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Plane shear waves in viscoelastic fluids as motions with proportional stretch history

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IT IS SHOWN that so-called circular and elliptic shearing flows of incompressible simple fluids can be treated as a subclass of motions with proportional stretch history [7]. In the case of circular shearings some of Carroll's results for circularly polarized plane waves [6] are rediscovered in a different way. In the case of elliptic shearings some new solutions for low frequency elliptically or linearly polarized plane waves are also discussed.

Pokazano, że tzw. kołowe i eliptyczne przepływy ścinające nieściśliwych cieczy prostych można traktować jako podklasę ruchów z proporcjonalną historią deformacji [7]. W przypadku kołowego ścinania uzyskano w odmienny sposób niektóre wyniki Carrolla dla kołowo spolaryzowanych płaskich fal [6]. W przypadku eliptycznego ścinania przedyskutowano pewne nowe rozwiązania dla eliptycznie lub liniowo spolaryzowanych płaskich fal o niskich częstościach.

Показано, что т. наз. круговые и эллиптические течения сдвига несжимаемых простых жидкостей можно трактовать как подкласс движений с пропорциональной историей деформации [7]. В случае кругового сдвига получены другим образом некоторые результаты Карролла для кругого поляризованных плоских волн [6]. В случае эллиптического сдвига обсуждаются некоторые новые решения для эллиптически или линейно поляризованных плоских волн с низкими частотами.

1. Introduction

IN CONTRAST to various problems concerned with finite amplitude elastic shear waves solved and discussed in many previous papers, similar wave problems for viscoelastic or dissipative media attracted much less attention. Some papers were devoted to a general theory of waves as propagating surfaces of discontinuity [1], while others presented certain types of solutions for oscillating viscoelastic and elastic bodies or fluids of various complexity [2, 3, 4, 5, 6].

In one of the recent papers CARROLL [6] derived conditions under which a class of motion called plane circular shearings led to the case of finite amplitude plane progressive or standing waves in fluids and solids.

In this work we try to treat plane circular shearings and more general elliptic shearings as motions with proportional stretch history discussed in our paper [7]. To this end the motions considered should be expressed in the form of complex variable functions. In the case of circular shearings some of Carroll's results for circularly polarized plane waves are rediscovered in a different way, while in the case of ellipic shearings certain new results are established. It is shown, among others, that for sufficiently low frequencies enabling essential simplifications in the constitutive equations, the form of governing equations for elliptically polarized plane shear waves is identical to that for circularly polarized waves. The only differences appear in the corresponding relations for normal stress components.

We want to emphasize, moreover, that the class of flows considered may be treated as a particular case of more general unsteady homothermal motions discussed by CARROLL [8]. These latter motions are, in general, equivalent to a simple superposition of two motions with proportional stretch history [7].

2. Plane circular shearings of simple fluids

Let us consider a class of motions called the plane circular shearings, the equations of which are the following (cf. [6]);

$$x = X+\varphi(Z)\cos\omega\tau+\psi(Z)\sin\omega\tau,$$
$$y = Y+\varphi(Z)\sin\omega\tau-\psi(Z)\cos\omega\tau, \tag{2.1}$$
$$z = Z,$$

where x, y, z denote the Cartesian coordinates of a particle at arbitrary times τ, X, Y, Z — the Cartesian coordinates of the same particle in a reference configuration at time τ_R, ω — denotes constant angular frequency, and φ, ψ are certain functions of Z only. The corresponding velocity as well as acceleration fields can easily be calculated from Eqs. (2.1).

Introducing the auxiliary notations;

$$\varphi' = \varkappa\cos\theta, \quad \psi' = \varkappa\sin\theta, \tag{2.2}$$

where primes denote derivatives with respect to z, we obtain the relations

$$(\varphi'^2+\psi'^2)^{\frac{1}{2}} = \varkappa, \quad \theta = \operatorname{arctg}\psi'/\varphi', \tag{2.3}$$

where $\varkappa$ is the amount of shear. In the class of motions considered the planes $z = \text{const}$ are material surfaces, and the paths of particles correspond to circles of radii $(\varphi^2+\psi^2)^{\frac{1}{2}}$.

The deformation gradient at time τ with respect to the reference configuration can be written either as

$$[\mathbf{F}(\tau)] = \begin{bmatrix} 1 & 0 & \varkappa\cos(\omega\tau-\theta) \\ 0 & 1 & \varkappa\sin(\omega\tau-\theta) \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.4}$$

or alternatively in the complex form

$$\mathbf{F}(\tau) = \operatorname{Re}\{\exp[\mathbf{M}e^{i(\omega\tau-\theta)}]\}, \quad [\mathbf{M}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{bmatrix}\varkappa, \tag{2.5}$$

where $i = \sqrt{-1}$ and only the real part of $\mathbf{F}(\tau)$ is meaningful.

On writing Eq. $(2.5)_1$ in the abbreviated form

$$\mathbf{F}(\tau) = \operatorname{Re}\{\exp(\mathbf{M}k(\tau))\}, \quad k(\tau) = \exp i(\omega\tau-\theta), \tag{2.6}$$

it can be proved that plane circular shearings belong to the class of motions with proportional stretch history (MPSH) discussed elsewhere [7](1).

(1) In [7] MPSH were defined using the deformation gradient with respect to the reference configuration at time 0, i.e. $\mathbf{F}_0(\tau)$. The function $k(\tau)$ was such that $k(0) = 0$. These differences are insignificant since using the reference configuration at time τ_R, we assume $k(\tau_R) = 0$.

Introducing the relative deformation gradient with respect to the reference configuration chosen at present time $t(\tau \leqslant t)$:

$$\mathbf{F}_t(\tau) = \mathbf{F}(\tau)\mathbf{F}^{-1}(t) = \exp[\mathbf{M}(k(\tau)-k(t))], \quad \mathbf{F}_t(t) = \mathbf{1}, \tag{2.7}$$

and the corresponding history (cf. e.g. [9]):

$$\mathbf{F}(s) \equiv \mathbf{F}_t(t-s) = \exp[g(s)\mathbf{M}], \quad g(s) = k(t-s)-k(t), \tag{2.8}$$

where $s \in [0, \infty)$, we arrive at the history of the right relative Cauchy-Green deformation tensor in the form

$$\mathbf{C}(s) = \mathbf{F}^T(s)\mathbf{F}(s) = \exp(g(s)\mathbf{M}^T)\exp(g(s)\mathbf{M}) = \mathbf{1}+g(s)(\mathbf{M}^T+\mathbf{M}). \tag{2.9}$$

In the above relation

$$g(s) = e^{i\zeta}(e^{-i\omega s}-1), \quad \zeta = \omega t-\theta \tag{2.10}$$

and, moreover, we have used

$$\mathbf{M}^2 = \mathbf{M}^{T2} = \mathbf{M}^T\mathbf{M} = \mathbf{0}. \tag{2.11}$$

Bearing in mind definitions of the spatial velocity gradient $\mathbf{L}_1$ and the rotated parametric tensor $\mathbf{L}$ (cf. [7, 9])

$$\mathbf{L}_1(t) = \dot{\mathbf{F}}(t)\mathbf{F}^{-1}(t) = \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t)+\mathbf{Q}(t)\mathbf{M}\dot{k}(t)\mathbf{Q}^T(t), \tag{2.12}$$

$$\mathbf{L}(t) = \mathbf{Q}(t)\mathbf{M}\dot{k}(t)\mathbf{Q}^T(t), \tag{2.13}$$

where $\mathbf{Q}(t)$ is an orthogonal tensor characterizing the rotation of a particle from the reference configuration to the configuration at time t, we arrive at

$$\mathbf{C}(s) = \exp\left(\frac{g(s)}{\dot{k}(t)}\mathbf{L}^T\right)\exp\left(\frac{g(s)}{\dot{k}(t)}\mathbf{L}\right) = 1+\frac{g(s)}{\dot{k}(t)}(\mathbf{L}^T+\mathbf{L}), \tag{2.14}$$

since in our case $\mathbf{Q} \equiv \mathbf{1}$, and

$$\mathbf{L}_1(t) = \mathbf{L}(t) = \mathbf{M}\dot{k}(t). \tag{2.15}$$

Equations (2.9) and (2.14) are equivalent definitions of a subclass of MPSH, if tensors $\mathbf{M}$ and $\mathbf{L}$ do not depend on s (they may depend on t) and $g(s)$ is of the form (2.10).

For MPSH, the constitutive equation of an incompressible simple fluid (cf. [9]),

$$\mathbf{S}(t) = \mathop{\mathscr{F}}_{s=0}^{\infty}(\mathbf{C}(s)), \quad \det\mathbf{C}(s) = 1, \tag{2.16}$$

where $\mathbf{S}$ is the extra-stress tensor (or deviatoric part of the stress tensor) at time t, and $\mathscr{F}$ denotes an isotropic constitutive functional, takes the following form:

$$\mathbf{S}(t) = \mathop{\mathscr{G}}_{s=0}^{\infty}(g(s); \mathbf{L}), \tag{2.17}$$

where $\mathscr{G}$ is a functional of the scalar function $g(s)$, and an isotropic function of the tensor $\mathbf{L}$. According to the representation theorem proved in [7], the tensor $\mathbf{L}$ may be determined at most by the first three Rivlin-Ericksen kinematic tensors defined as follows (cf. [9]);

$$\mathbf{A}_n(t) = (-1)^n \frac{d^n\mathbf{C}(s)}{ds^n}\bigg|_{s=0}, \quad n = 1, 2, 3, \ldots. \tag{2.18}$$

In the present consideration we apply a slightly different approach but the final results are quite equivalent.

On the basis of Eqs. (2.9) and (2.18) we have, in particular,

$$\text{(2.19)}\qquad \mathbf{A}_1 = i\omega e^{i\zeta}(\mathbf{M}^T+\mathbf{M}), \quad \mathbf{A}_2 = -\omega^2 e^{i\zeta}(\mathbf{M}^T+\mathbf{M}),$$

what implies that $\mathbf{A}_2 = i\omega\mathbf{A}_1$. Taking into account Eqs. (2.9), (2.10) and (2.19), we obtain the following relation:

$$\text{(2.20)}\qquad \mathbf{C}(s) = 1-\frac{1}{\omega}\mathbf{A}_1\sin\omega s+\frac{1}{\omega^2}\mathbf{A}_2(1-\cos\omega s).$$

Substituting the above relation into Eq. (2.16), we arrive at

$$\text{(2.21)}\qquad \mathbf{S}(t) = \mathop{\mathscr{H}}_{s=0}^{\infty}\left(\frac{1}{\omega}\sin\omega s, \frac{1}{\omega^2}(1-\cos\omega s); \mathbf{A}_1(t), \mathbf{A}_2(t)\right) = f(\omega^2; \mathbf{A}_1(t), \mathbf{A}_2(t)),$$

where f is a tensor function even in ω, and isotropic with respect to $\mathbf{A}_1$ and $\mathbf{A}_2$.(2)

Taking into account the well-known Rivlin-Ericksen representation of an isotropic tensor function of two arguments (cf. [9]),

$$\text{(2.22)}\qquad \mathbf{S} = f(\omega^2; \mathbf{A}_1, \mathbf{A}_2) = \alpha_1\mathbf{A}_1+\alpha_2\mathbf{A}_2+\alpha_3\mathbf{A}_1^2+\alpha_4\mathbf{A}_2^2+\alpha_5(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1)$$
$$+\alpha_6(\mathbf{A}_1^2\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1^2)+\alpha_7(\mathbf{A}_1\mathbf{A}_2^2+\mathbf{A}_2^2\mathbf{A}_1)+\alpha_8(\mathbf{A}_1^2\mathbf{A}_2^2+\mathbf{A}_2^2\mathbf{A}_1^2),$$

as well as the properties (2.11), and the fact that $\mathbf{A}_2 = i\omega\mathbf{A}_1$, we obtain

$$\text{(2.23)}\qquad \mathbf{S} = (\alpha_1+i\omega\alpha_2)\mathbf{A}_1+(\alpha_3-\omega^2\alpha_4+2i\omega\alpha_5)\mathbf{A}_1^2,$$

where $\alpha_i(i = 1, \ldots 5)$ are functions of ω^2 and all invariants of the tensor $\mathbf{A}_1$.

Since tensor $\mathbf{A}_1$ is complex, its invariants should be composed of $\mathrm{Re}\mathbf{A}_1$ and $\mathrm{Im}\mathbf{A}_1$ or, equivalently, of $\mathbf{A}_1$ and $\mathbf{A}_1^*$, where $\mathbf{A}_1^*$ denotes the Hermitian-conjugate of $\mathbf{A}_1$. In the case considered these invariants are as follows(3):

$$\begin{aligned}
&\mathrm{tr}\mathbf{A}_1 = \mathrm{tr}\mathbf{A}_1^* = 0,\\
&\mathrm{tr}\mathbf{A}_1^2 = \mathrm{tr}\mathbf{A}_1^{*2} = 0, \quad \mathrm{tr}\mathbf{A}_1\mathbf{A}_1^* = 4\omega^2\varkappa^2,\\
\text{(2.24)}\qquad &\mathrm{tr}\mathbf{A}_1^2\mathbf{A}_1^* = \mathrm{tr}\mathbf{A}_1^{*2}\mathbf{A}_1 = \mathrm{tr}\mathbf{A}_1^3 = \mathrm{tr}\mathbf{A}_1^{*3} = 0,\\
&\mathrm{tr}\mathbf{A}_1^2\mathbf{A}_1^{*2} = 4\omega^4\varkappa^4 = \frac{1}{4}(\mathrm{tr}\mathbf{A}_1\mathbf{A}_1^*)^2.
\end{aligned}$$

The above results imply that the coefficients

$$\text{(2.25)}\qquad \alpha_i = \alpha_i(\omega^2, \omega^2\varkappa^2), \quad i = 1, 2, \ldots 5$$

are real functions of the real arguments ω^2, $\varkappa^2$. Thus, we can rewrite Eq. (2.23) in the form

$$\text{(2.26)}\qquad \mathrm{Re}\mathbf{S} = \alpha_1\mathrm{Re}\mathbf{A}_1-\omega\alpha_2\mathrm{Im}\mathbf{A}_1+(\alpha_3-\omega^2\alpha_4)\mathrm{Re}\mathbf{A}_1^2-2\omega\alpha_5\mathrm{Im}\mathbf{A}_1^2.$$

On using Eq. $(2.19)_1$ we arrive at the following real shear stress components:

$$\begin{aligned}
&\mathrm{Re}S^{13} = -\alpha_1\omega\varkappa\sin\xi-\alpha_2\omega^2\varkappa\cos\zeta,\\
\text{(2.27)}\qquad &\mathrm{Re}S^{23} = \alpha_1\omega\varkappa\cos\zeta-\alpha_2\omega^2\varkappa\sin\zeta,\\
&\mathrm{Re}S^{12} = -(\alpha_3-\omega^2\alpha_4)\omega^2\varkappa^2\sin2\zeta-2\alpha_5\omega^3\varkappa^2\cos2\zeta,
\end{aligned}$$

(2) Although $\mathbf{A}_2$ may formally be replaced by $i\omega\mathbf{A}_1$, it is more useful for the time being to treat f as a function of two tensor arguments.

(3) Only the products $\mathbf{A}_1^*\mathbf{A}_1 = \mathbf{A}_1^*\mathbf{A}_1$ and $\mathbf{A}_1^{*2}\mathbf{A}_1^2 = \mathbf{A}_1^2\mathbf{A}_1^{*2}$ are Hermitian (or self-conjugate), i.e. $(\mathbf{A}_1^*\mathbf{A}_1)^* = \mathbf{A}_1^*\mathbf{A}_1$ and $(\mathbf{A}_1^{*2}\mathbf{A}_1^2)^* = \mathbf{A}_1^{*2}\mathbf{A}_1^2$. Diagonal elements of Hermitian matrices are always real.

where $\zeta = \omega t - \theta$. In a similar way the normal extra-stress components are expressed as

$$\begin{aligned} \mathrm{Re}S^{11} &= -(\alpha_3-\omega^2\alpha_4)\omega^2\varkappa^2\cos 2\zeta + 2\alpha_5\omega^3\varkappa^2\sin 2\zeta, \\ \mathrm{Re}S^{22} &= (\alpha_3-\omega^2\alpha_4)\omega^2\varkappa^2\cos 2\zeta - 2\alpha_5\omega^3\varkappa^2\sin 2\zeta, \\ \mathrm{Re}S^{33} &= 0. \end{aligned} \tag{2.28}$$

It is seen that the real parts of S^{13}, S^{23} are odd functions of $\varkappa$, while S^{12}, S^{11} and S^{22} are even. On the other hand, only S^{23} and S^{12} are odd functions of ω and the remaining components are even.

The dynamical equations of equilibrium can be written in the form

$$\mathrm{div}(\mathbf{S}+p\mathbf{1}) - \varrho\,\mathrm{grad}\,\eta = \varrho\ddot{\mathbf{x}}, \tag{2.29}$$

where p is hydrostatic pressure, $\ddot{\mathbf{x}}$ — the acceleration vector, ϱ — density of a fluid, and η denotes a potential of conservative body forces. Since all the stress components depend on the variable z only (through the function $\varkappa(z)$), it is reasonable to assume that also $\eta = \eta(z)$. Taking into account Eqs. (2.29), (2.27), (2.28) and (2.2), we arrive at the following system of linear differential equations:

$$\begin{aligned} (\alpha_1\varphi' + \alpha_2\omega\psi')' - \varrho\omega\psi &= 0, \\ (\alpha_1\psi' - \alpha_2\omega\varphi')' + \varrho\omega\varphi &= 0, \\ (p+\eta)' &= 0, \end{aligned} \tag{2.30}$$

where primes denote derivatives with respect to z. The first two equations in the set (2.30) may be solved for appropriate boundary conditions, at least in a numerical way, if a dependence of α_1 and α_2 on ω^2 and $\varkappa^2$ is known from other considerations or experiments. The third equation in the set (2.30) gives the function of hydrostatic pressure p.

The system of equations (2.30) is fully equivalent to that derived in a different way by CARROLL [6]. To prove this, we may put into Eqs. (2.30) the following relations:

$$\alpha_1 = \gamma_4, \quad \alpha_2 = -\frac{1}{\omega^2}\gamma_5, \tag{2.31}$$

where γ_4, γ_5 are the functions used by Carroll. On the other hand, the material functions α_1 and α_2 are easily interpreted on the basis of Eqs. (2.22); α_1 is an apparent viscosity depending on ω^2 and $\varkappa^2$, α_2 — a function responsible for the elastic properties of a fluid (the ratio $T = |\alpha_2/\alpha_1|$ gives the characteristic time of a fluid).

3. Circularly polarized plane shear waves

If solutions $\varphi(z)$ and $\psi(z)$ of Eqs. (2.30) are periodic or, in particular, sinusoidal, the motion described by Eqs. (2.1) corresponds to the case of circularly polarized plane progressive or standing waves. It may be shown, in a way similar to [6], that if α_1 are independent of z (or $\varkappa(z)$), Eqs. (2.30) take the simplified form

$$\begin{aligned} \alpha_1\varphi'' + \alpha_2\omega\psi'' - \varrho\omega\psi &= 0, \\ \alpha_1\psi'' - \alpha_2\omega\varphi'' + \varrho\omega\varphi &= 0. \end{aligned} \tag{3.1}$$

The general solution of these equations is

$$\begin{aligned}\varphi(z) &= Ae^{-\alpha z}\cos(\beta z+\lambda)+Be^{\alpha z}\cos(\beta z+\mu),\\ \psi(z) &= Ae^{-\alpha z}\sin(\beta z+\lambda)-Be^{\alpha z}\sin(\beta z+\mu),\end{aligned} \tag{3.2}$$

where A, B, λ and μ are integration constants consistent with appropriate boundary conditions, and

$$(\alpha+i\beta)^2 = \frac{\varrho\omega}{\omega\alpha_2 \pm i\alpha_1}, \quad \alpha \geqslant 0, \quad \beta > 0. \tag{3.3}$$

The constant α characterizes an exponential decay or growth of the wave amplitude, while the constant β is simply related to the wave length. For Newtonian fluids ($\alpha_2 \equiv 0$), we have for example

$$\alpha^2 = \beta^2 = \frac{\varrho\omega}{2\alpha_1}. \tag{3.4}$$

The simplest boundary conditions for Eqs. (3.2) were discussed by CARROLL [6]. We present only some of his results.

For a semi-infinite fluid bounded by a rigid plate at $z = 0$, oscillating with the velocity components; $\dot{x} = V\cos\omega t$, $\dot{y} = V\sin\omega t$, $\dot{z} = 0$, the boundary conditions satisfied by φ and ψ are as follows:

$$\varphi(0) = \varphi(\infty) = \psi(\infty) = 0, \quad \psi(0) = \frac{V}{\omega}. \tag{3.5}$$

We have the case of a circularly polarized plane progressive wave if

$$A = \frac{V}{\omega}, \quad B = 0, \quad \lambda = \frac{\pi}{2}. \tag{3.6}$$

For a fluid contained between two plates, one fixed at $z = 0$ and the other oscillating with the same velocity components at the distance $z = l$, the boundary conditions are as follows:

$$\varphi(0) = \varphi(l) = \psi(0) = 0, \quad \psi(l) = \frac{V}{\omega}. \tag{3.7}$$

We have the case of a circularly polarized plane standing wave if

$$\begin{aligned}A &= -B = \frac{V}{2\omega}[\mathrm{sh}^2\alpha l\cos^2\beta l+\mathrm{ch}^2\alpha l\sin^2\beta l]^{\frac{1}{2}},\\ \mu &= -\lambda = \mathrm{arctg}[\mathrm{th}\,\alpha l\,\mathrm{ctg}\beta l].\end{aligned} \tag{3.8}$$

Let us briefly discuss the conditions under which the material functions α_1 and α_2 do not depend on the amount of shear $\varkappa$ (or equivalently on z). Bearing in mind Eqs. (2.22) and the fact that only the shear stress components S^{13} and S^{23} are involved in the corresponding equations of equilibrium (2.29), the following cases may be distinguished;

1. The case of Newtonian or purely viscous fluids for which only $\alpha_1 \neq 0$. This quantity may or not depend on the angular frequency ω;

2. The case of fluids with linear shear response or second-order fluids for which α_1 as well as α_2 do not depend on the amount of shear $\varkappa$.

3. The case of sufficiently slow oscillations, i.e. flows with moderately low angular frequencies ω as compared with the inverse of a fluid characteristic time T. Then α_1 and α_2 can be treated as material constants. This point will be clarified in the next section.

4. Plane elliptic shearings and elliptically polarized shear waves

Let us consider a class of motions which may be called the plane elliptic shearings;

$$
\begin{aligned}
x &= X+a\varphi(Z)\cos\omega\tau+a\psi(Z)\sin\omega\tau,\\
y &= Y+b\varphi(Z)\sin\omega\tau-b\psi(Z)\cos\omega\tau, \qquad (4.1)\\
z &= Z,
\end{aligned}
$$

where capital letters denote the Cartesian coordinates in a reference configuration at time τ_R, φ and ψ are certain functions of Z only, and $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$ are dimensionless constant parameters describing ellipticity of a motion.

On using the notations determined by Eqs. (2.2) and (2.3), we define the amount of shear $\varkappa$ and the angle θ. In the class of motions considered the planes $z =$ const are material surfaces, and the paths of particles are the ellipses $\varphi^2+\psi^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}$, where a and b are proportional to the corresponding axes.

The deformation gradient at time τ with respect to the reference configuration can be written in the form

$$
(4.2) \qquad [\mathbf{F}(\tau)] = \begin{bmatrix} 1 & 0 & a\varkappa\cos(\omega\tau-\theta) \\ 0 & 1 & b\varkappa\sin(\omega\tau-\theta) \\ 0 & 0 & 1 \end{bmatrix},
$$

or, alternatively,

$$
(4.3) \qquad \mathbf{F}(\tau) = \mathrm{Re}\{\exp[\mathbf{M}e^{i(\omega\tau-\theta)}]\}, \qquad [\mathbf{M}] = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & -ib \\ 0 & 0 & 0 \end{bmatrix}\varkappa,
$$

where again the real part of $\mathbf{F}(\tau)$ is meaningful.

It can be proved that plane elliptic shearings belong to the class of motions with proportional stretch history (MPSH) discussed elsewhere [7]. Considerations similar to those presented in Sect. 2 lead to the result

$$
(4.4) \qquad \mathbf{C}(s) = \mathbf{F}^T(s)\mathbf{F}(s) = \exp\big(g(s)\mathbf{M}^T\big)\exp\big(g(s)\mathbf{M}\big) = 1+g(s)(\mathbf{M}^T+\mathbf{M})+g^2(s)\mathbf{M}^T\mathbf{M},
$$

where the function $g(s)$ is defined by Eq. (2.10), and the following relations are used:

$$
(4.5) \qquad \mathbf{M}^2 = \mathbf{M}^{T2} = 0, \qquad [\mathbf{M}^T\mathbf{M}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a^2-b^2 \end{bmatrix}\varkappa^2 \neq 0.
$$

All the equations presented from (2.12) to (2.18) remain valid if the matrix [M] resulting from Eq. $(4.3)_2$ is used instead of Eq. $(2.5)_2$.

For plane elliptic shearings the first two Rivlin-Ericksen kinematic tensors defined by Eq. (2.18) take the form

$$
(4.6) \qquad \mathbf{A}_1 = i\omega e^{i\zeta}(\mathbf{M}^T+\mathbf{M}), \qquad \mathbf{A}_2 = -\omega^2 e^{i\zeta}(\mathbf{M}^T+\mathbf{M})-2\omega^2 e^{2i\zeta}\mathbf{M}^T\mathbf{M},
$$

what means that there is no simple relation between $\mathbf{A}_1$ and $\mathbf{A}_2$. Equation (4.4), after taking into account Eqs. (2.10) and (4.6), leads to

$$\mathbf{C}(s) = 1 - \frac{i}{\omega}(e^{-i\omega s}-1)\mathbf{A}_1 + \frac{1}{2\omega}(e^{-i\omega s}-1)^2\left(\mathbf{A}_1 - \frac{1}{\omega}\mathbf{A}_2\right). \tag{4.7}$$

Substituting the above relation into the constitutive equation (2.16), we arrive at

$$\mathbf{S}(t) = \mathop{\mathscr{K}}_{s=0}^{\infty}\left(\frac{1}{\omega}(e^{-i\omega s}-1), \frac{1}{\omega}(e^{-i\omega s}-1)^2; \mathbf{A}_1(t), \mathbf{A}_2(t)\right) = \mathscr{h}(\omega; \mathbf{A}_1(t), \mathbf{A}_2(t)), \tag{4.8}$$

where $\mathscr{h}$ is a tensor function of ω, isotropic with respect to $\mathbf{A}_1$ and $\mathbf{A}_2$. The function $\mathscr{h}$ is not even in ω since two scalar arguments appearing in Eq. $(4.8)_1$ are neither even nor odd.

Because of no simple relation between $\mathbf{A}_1$ and $\mathbf{A}_2$, the full representation (2.22) for two tensor arguments must be applied. The resulting constitutive equations are too complex for any effective solution of the problem. For example, the fact that

$$\mathbf{A}_2 = i\omega\mathbf{A}_1 - 2\omega^2 e^{2i\zeta}\mathbf{M}^T\mathbf{M}, \tag{4.9}$$

where $\mathbf{M}^T\mathbf{M}$ is determined by Eq. $(4.5)_3$, does not lead to essential simplifications. To achieve more progress in the flow considered, we shall try to aply an expansion procedure, similar to that proposed by NIILER and PIPKIN [2] for shear waves in some non-Newtonian fluids.

Starting from Eqs. (2.22), we shall seek approximate constitutive equations which approach Newtonian equations at very low angular frequencies. The angular frequency ω will enter into the constitutive equation through a dimensionless parameter ε defined as follows:

$$\varepsilon^2 = \omega T, \tag{4.10}$$

where T is the characteristic time of a fluid. The finite amplitude of a Newtonian solution A is expressed by the dimensionless parameter Q;

$$Q = A\left(\frac{\varrho}{\alpha_1 T}\right)^{\frac{1}{2}}. \tag{4.11}$$

Moreover, we introduce the following dimensionless quantities:

$$\overline{\mathbf{S}} = \frac{\mathbf{S}}{Ak\omega\alpha_1}, \quad \overline{\mathbf{A}}_n = \frac{\mathbf{A}_n}{Ak\omega^n}, \quad n = 1, 2, \ldots, \tag{4.12}$$

where $k = (\varrho\omega/\alpha_1)^{\frac{1}{2}}$. The first material coefficients α_i occurring in Eq. (2.22) can be written as follows;

$$\alpha_1 = \mu_0, \quad \alpha_2 = \beta_2\mu_0 T, \quad \alpha_3 = \beta_3\mu_0 T, \quad \text{etc.}, \tag{4.13}$$

where μ_0 denotes the apparent viscosity at zero shear rate, and $\beta_2, \beta_3, \ldots$ are dimensionless material coefficients.

Assuming that the product $\varepsilon^2 = \omega T$ is sufficiently small as compared with unity, and Q is constant, the constitutive equations may be expanded into a dimensionless form:

$$\overline{\mathbf{S}} = \overline{\mathbf{A}}_1 + \varepsilon^2\beta_2\overline{\mathbf{A}}_2 + \varepsilon^3\beta_3 Q\overline{\mathbf{A}}_1^2 + O(\varepsilon^4). \tag{4.14}$$

If all terms of order greater than ε^3 may be disregarded, the dimensionless material coefficients β_2, β_3 are constants independent of ω (cf. $(4.8)_2$).

The above procedure shows that for moderately low angular frequencies ω or, strictly speaking, for sufficiently small $\varepsilon^2 = \omega T$, where T is the characteristic time of a fluid, the approximate constitutive equations of a second-order incompressible fluid may be applied. These are in the form (cf. [9])

$$\mathbf{S} = \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{A}_1^2, \quad \mathrm{tr}\mathbf{A}_1 = 0, \tag{4.15}$$

where $\alpha_1, \alpha_2, \alpha_3$ denote material constants.

Thus, taking into account Eqs. $(4.5)_3$, (4.6) and (4.15), we obtain the following real parts of shear stress components;

$$\begin{aligned} \mathrm{Re}S^{13} &= -\alpha_1 \omega a\varkappa \sin\zeta - \alpha_2 \omega^2 a\varkappa \cos\zeta, \\ \mathrm{Re}S^{33} &= \alpha_1 \omega b\varkappa \cos\zeta - \alpha_2 \omega^2 b\varkappa \sin\zeta, \\ \mathrm{Re}S^{12} &= -\alpha_3 \omega^2 ab\varkappa^2 \sin 2\zeta, \end{aligned} \tag{4.16}$$

and the normal extra-stress components:

$$\begin{aligned} \mathrm{Re}S^{11} &= -\alpha_3 \omega^2 a^2 \varkappa^2 \cos 2\zeta, \\ \mathrm{Re}S^{22} &= \alpha_3 \omega^2 b^2 \varkappa^2 \cos 2\zeta, \\ \mathrm{Re}S^{33} &= -(2\alpha_2 + \alpha_3)\omega^2 (a^2 - b^2)\varkappa^2 \cos 2\zeta, \end{aligned} \tag{4.17}$$

where $\zeta = \omega t - \theta$. We may easily observe that S^{13}, S^{23} are odd in $\varkappa$, while the remaining stresses are even. The shear stresses S^{13}, S^{23} oscillate with the angular frequency ω while the remaining components with 2ω.

Since all the stress components depend on the variable z only (through the function $\varkappa(z)$), the dynamical equations of equilibrium (2.24) lead to the following system of linear differential equations:

$$\begin{aligned} \alpha_1 \varphi'' + \alpha_2 \omega \psi'' - \varrho \omega \psi &= 0, \\ \alpha_1 \psi'' - \alpha_2 \omega \varphi'' + \varrho \omega \varphi &= 0, \\ (p+\eta)' + \omega^2 (2\alpha_2 + \alpha_3)(a^2 - b^2)[(\varphi'^2 - \psi'^2)\cos 2\omega t + 2\varphi'\psi' \sin 2\omega t]' &= 0, \end{aligned} \tag{4.18}$$

where primes denote derivatives with respect to z, and $\eta = \eta(z)$ is a potential of conservative body forces. The first two equations in the set (4.18) are identical to Eqs. (3.1), the solution of which can be expressed by Eqs. (3.2) and (3.3). The third equation in the set (4.18) determines the hydrostatic pressure p if φ and ψ are known.

In full analogy to our previous considerations in Sect. 3, we claim that under the assumed order of approximation the motion described by Eqs. (4,1) with periodic φ and ψ determined by Eqs. (3.2) corresponds to the case of elliptically polarized plane progressive or standing waves. The examples briefly discussed in Sect. 3 can easily be solved with slightly modified boundary conditions, i.e. for plates oscillating with the following velocity components: $\dot{x} = aV\cos\omega t$, $\dot{y} = bV\sin\omega t$, $\dot{z} = 0$.

It is worthwhile to note that the governing equations for low frequency elliptically polarized shear waves do not differ at all from those for low frequency circularly polarized shear waves. Certain essential differences exist in the form of the hydrostatic pressure function as well as in the normal extra-stress components. These observations may be of importance in such cases in which circularly polarized plane progressive waves are refracted on an interface between two non-mixing fluids. If the direction of propagation of a primary circularly polarized wave is not perpendicular to the interface, a resulting refracted wave will be elliptically polarized with the parameters a and b depending on the corresponding angles of incidence and refraction.

5. Linearly polarized plane shear waves

It is easy to see that the results obtained for low frequency elliptically polarized plane shear waves are, in particular, valid for low frequency linearly polarized waves. If waves are linearly polarized in the plane xz, we take in Eqs. (4.1) $a = 1$ and $b = 0$, if in the plane yz, then $a = 0$, $b = 1$.

For example, if $a = 1$, $b = 0$, the only non-vanishing extra stress components are as follows;

$$\begin{aligned} \mathrm{Re} S^{13} &= -\alpha_1 \omega \varkappa \sin\zeta - \alpha_2 \omega^2 \varkappa \cos\zeta, \\ \mathrm{Re} S^{11} &= -\alpha_3 \omega^2 \varkappa^2 \cos 2\zeta, \\ \mathrm{Re} S^{33} &= -(2\alpha_2 + \alpha_3)\omega^2 \varkappa^2 \cos 2\zeta, \end{aligned} \tag{5.1}$$

where $\zeta = \omega t - \theta$. The differential equations describing linearly polarized shear waves are again int the form of the set (4.18), and all further remarks remain valid.

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