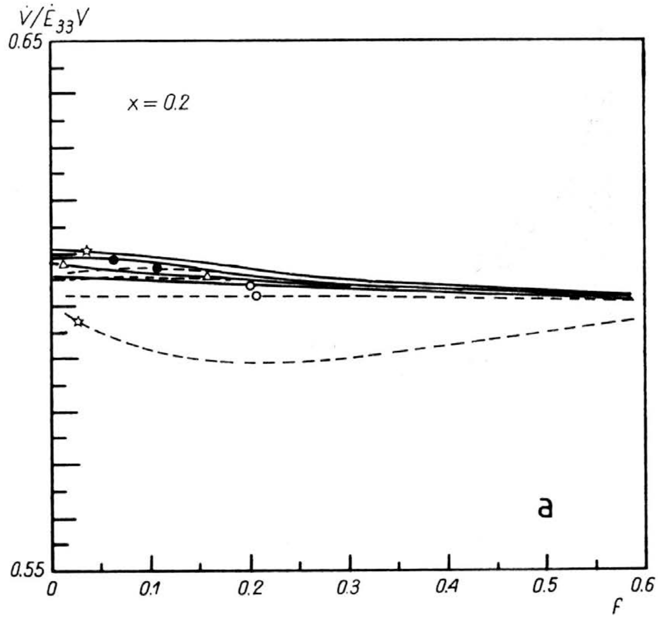
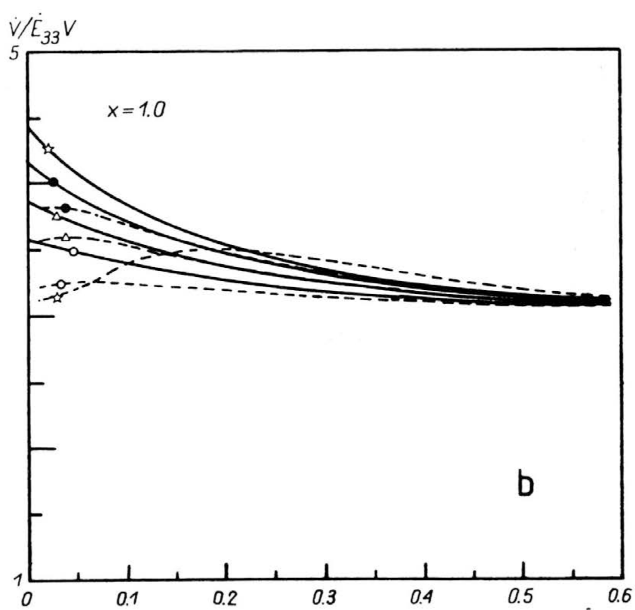


FIG. 3. Variations of the absolute growth rate with the void volume fraction f.
—— exact, ----- interpolation formula (4.5).
Influence of material nonlinearity. Influence of the stress ratio $x = T/(S-T)$. a) $x = 0.2$, b) $x = 1$, c) $x = 3$, d) $x = 5$.
○ — $s = 1.5$, △ — $s = 1.33$, ● — $s = 1.2$, ✡ — $s = 1.1$.

5. Conclusion

A simple model of cylindrical cavities growing in a nonlinear viscous matrix material has been studied. Examination of the exact void growth-rates clearly demonstrates the very strong influence of the existing void volume fraction. The growth-rate is enhanced by stress triaxiality and by material nonlinearity.





(FIG. 4a, b)

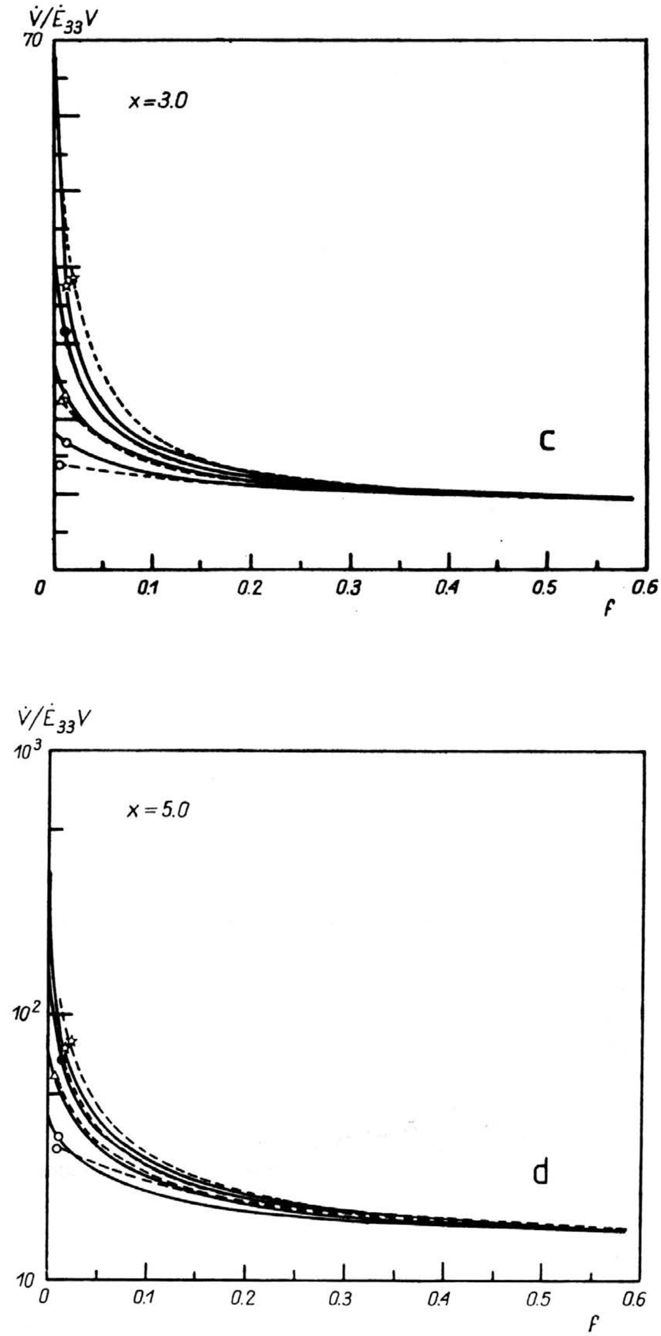


FIG. 4. Variations of the relative growth rate with the void volume fraction f.
—— exact, - - - - - interpolation formula (4.6).
Influence of material nonlinearity. Influence of the stress ratio $x = T/(S-T)$, a) $x = 0.2$, b) $x = 1$, c) $x = 3$, d) $x = 5$.
○ — $s = 1.5$, △ — $s = 1.33$, ● — $s = 1.2$, ✡ — $s = 1.1$.

An interpolation formula given in a closed form provides a reasonable approximation of the exact growth-rate, and its reliability allows to use it in macroscopic damage laws.

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