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Some existence results in dynamical thermoelasticity
Part I. Nonlinear case

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THE PURPOSE of this paper is to show that classical energy methods can be used in the proof of existence solutions in nonlinear dynamical thermoelasticity. We show that a solution of the theory exists for sufficiently small times but is global in the space and belongs to Sobolev spaces of orders sufficiently large to be smooth and to satisfy the equations, boundary and initial conditions in the classical sense. Similar methods for nonlinear elastodynamics were used in [1].

W pracy zastosowano klasyczne metody energetyczne oraz twierdzenie Banacha o punkcie stałym do dowodu istnienia rozwiązań dla nieliniowej dynamicznej termosprężystości. Udowodniono istnienie rozwiązań lokalnych w czasie, globalnych przestrzennie i należących do przestrzeni Sobolewa rzędów dostatecznie wysokich, aby równania oraz warunki brzegowe i początkowe spełnione były w klasycznym sensie.

В работе применены классические энергетические методы и теорема Банаха о неподвижной точке для доказательства существования решений в нелинейной динамической термоупругости. Полученные решения являются локальными во времени, глобальными в пространстве и принадлежат пространствам Соболева достаточно высоких порядков, чтобы уравнения, граничные и начальные условия были выполнены в классическом смысле.

1. Introduction

LET A HOMOGENEOUS thermoelastic body $\mathscr{B}$ be identified with the bounded domain of space it occupies in a fixed reference configuration $\varkappa$ and let $]0, T[$ denote the finite interval of time. Let $\chi_\varkappa(X, t)$ denote the position of the point $X \in \mathscr{B}$ at the time $t \in]0, T[$. If we use the standard notations (see [2], Chapt. XV) $T_\varkappa$ — the Piola stress tensor, $\varrho_\varkappa$ — the mass density, $\mathbf{b}$ — the body force, η — the entropy, $\boldsymbol{\theta}$ — the temperature, $h_\varkappa$ — the heat flux, s — the heat supply, ψ — the Helmholtz free energy (all the quantities are considered with respect to the reference configuration), then the equations of thermoelasticity can be written in the following form (see [2], form. VII. 2–6, XV. 2–10, XV. 3–15, XV. 3–18)

$$(1.1)\qquad \varrho_\varkappa \ddot{\chi}_\varkappa = \operatorname{Div} T_\varkappa + \varrho_\varkappa \mathbf{b},$$

$$(1.2)\qquad \boldsymbol{\theta}\dot{\eta} = \frac{1}{\varrho_\varkappa}\operatorname{Div} h_\varkappa + s, \qquad \text{in} \quad \mathscr{B}\times]0, T[,$$

$$(1.3)\qquad \begin{aligned} \psi &= \psi(\nabla\chi_\varkappa, \boldsymbol{\theta}), & \eta &= -\partial_{\boldsymbol{\theta}}\psi(\nabla\chi_\varkappa, \boldsymbol{\theta}),\\ T_\varkappa &= \varrho_\varkappa \partial_{\nabla\chi_\varkappa}\psi(\nabla\chi_\varkappa, \boldsymbol{\theta}), & h_\varkappa &= h_\varkappa(\nabla\chi_\varkappa, \boldsymbol{\theta}, \nabla\boldsymbol{\theta}). \end{aligned}$$

To obtain Eqs. (1.1) and (1.2) in Cartesian coordinates we put for arbitrary α, β, p, $q = 1, 2, 3$,

$$(1.4)\qquad \begin{aligned} A^{\alpha\beta}_{pq} &= \partial_{F_{q\beta}}\partial_{F_{p\alpha}}\psi, & A^{\alpha}_{p} &= \partial_{\theta}\partial_{F_{p\alpha}}\psi,\\ b_0 &= -\theta\partial^2_{\theta}\psi, & B^{\alpha\beta} &= \frac{1}{\varrho_{\varkappa}}\partial_{G_{\alpha}}h_{\varkappa\beta},\\ B^{\alpha} &= \frac{1}{\varrho_{\varkappa}}\partial_{\theta}h_{\varkappa\alpha}, & B^{\alpha\beta}_{q} &= \frac{1}{\varrho_{\varkappa}}\partial_{F_{q\alpha}}h_{\varkappa\beta},\\ & & B^{\alpha}_{q} &= -\theta\partial_{F_{q\alpha}}\partial_{\theta}\psi, \end{aligned}$$

where

$$F_{p\alpha} = \partial_{X^{\alpha}}\chi_{\varkappa p}, \qquad G_{\alpha} = \partial_{X^{\alpha}}\theta$$

and

$$\chi_{\varkappa} = \{\chi_{\varkappa p}\}_{p=1,2,3}, \qquad h_{\varkappa} = \{h_{\varkappa p}\}_{p=1,2,3}, \qquad X = \{X^{\alpha}\}_{\alpha=1,2,3}.$$

In consequence the system (1.1), (1.2) takes the form

$$(1.5)\qquad \ddot{\chi}_{\varkappa p} = A^{\alpha\beta}_{pq}\chi_{\varkappa q,\alpha\beta}+A^{\alpha}_{p}\theta_{,\alpha}+b_p, \qquad p = 1, 2, 3,$$

$$(1.6)\qquad b_0\dot{\theta} = B^{\alpha\beta}\theta_{,\alpha\beta}+B^{\alpha}\theta_{,\alpha}+B^{\alpha\beta}_{q}\chi_{\varkappa q,\alpha\beta}+B^{\alpha}_{q}\dot{\chi}_{\varkappa q,\alpha}+s,$$

where $\mathsf{b} = \{b_p\}_{p=1,2,3}$; moreover the summation over repeated indices α, β, $q = 1, 2, 3$ is implied and the notations $\varphi_{,\alpha} \equiv \partial_{X^{\alpha}}\varphi$, $\varphi_{,\alpha\beta} \equiv \partial_{X^{\beta}}\partial_{X^{\alpha}}\varphi$ are used. Let us note that the coefficients $A^{\alpha\beta}_{pq}$, A^{α}_{p}, b_0, B^{α}_{q} depend on $\nabla\chi_{\varkappa}$, θ only and $B^{\alpha\beta}$, B^{α}, $B^{\alpha\beta}_{q}$ depend additionally on $\nabla\theta$.

In the present paper we consider Eqs. (1.5) and (1.6) together with the initial conditions

$$(1.7)\qquad \chi_{\varkappa}(0) = \overset{0}{\chi}_{\varkappa}, \quad \dot{\chi}_{\varkappa}(0) = \overset{1}{\chi}_{\varkappa}, \quad \theta(0) = \overset{0}{\theta} \quad \text{on } \mathscr{B}$$

and the boundary conditions

$$(1.8)\qquad \chi_{\varkappa} = \chi_{\varkappa 0}, \quad \theta = \theta_0 \quad \text{on } \partial\mathscr{B}\times[0, T].$$

Here and in the sequel the dependence of all functions on $X \in \mathscr{B}$ is consequently omitted but the dependence on $t \in [0, T]$ is sometimes explicitly written, as for example in the conditions (1.7) where $t = 0$.

Having $\overset{0}{\chi}_{\varkappa}$, $\overset{1}{\chi}_{\varkappa}$, $\overset{0}{\theta}$, we can define the new symbols $\overset{2}{\chi}_{\varkappa}, \ldots, \overset{m}{\chi}_{\varkappa}$, $\overset{1}{\theta}, \ldots, \overset{m-1}{\theta}$, $m \geqslant 4$ using the recursive formulas

$$\overset{k+2}{\chi}_{\varkappa p} = \sum_{i=0}^{k}\binom{k}{i}\overset{(i)}{A}{}^{\alpha\beta}_{pq}(0)\overset{k-i}{\chi}_{\varkappa q,\alpha\beta}+\sum_{i=0}^{k}\binom{k}{i}\overset{(i)}{A}{}^{\alpha}_{p}(0)\overset{k-i}{\theta}_{,\alpha}+\overset{(k)}{b}_p(0), \qquad p = 1, 2, 3,$$

$$(1.9)\qquad b_0(0)\overset{k+1}{\theta} = \sum_{i=0}^{k}\binom{k}{i}\overset{(i)}{B}{}^{\alpha\beta}(0)\overset{k-i}{\theta}_{,\alpha\beta}+\sum_{i=0}^{k}\binom{k}{i}\overset{(i)}{B}{}^{\alpha}(0)\overset{k-i}{\theta}_{,\alpha}+\sum_{i=0}^{k}\binom{k}{i}\overset{(i)}{B}{}^{\alpha\beta}_{q}(0)\overset{k-i}{\chi}_{\varkappa q,\alpha\beta}$$

$$+\sum_{i=0}^{k}\binom{k}{i}\overset{(i)}{B}{}^{\alpha}_{q}(0)\overset{k-i+1}{\chi}_{\varkappa q,\alpha}+\overset{(k)}{s}(0)-\sum_{i=1}^{k}\binom{k}{i}\overset{(i)}{b}_0(0)\overset{k-i+1}{\theta}, \qquad k = 0, \ldots, m-2.$$

In the formulas (1.9) the symbol $\overset{(i)}{A}{}^{\alpha\beta}_{pq}(0)$ denotes the i-th time derivative of the function $t \to A^{\alpha\beta}_{pq}(\nabla\chi_\varkappa(t), \boldsymbol{\theta}(t))$ taken at $t = 0$ and similar abbreviations are used for other symbols of this type.

The functions $\overset{2}{\chi}_\varkappa, \ldots, \overset{m}{\chi}_\varkappa, \overset{1}{\boldsymbol{\theta}}, \ldots, \overset{m-1}{\boldsymbol{\theta}}$ will play an important role in a formulation of our main theorem. Before we give this formulation we recall some notations concerning function spaces.

2. Function spaces

In general, we use the notations of the paper [1] but some modifications are necessary. If γ is any real number, then the symbols $\mathsf{H}_\gamma \equiv W^{\gamma,2}(\mathscr{B}, R^3)$ and $H_\gamma \equiv W^{\gamma,2}(\mathscr{B}, R^1)$ denote the Sobolev spaces of functions with values in R^3 and R^1, respectively. In both cases the corresponding norm will be denoted by $\|\cdot\|_\gamma$. As usual the spaces $\mathsf{V} \equiv W_0^{1,2}(\mathscr{B}, R^3)$ and $V \equiv W_0^{1,2}(\mathscr{B}, R^1)$ consist of functions vanishing on the boundary $\partial\mathscr{B}$ of $\mathscr{B}$. In the paper we shall frequently use the abbreviations $X_\gamma = \mathsf{H}_\gamma \cap \mathsf{V}$, $Y_\gamma \equiv H_\gamma \cap V$.

For the Banach spaces Z^1, Z^2 the symbol $\mathscr{L}(Z^1, Z^2)$ will denote the space of bounded linear opaerators from Z^1 to Z^2. Moreover, we set

$$\mathscr{L}_1 = \bigcap_{j=-1}^{m-1} \mathscr{L}(Z_{j+1}, Z_j), \qquad \mathscr{L}_2 = \bigcap_{j=-1}^{m-2} \mathscr{L}(Z_{j+2}, Z_j),$$
$$\mathscr{L}_k = \bigcap_{j=0}^{m-k} \mathscr{L}(Z_{j+k}, Z_j), \qquad k = 3, \ldots, m, \qquad m \geqslant 4,$$

where $Z_j = \mathsf{H}_j$ or $Z_j = H_j$, $j = -1, 0, \ldots, m$. The spaces $\mathscr{L}_1, \ldots, \mathscr{L}_m$ are equipped with standard operator norms denoted by $\|\cdot\|_{\mathscr{L}_k}$.

For $T > 0$, $1 \leqslant p \leqslant \infty$, $k = 0, 1, \ldots$ and for the Banach space Z we use the symbol $W^{k,p}([0, T], Z)$ to denote the Sobolev spaces of functions defined on $[0, T]$ and with values in Z (see [3], Chapt. I, Sect. 1.3). If $k = 0$, we obtain the space $L^p([0, T], Z)$ of Z-valued functions strongly measurable and integrable over $[0, T]$ with the p-th power for $1 \leqslant p < \infty$ and essentially bounded on $[0, T]$ if $p = \infty$. The symbol $C^k([0, T], Z)$ denotes the space of k-times continuously differentiable functions from $[0, T]$ to Z.

Finally, for an open subset $\mathcal{O}$ of R^n the symbol $C_b^k(\mathcal{O})$ will denote the set of all functions possessing continuous and bounded derivatives to the order k.

3. Formulation of the main theorem

Let m be any integer such that $m \geqslant 4$. Assume that the reference configuration of the thermoelastic body is a bounded domain $\mathscr{B} \subset R^3$ with the boundary of class C^m.

Suppose that $\mathcal{O}_i$, $i = 1, 2, 3$ are open sets such that

$$\begin{aligned} &\mathcal{O}_1 \subseteq \{F \in R^{3\times 3} : \det F > 0\}, \quad \text{(we identify the set of } 3\times 3 \text{ matrices with } R^{3\times 3}),\\ (3.1)\qquad &\mathcal{O}_2 \subseteq\,]0, +\infty[,\\ &\mathcal{O}_3 \subseteq R_3. \end{aligned}$$

Let the coefficients of Eqs. (1.5) and (1.6) satisfy for any $\alpha, \beta, p, q = 1, 2, 3$, $F \in \mathcal{O}_1$, $\boldsymbol{\theta} \in \mathcal{O}_2$, $G \in \mathcal{O}_3$, $\xi = \{\xi_p\}_{p=1,2,3} \neq 0$ and $\eta = \{\eta_p\}_{p=1,2,3} \neq 0$ the following relations:

$$\begin{aligned} &A^{\alpha\beta}_{pq}(\cdot, \cdot);\ A^{\alpha}_{p}(\cdot, \cdot);\ b_0(\cdot, \cdot);\ B^{\alpha}_{q}(\cdot, \cdot) \in C^m_b(\mathcal{O}_1 \times \mathcal{O}_2),\\ &B^{\alpha\beta}(\cdot, \cdot, \cdot);\ B^{\alpha}(\cdot, \cdot, \cdot);\ B^{\alpha\beta}_{q}(\cdot, \cdot, \cdot) \in C^m_b(\mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3), \end{aligned} \tag{3.2}$$

$$\begin{aligned} A^{\alpha\beta}_{pq}(F, \theta) &= A^{\beta\alpha}_{qp}(F, \theta),\\ B^{\alpha\beta}(F, \theta, G) &= B^{\beta\alpha}(F, \theta, G), \end{aligned} \tag{3.3}$$

$$\varrho_{\varkappa} > 0, \quad b_0(F, \theta) > 0, \tag{3.4}$$

$$\begin{aligned} A^{\alpha\beta}_{pq}(F, \theta)\xi_p\xi_q\eta_\alpha\eta_\beta > 0,\\ B^{\alpha\beta}(F, \theta, G)\eta_\alpha\eta_\beta > 0. \end{aligned} \tag{3.5}$$

Let the body forces, the heat sources and the initial functions satisfy the inclusions

$$\begin{aligned} &\mathsf{b} \in \bigcap_{k=0}^{m-2} C^k([0, T], \mathsf{H}_{m-k-2}), \qquad \overset{(m-1)}{\mathsf{b}} \in L^2([0, T], \mathsf{H}_0),\\ &s \in \bigcap_{k=0}^{m-2} C^k([0, T], H_{m-k-2}), \qquad \overset{(m-1)}{s} \in L^2([0, T], H_{-1}), \end{aligned} \tag{3.6}$$

$$\overset{0}{\chi}_{\varkappa} \in \mathsf{H}_m, \quad \overset{1}{\chi}_{\varkappa} \in \mathsf{H}_{m-1}, \quad \overset{0}{\boldsymbol{\theta}} \in H_m, \tag{3.7}$$

$$\nabla\overset{0}{\chi}_{\varkappa} \in \mathcal{O}_1, \quad \overset{0}{\boldsymbol{\theta}} \in \mathcal{O}_2, \quad \nabla\overset{0}{\boldsymbol{\theta}} \in \mathcal{O}_3 \quad \text{on } \mathcal{B}. \tag{3.8}$$

Furthermore let us assume that the boundary functions $\chi_{\varkappa 0}, \boldsymbol{\theta}_0$ are the traces on $\partial\mathcal{B} \times [0, T]$ of some functions $\bar{\chi}_{\varkappa 0}, \bar{\boldsymbol{\theta}}_0$ satisfying the relations

$$\bar{\chi}_{\varkappa 0} \in \bigcap_{k=0}^{m} C^k([0, T], \mathsf{H}_{m-k}), \qquad \bar{\boldsymbol{\theta}}_0 \in \bigcap_{k=0}^{m-1} C^k([0, T], H_{m-k}), \tag{3.9}$$

$$\nabla\bar{\chi}_{\varkappa 0} \in \mathcal{O}_1, \quad \bar{\boldsymbol{\theta}}_0 \in \mathcal{O}_2, \quad \nabla\bar{\boldsymbol{\theta}}_0 \in \mathcal{O}_3 \quad \text{on } \bar{\mathcal{B}} \times [0, \bar{T}], \tag{3.10}$$

where $\bar{T} \in [0, T]$ is sufficiently small.

Finally let the following compatibility conditions be satisfied:

$$\overset{k}{\chi}_{\varkappa} - \overset{(k)}{\bar{\chi}}_{\varkappa 0}(0) \equiv \overset{k}{\chi} \in \mathsf{V}, \quad k = 0, \ldots, m, \tag{3.11}$$

$$\overset{k}{\boldsymbol{\theta}} - \overset{(k)}{\bar{\boldsymbol{\theta}}}_0(0) \equiv \overset{k}{\theta} \in V, \quad k = 0, \ldots, m-1, \tag{3.12}$$

where the functions $\overset{k}{\chi}_{\varkappa}, \overset{k}{\boldsymbol{\theta}}$ are defined in the formulas (1.9).

Now we are ready to formulate our main result.

THEOREM 1. *If the assumptions* (3.1)–(3.12) *are satisfied, then for T sufficiently small there exists a unique solution of the initial-boundary value problem* (1.5)–(1.8) *with the properties*

$$\begin{aligned} &\chi_{\varkappa} \in \bigcap_{k=0}^{m} C^k([0, T], \mathsf{H}_{m-k}), \qquad \boldsymbol{\theta} \in \bigcap_{k=0}^{m-2} C^k([0, T], H_{m-k}),\\ &\qquad\qquad \overset{(m-1)}{\vartheta} \in C^0([0, T], H_0) \cap L^2([0, T], H_1), \end{aligned} \tag{3.13}$$

$$\nabla\chi_{\varkappa} \in \mathcal{O}_1, \quad \boldsymbol{\theta} \in \mathcal{O}_2, \quad \nabla\boldsymbol{\theta} \in \mathcal{O}_3 \quad \text{on } \mathcal{B} \times [0, T]. \tag{3.14}$$

REMARK. The assumption $m \geqslant 4$ implies that the solutions satisfying the properties (3.13) are sufficiently smooth to satisfy Eqs. (1.5)–(1.8) in the classical sense.

In the next two sections we make some simplifications and sketch the strategy of the proof.

4. Simplifications of the problem

To avoid the difficulties connected with the conditions (3.14) we introduce two non-negative functions $\varphi_1 \in C^\infty(R^{10})$ and $\varphi_2 \in C^\infty(R^{13})$ with compact supports contained in $\mathcal{O}_1 \times \mathcal{O}_2$ and $\mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3$, respectively, such that $\varphi_1(F, \theta) = 1$ on $\mathcal{P}_1$ and $\varphi_2(F, \theta, G) = 1$ on $\mathcal{P}_2$ for some open neighbourhoods $\mathcal{P}_1$ and $\mathcal{P}_2$ of $(\nabla \overset{0}{\chi}_\varkappa, \overset{0}{\boldsymbol{\theta}})$ and $(\nabla \overset{0}{\chi}_\varkappa, \overset{0}{\boldsymbol{\theta}}, \nabla \overset{0}{\boldsymbol{\theta}})$, repectively. Next we define the functions

$$(4.1)\qquad \bar{A}^{\alpha\beta}_{pq}(F, \theta) = \begin{cases} \varphi_1(F, \theta) A^{\alpha\beta}_{pq}(F, \theta) + (1 - \varphi_1(F, \theta))\delta_{\alpha\beta}\delta_{pq} & \text{for } (F, \theta) \in \mathcal{O}_1 \times \mathcal{O}_2, \\ \delta_{\alpha\beta}\delta_{pq} & \text{for } (F, \theta) \notin \mathcal{O}_1 \times \mathcal{O}_2, \end{cases}$$

$$(4.2)\qquad \bar{B}^{\alpha\beta}(F, \theta, G) = \begin{cases} \varphi_2(F, \theta, G) B^{\alpha\beta}(F, \theta, G) + (1 - \varphi_2(F, \theta, G))\delta_{\alpha\beta} \\ \qquad\qquad \text{for } (F, \theta, G) \in \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3, \\ \delta_{\alpha\beta} \qquad \text{for } (F, \theta, G) \notin \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3, \end{cases}$$

$$(4.3)\qquad \bar{b}_0(F, \theta) = \begin{cases} \varphi_1(F, \theta) b_0(F, \theta) + (1 - \varphi_1(F, \theta)) b_{\min} & \text{for } (F, \theta) \in \mathcal{O}_1 \times \mathcal{O}_2, \\ b_{\min} & \text{for } (F, \theta) \notin \mathcal{O}_1 \times \mathcal{O}_2, \end{cases}$$

where $b_{\min} = \min\{b(F, \theta) : (F, \theta) \in \text{support of } \varphi_1\}$ and we put the coefficients $\bar{A}^{\alpha}_{p}(F, \theta)$, $\bar{B}^{\alpha}(F, \theta, G)$, $\bar{B}^{\alpha\beta}_{q}(F, \theta, G)$ and $\bar{B}^{\alpha}_{q}(F, \theta)$ equal to $\varphi_1(F, \theta) A^{\alpha}_{p}(F, \theta)$, $\varphi_2(F, \theta, G) B^{\alpha}(F, \theta, G)$, $\varphi_2(F, \theta, G) B^{\alpha\beta}_{q}(F, \theta, G)$ and $\varphi_1(F, \theta) B^{\alpha}_{q}(F, \theta)$, respectively, for $(F, \theta, G) \in \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3$ and equal to zero for $(F, \theta, G) \notin \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3$. Now, if $\{\chi^*, \theta^*\}$ is a solution of the problem (1.5)–(1.8) with these new coefficients and without the condition (3.14), then, by the relations (3.8) and by continuity, for sufficiently small T we obtain the inclusions $(\nabla\chi^*, \theta^*) \in \mathcal{P}_1$ and $(\nabla\chi^*, \theta^*, \nabla\theta^*) \in \mathcal{P}_2$. This implies that $\{\chi^*, \theta^*\}$ is also a solution of the original problem with the conditions (3.14). Thus it is sufficient to solve the problem (1.5)–(1.8) with the new coefficients and without the conditions (3.14).

To obtain the more compact form of the problem (1.5)–(1.8) ([1]), we write these equations as follows:

$$(4.4)\qquad \ddot{\chi} + A_1(\chi, \theta)\chi + A_2(\chi, \theta) = f(\chi, \theta),$$

$$(4.5)\qquad b(\chi, \theta)\dot{\theta} + B_1(\chi, \theta)\theta + B_2(\chi, \theta)\chi + B_3(\chi, \theta)\dot{\chi} = g(\chi, \theta),$$

where

$$\chi = \chi_\varkappa - \bar{\chi}_{\varkappa 0}, \qquad \theta = \boldsymbol{\theta} - \bar{\boldsymbol{\theta}}_0$$

and

$$A_1(\chi, \theta)\tilde{\chi} = \{-\bar{A}^{\alpha\beta}_{pq}(\nabla\chi + \nabla\bar{\chi}_{\varkappa 0}, \theta + \bar{\boldsymbol{\theta}}_0)\tilde{\chi}_{q,\alpha\beta}\}_{p=1,2,3},$$

$$A_2(\chi, \theta)\tilde{\theta} = \{-\bar{A}^{\alpha}_{p}(\nabla\chi + \nabla\bar{\chi}_{\varkappa 0}, \theta + \bar{\boldsymbol{\theta}}_0)\tilde{\theta}_{,\alpha}\}_{p=1,2,3},$$

([1]) with the new coefficients

$$f(\chi,\theta) = -\ddot{\bar{\chi}}_{\varkappa 0} - A_1(\chi,\theta)\bar{\chi}_{\varkappa 0} - A_2(\chi,\theta)\bar{\boldsymbol{\theta}}_0 + \mathsf{b},$$
$$b(\chi,\theta) = \bar{b}_0(\nabla\chi + \nabla\bar{\chi}_{\varkappa 0}, \theta + \bar{\boldsymbol{\theta}}_0),$$
$$B_1(\chi,\theta)\tilde{\theta} = -\bar{B}^{\alpha\beta}(\nabla\chi + \nabla\bar{\chi}_{\varkappa 0}, \theta + \bar{\boldsymbol{\theta}}_0, \nabla\theta + \nabla\bar{\boldsymbol{\theta}}_0)\tilde{\theta}_{,\alpha\beta} - B^{\alpha}(\nabla\chi + \nabla\bar{\chi}_{\varkappa 0}, \theta + \bar{\boldsymbol{\theta}}_0, \nabla\theta + \nabla\bar{\boldsymbol{\theta}}_0)\tilde{\theta}_{,\alpha},$$
$$B_2(\chi,\theta)\tilde{\chi} = -\bar{B}^{\alpha\beta}_q(\nabla\chi + \nabla\bar{\chi}_{\varkappa 0}, \theta + \bar{\boldsymbol{\theta}}_0, \nabla\theta + \nabla\bar{\boldsymbol{\theta}}_0)\tilde{\chi}_{q,\alpha\beta},$$
$$B_3(\chi,\theta)\dot{\tilde{\chi}} = -\bar{B}^{\alpha}_q(\nabla\chi + \nabla\bar{\chi}_{\varkappa 0}, \theta + \bar{\boldsymbol{\theta}}_0)\dot{\tilde{\chi}}_{q,\alpha},$$
$$g(\chi,\theta) = -b(\chi,\theta)\dot{\bar{\boldsymbol{\theta}}}_0 - B_1(\chi,\theta)\bar{\boldsymbol{\theta}}_0 - B_2(\chi,\theta)\bar{\chi}_{\varkappa 0} - B_3(\chi,\theta)\dot{\bar{\chi}}_{\varkappa 0} + s.$$

The functions χ, θ satisfy the homogeneous boundary conditions

$$\chi = 0, \quad \theta = 0 \quad \text{on } \partial\mathscr{B}\times[0,T] \tag{4.6}$$

and modified initial conditions

$$\begin{aligned} \chi(0) &= \overset{0}{\chi}_{\varkappa} - \bar{\chi}_{\varkappa 0}(0) \equiv \overset{0}{\chi}, \\ \dot{\chi}(0) &= \overset{1}{\chi}_{\varkappa} - \dot{\bar{\chi}}_{\varkappa 0}(0) \equiv \overset{1}{\chi}, \\ \theta(0) &= \overset{0}{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}_0(0) \equiv \overset{0}{\theta}, \quad \text{on } \mathscr{B}. \end{aligned} \tag{4.7}$$

The conditions (1.9) take now the form

$$\overset{k+2}{\chi} = -\sum_{i=0}^{k}\binom{k}{i}\left[\overset{(i)}{A_1}(\chi,\theta)(0)\overset{k-i}{\chi} + \overset{(i)}{A_2}(\chi,\theta)(0)\overset{k-i}{\theta}\right] + \overset{(k)}{f}(\chi,\theta)(0), \tag{4.8}$$

$$\begin{aligned} b(\chi,\theta)(0)\overset{k+1}{\theta} = &-\sum_{i=1}^{k}\binom{k}{i}\overset{(i)}{b}(\chi,\theta)(0)\overset{k+1-i}{\theta} - \sum_{i=0}^{k}\binom{k}{i}\overset{(i)}{B_1}(\chi,\theta)(0)\overset{k-i}{\theta} \\ &-\sum_{i=0}^{k}\binom{k}{i}\left[\overset{(i)}{B_2}(\chi,\theta)(0)\overset{k-i}{\chi} + \overset{(i)}{B_3}(\chi,\theta)(0)\overset{k+1-i}{\chi}\right] + \overset{(k)}{g}(\chi,\theta)(0), \end{aligned} \tag{4.9}$$

where

$$\overset{(i)}{A_1}(\chi,\theta)(0)\psi = \left\{-\frac{\partial^i}{\partial t^i}\left[\bar{A}^{\alpha\beta}_{pq}(\nabla\chi(t) + \nabla\bar{\chi}_{\varkappa 0}(t), \theta(t) + \bar{\boldsymbol{\theta}}_0(t))\right](0)\psi_{q,\alpha\beta}\right\}_{p=1,2,3}$$

and similar notations are used for other operators.

5. The basic idea of the proof

From Sect. 4 it follows that it is sufficient to prove the existence of a solution for the problem (4.4)–(4.7). To this end, following [1] we introduce the set $Z(M,T)$ of pairs of functions $\{\bar{\chi},\bar{\theta}\}$ which satisfy the conditions

$$\begin{aligned} &\bar{\chi} \in \bigcap_{k=0}^{m} W^{k,\infty}([0,T], \mathsf{H}_{m-k}), \\ &\bar{\theta} \in \bigcap_{k=0}^{m-2} W^{k,\infty}([0,T], H_{m-k}), \qquad \overset{(m-1)}{\bar{\theta}} \in L^{\infty}([0,T], H_0) \cap L^2([0,T], H_1), \end{aligned} \tag{5.1}$$

$$\overset{(k)}{\bar{\chi}}(0) = \overset{k}{\chi}, \quad k = 0, 1, \ldots, m-1, \tag{5.2}$$
$$\overset{(k)}{\bar{\theta}}(0) = \overset{k}{\theta}, \quad k = 0, 1, \ldots, m-2,$$

$$\operatorname*{ess\,sup}_{t\in[0,T]}\Big(\sum_{k=0}^{m} \|\overset{(k)}{\bar{\chi}}(t)\|_{m-k}^2 + \sum_{k=0}^{m-2} \|\overset{(k)}{\bar{\theta}}(t)\|_{m-k}^2 + \|\overset{(m-1)}{\bar{\theta}}(t)\|_0^2\Big) + \int_0^T \|\overset{(m-1)}{\bar{\theta}}(t)\|_1^2 dt \leqslant M^2. \tag{5.3}$$

where M is a constant independent of $(\bar{\chi}, \bar{\theta})$. For given $\{\bar{\chi}, \bar{\theta}\} \in Z(M, T)$ let us consider the linear problem

$$\ddot{\chi} + A_1(\bar{\chi}, \bar{\theta})\chi + A_2(\bar{\chi}, \bar{\theta})\theta = f(\bar{\chi}, \bar{\theta}), \tag{5.4}$$

$$b(\bar{\chi}, \bar{\theta})\dot{\theta} + B_1(\bar{\chi}, \bar{\theta})\theta + B_2(\bar{\chi}, \bar{\theta})\chi + B_3(\bar{\chi}, \bar{\theta})\dot{\chi} = g(\bar{\chi}, \bar{\theta}), \tag{5.5}$$

with the boundary and initial conditions (4.6), (4.7).

We shall prove that for every pair $\{\bar{\chi}, \bar{\theta}\} \in Z(M, T)$ there exists a unique solution $\{\chi, \theta\}$ of the problem (5.4), (5.5), (4.6), (4.7) and that for M sufficiently large and T sufficiently small the operator $\mathcal{T}: \{\bar{\chi}, \bar{\theta}\} \to \{\chi, \theta\}$ defines a contractive mapping of $Z(M, T)$ into itself. The contraction mapping principle will imply that $\mathcal{T}$ has a unique fixed point which is a solution of Eqs. (4.4)–(4.7).

6. Some results concerning the linear problem (5.4), (5.5), (4.6), (4.7)

To implement the program outlined above we shall need some results concerning the linear problem (5.4), (5.5), (4.6), (4.7). The present section contains the formulation of these results.

LEMMA 1. Let $\{\bar{\chi}, \bar{\theta}\} \in Z(M, T)$ and $k = 0, 1, \ldots, m-2$. The conditions $\chi \in X_k$, $A_1(\bar{\chi}, \bar{\theta})\chi \in \mathrm{H}_k$(resp. $\theta \in Y_k$, $B_1(\bar{\chi}, \bar{\theta})\theta \in H_k$ imply $\chi \in X_{k+2}$ (resp. $\theta \in Y_{k+2}$). Furthermore there exist continuous functions $\bar{\mu}_i$, $\bar{\varkappa}_i$, $\bar{\lambda}_i$, $\bar{c}_0$, $i = 1, 2$, defined on $[0, +\infty[\times[0, +\infty[$ with values in$]0, +\infty[$ such that for sufficiently small $\varepsilon \in]0, \frac{1}{2}[$, $T \in]0, +\infty[$ and for arbitrary $\{\bar{\chi}, \bar{\theta}\} \in Z(M, T)$ we have

$$\|\chi\|_{k+2} \leqslant \bar{\mu}_1(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})(\|\chi\|_k + \|A_1(\bar{\chi}, \bar{\theta})\chi\|_k) \quad \text{for all } \chi \in X_{k+2}, \tag{6.1}$$
$$\|\theta\|_{k+2} \leqslant \bar{\mu}_2(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})(\|\theta\|_k + \|B_1(\bar{\chi}, \bar{\theta})\theta\|_k) \quad \text{for all } \theta \in Y_{k+2},$$

$$\langle A_1(\bar{\chi}, \bar{\theta})\chi, \chi\rangle + \bar{\varkappa}_1(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})\|\chi\|_0^2 \geqslant \bar{\lambda}_1(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})\|\chi\|_1^2 \tag{6.2}$$
$$\langle B_1(\bar{\chi}, \bar{\theta})\theta, \theta\rangle + \bar{\varkappa}_2(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})\|\theta\|_0^2 \geqslant \bar{\lambda}_2(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})\|\theta\|_1^2$$

for all $\chi \in X_1$ and $\theta \in Y_1$,

$$b(\bar{\chi}, \bar{\theta}) \geqslant \bar{c}_0(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon}) > 0. \tag{6.3}$$

Sketch of the proof of Lemma 1

Let us introduce the expressions $Q_1(F^*, \theta^*, \xi, \eta) = \bar{A}^{\alpha\beta}_{pq}(F^*, \theta^*)\xi_\alpha\xi_\beta\eta_p\eta_q$, $Q_2(F^*, \theta^*, G^*, \eta) = \bar{B}^{\alpha\beta}(F^*, \theta^*, G^*)\eta_\alpha\eta_\beta$ for arbitrary $F^* \in R^9$, $\theta^* \in R^1$, $\xi, \eta, G^* \in R^3$ and let us define the functions

$$\tilde{\lambda}_1(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon}) = \min\{Q_1(F^*, \theta^*, \xi, \eta) : |\xi| = |\eta| = 1,$$
$$|F^*|^2 + |\theta^*|^2 \leqslant \tilde{C}_1^2\|\bar{\chi}\|^2_{m-\varepsilon} + \tilde{C}_2^2\|\bar{\theta}\|^2_{m-\varepsilon}\},$$
$$\tilde{\lambda}_2(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon}) = \min\{Q_2(F^*, \theta^*, G^*, \eta) : |\eta| = 1$$
$$|F^*|^2 + |\theta^*|^2 + |G^*|^2 \leqslant \tilde{C}_1^2\|\bar{\chi}\|^2_{m-\varepsilon} + (\tilde{C}_2^2 + \tilde{C}_3^2)\|\bar{\theta}\|^2_{m-\varepsilon}\},$$

where $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ are constants from the Sobolev inequalities

$$\max_{X\in\mathscr{B}}|\nabla\psi(X)| \leqslant \tilde{C}_1\|\psi\|_{m-\varepsilon}, \quad \max_{X\in\mathscr{B}}|\varphi(X)| \leqslant \tilde{C}_2\|\varphi\|_{m-\varepsilon}$$
$$\max_{X\in\mathscr{B}}|\nabla\varphi(X)| \leqslant \tilde{C}_3\|\varphi\|_{m-\varepsilon}, \quad \text{for any} \quad \psi \in \mathsf{H}_{m-\varepsilon}, \quad \varphi \in H_{m-\varepsilon}.$$

It is not difficult to prove that the functions $\tilde{\lambda}_i$, $i = 1, 2$ depend continuously on their arguments and that the following inequalities hold:

$$\text{(6.4)} \qquad \begin{aligned} Q_1(\nabla\bar{\chi}, \bar{\theta}, \xi, \eta) &\geqslant \bar{\lambda}_1(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})|\xi|^2|\eta|^2, \\ Q_2(\nabla\bar{\chi}, \bar{\theta}, \nabla\bar{\theta}, \eta) &\geqslant \bar{\lambda}_2(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})|\eta|^2, \end{aligned}$$

for arbitrary $\{\bar{\chi}, \bar{\theta}\} \in Z(M, T)$, $\xi, \eta \in R^3$.

The inequalities (6.4) are the starting points in the proof of the Garding type inequalities (6.2) and the elliptic regularity results (6.1). The detailed proofs can be obtained for example by modification of corresponding proofs of the book [4].

To prove the inequality (6.3) let us introduce the symbol $\bar{\boldsymbol{\theta}} \equiv \bar{\theta} + \boldsymbol{\theta}_0$ and remark that the inequality

$$\bar{\boldsymbol{\theta}}(t) = \bar{\boldsymbol{\theta}}(t) - \bar{\boldsymbol{\theta}}(0) + \bar{\boldsymbol{\theta}}(0) \geqslant -T\max_{t\in[0,T]}|\dot{\bar{\boldsymbol{\theta}}}(t)| + \bar{\boldsymbol{\theta}}(0) \geqslant -T\check{C}_1(M + \check{C}_2) + \bar{\boldsymbol{\theta}}(0),$$

which holds for some positive constants $\check{C}_1$, $\check{C}_2$ implies the existence of T such that $\bar{\boldsymbol{\theta}}(t)$ is positive for all $t \in [0, T]$ and all $\bar{\theta}$ such that $\{\bar{\chi}, \bar{\theta}\} \in Z(M, T)$. Therefore the coefficient $b(\bar{\chi}, \bar{\theta})$ is positive for these $\{\bar{\chi}, \bar{\theta}\}$ and the function

$$\bar{c}_0(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon}) = \min\{b(F^*, \theta^*) : |F^*|^2 + |\theta^*|^2 \leqslant \tilde{C}_1^2\|\bar{\chi}\|^2_{m-\varepsilon} + \tilde{C}_2^2\|\theta\|^2_{m-\varepsilon}\}$$

can be used to satisfy the inequality (6.3). This completes the sketch of proof of Lemma 1.

In the formulation of the theorem describing the properties of a solution of the problem (5.4), (5.5), (4.6), (4.7) we shall use the notations

$$\text{(6.5)} \qquad \begin{aligned} \varkappa &= \{\bar{\varkappa}_i(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})\}_{i=1,2}, & \lambda &= \{\bar{\lambda}_i(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})\}_{i=1,2}, \\ \mu &= \{\bar{\mu}_i(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon})\}_{i=1,2}, & c_0 &= \bar{c}_0(\|\bar{\chi}\|_{m-\varepsilon}, \|\bar{\theta}\|_{m-\varepsilon}), \end{aligned}$$

$$\text{(6.6)} \qquad L^0 = \|A_1(\bar{\chi}, \bar{\theta})(0)\|_{\mathscr{L}_2} + \|b(\bar{\chi}, \bar{\theta})(0)\|_{m-\varepsilon},$$

$$\text{(6.7)} \qquad L = \operatorname*{ess\,sup}_{t\in[0,T]}\Big\{\|A_1(\bar{\chi}, \theta)(t)\|_{\mathscr{L}_2} + \sum_{k=1}^{m-1}\|\overset{(k)}{A_1}(\bar{\chi}, \bar{\theta})(t)\|_{\mathscr{L}_{k+1}} + \|A_2(\bar{\chi}, \theta)(t)\|_{\mathscr{L}_1}$$
$$+ \sum_{k=1}^{m-1}\|\overset{(k)}{A_2}(\bar{\chi}, \bar{\theta})(t)\|_{\mathscr{L}_k} + \|B_1(\bar{\chi}, \bar{\theta})(t)\|_{\mathscr{L}_2} + \sum_{k=1}^{m-2}\|\overset{(k)}{B_1}(\bar{\chi}, \theta)(t)\|_{\mathscr{L}_{k+1}}$$
$$+ \|B_2(\bar{\chi}, \bar{\theta})(t)\|_{\mathscr{L}_2} + \sum_{k=1}^{m-2}\|\overset{(k)}{B_2}(\bar{\chi}, \bar{\theta})(t)\|_{\mathscr{L}_{k+1}} + \|B_3(\bar{\chi}, \bar{\theta})(t)\|_{\mathscr{L}_1}$$

(6.7) [cont.]
$$+\sum_{k=1}^{m-1}\|\overset{(k)}{B_3}(\bar\chi,\bar\theta)(t)\|_{\mathscr{L}_k}+\|\overset{(m-1)}{b}(\bar\chi,\bar\theta)(t)\|_0+\sum_{k=0}^{m-2}\|\overset{(k)}{b}(\bar\chi,\bar\theta)\|_{m-k-1-\varepsilon}\Big\}$$
$$+\Big(\int_0^T\big(\|\overset{(m-1)}{B_1}(\bar\chi,\bar\theta)(t)\|^2_{\mathscr{L}(H_m,H_0)}+\|\overset{(m-1)}{B_1}(\bar\chi,\bar\theta)(t)\|^2_{\mathscr{L}(H_{m-1},H_{-1})}$$
$$+\|\overset{(m-1)}{B_2}(\bar\chi,\bar\theta)(t)\|^2_{\mathscr{L}(H_m,H_0)}\big)\,dt\Big)^{1/2},$$

(6.8)
$$N=\sum_{k=0}^{m}\|\overset{k}{\chi}\|^2_{m-k}+\max_{t\in[0,T]}\sum_{k=0}^{m-2}\|\overset{(k)}{\mathrm{b}}(t)\|^2_{m-2-k}+\int_0^T\|\overset{(m-1)}{\mathrm{b}}(t)\|^2_0\,dt$$
$$+\max_{t\in[0,T]}\sum_{k=0}^{m}\|\overset{(k)}{\bar\chi}_{\varkappa 0}(t)\|^2_{m-k}+\int_0^T\|\overset{(m+1)}{\bar\chi}_{\varkappa 0}(t)\|^2_0\,dt+\sum_{k=0}^{m-1}\|\overset{k}{\theta}\|^2_{m-k}+\max_{t\in[0,T]}\sum_{k=0}^{m-2}\|\overset{(k)}{s}(t)\|^2_{m-2-k}$$
$$+\int_0^T\|\overset{(m-1)}{s}(t)\|^2_{-1}\,dt+\max_{t\in[0,T]}\sum_{k=0}^{m-1}\|\overset{(k)}{\bar{\boldsymbol\theta}}_0(t)\|^2_{m-k}+\int_0^T\|\overset{(m)}{\bar{\boldsymbol\theta}}_0(t)\|^2_{-1}\,dt.$$

The introduction of the symbol L (see Eq. (6.7)) is justified by the following lemma.

LEMMA 2. If $\{\bar\chi,\bar\theta\}\in Z(M,T)$, then the corresponding operators $A_i(\bar\chi,\bar\theta)$, $i=1,2$, $B_i(\bar\chi,\bar\theta)$, $i=1,2,3$ and the function $b(\bar\chi,\bar\theta)$ have the properties

(6.9)
$$A_1(\bar\chi,\bar\theta)\in\bigcap_{k=1}^{m-1}W^{k,\infty}([0,T],\mathscr{L}_{k+1}),$$
$$B_1(\bar\chi,\bar\theta),\ B_2(\bar\chi,\bar\theta)\in\bigcap_{k=1}^{m-2}W^{k,\infty}([0,T],\mathscr{L}_{k+1}),$$

$$\overset{(m-1)}{B_1}(\bar\chi,\bar\theta)\in L^2([0,T],\mathscr{L}(H_{m-1},H_{-1}))\cap L^2([0,T],\mathscr{L}(H_m,H_0)),$$
$$\overset{(m-1)}{B_2}(\bar\chi,\bar\theta)\in L^2([0,T],\mathscr{L}(H_m,H_0)),$$
$$A_2(\bar\chi,\bar\theta),\ B_3(\bar\chi,\bar\theta)\in\bigcap_{k=1}^{m-1}W^{k,\infty}([0,T],\mathscr{L}_k),$$
$$b(\bar\chi,\bar\theta)\in\bigcap_{k=0}^{m-1}W^{k,\infty}([0,T],H_{m-1-k})$$

and the estimate

(6.10)
$$L\leqslant K_1\operatorname*{ess\,sup}_{t\in[0,T]}\|\overset{(m-1)}{\bar\chi}(t)\|_1+K_2\Big(\int_0^T\|\overset{(m-1)}{\bar\theta}(t)\|_1^2\,dt\Big)^{1/2}+K_3$$

holds, where K_1, K_2 are positive constants and K_3 is a positive function continuously depending on the variables $\|\overset{(i)}{\bar\chi}\|_{m-i-\varepsilon}$, $\|\overset{(i)}{\bar\theta}\|_{m-i-\varepsilon}$, $i=0,1,\ldots,m-2$ with sufficiently small $\varepsilon\in\,]0,\frac{1}{2}[$.

In the proof of this lemma one is to use the definition of the spaces $\mathscr{L}_k$, $k=1,\ldots,m$, the chain rule and the properties of the Sobolev spaces mentioned e.g. in Sect. 5 of the paper [1].

Now, using the notations (6.5)–(6.8) we formulate the following well-posedness result for the problem (5.4), (5.5), (4.6), (4.7).

THEOREM 2. *If the assumptions of Theorem 1 are satisfied and* $\{\bar{\chi}, \bar{\theta}\}$ *is an arbitrary element of the space* $Z(M, T)$ *then for any positive* T *less than some* $\bar{T}$ *and for sufficiently large* M *the problem* (5.4), (5.5), (4.6), (4.7) *has a unique solution* $\{\chi, \theta\}$ *with the properties*

$$(6.11)\qquad \chi \in \bigcap_{k=0}^{m} C^k([0, T], \mathsf{H}_{m-k}), \qquad \theta \in \bigcap_{k=0}^{m-2} C^k([0, T], H_{m-k}),$$
$$\overset{(m-1)}{\theta} \in C^0([0, T], H_0) \cap L^2(0, T], H_1).$$

Furthermore the following estimate holds:

$$(6.12)\qquad \operatorname*{ess\,sup}_{t\in[0,T]}\Big(\sum_{k=0}^{m} \|\overset{(k)}{\chi}(t)\|^2_{m-k} + \sum_{k=0}^{m-2} \|\overset{(k)}{\theta}(t)\|^2_{m-k} + \|\overset{(m-1)}{\theta}(t)\|^2_0\Big)$$
$$+\int_0^T \|\overset{(m-1)}{\theta}(t)\|^2_1 dt \leqslant (C_1 N + C_2 LN)\big(1 + (1 + TC_3)\exp(TC_3)\big),$$

where

$$C_i = \hat{C}_i(c_0, \varkappa, \mu, \lambda, L^0) \quad \text{for} \quad i = 1, 2, \qquad C_3 = \hat{C}_3(c_0, \varkappa, \mu, \lambda, L^0, L)$$

and the functions $\hat{C}_1, \hat{C}_2, \hat{C}_3$ *depend continuously on their arguments.*

The proof of Theorem 2 will be given in the second part of the present paper (see [5]).

Now we are ready to implement the idea sketched in Sect. 5.

7. Proof of the Theorem 1

In the first step, we prove that the operator $\mathscr{T}: \{\bar{\chi}, \bar{\theta}\} \to \{\chi, \theta\}$ defined in Sect. 5 and existing due to Theorem 2, maps $Z(M, T)$ into $Z(M, T)$ for sufficiently small T and sufficiently large M. To this end, let us note that by virtue of the interpolation inequality (see [3], Chapt. I, Sect. 2.5) we have for some positive constants $\bar{C}_i, i = 1, 2, 3$ and $k = 0, 1, \ldots, m-2$,

$$(7.1)\qquad \|\overset{(k)}{\varphi}(t) - \overset{(k)}{\varphi}(0)\|_{m-k-1-\varepsilon} \leqslant \bar{C}_1 \|\overset{(k)}{\varphi}(t) - \overset{(k)}{\varphi}(0)\|^{1-\varepsilon}_{m-k-1} \|\overset{(k)}{\varphi}(t) - \overset{(k)}{\varphi}(0)\|^{\varepsilon}_{m-k-2}$$
$$\leqslant \bar{C}_2 M^{1-\varepsilon}\big(\operatorname*{ess\,sup}_{t\in[0,T]} \|\overset{(k+1)}{\varphi}(t)\|_{m-k-2}\, T\big)^{\varepsilon} \leqslant \bar{C}_3 M T^{\varepsilon},$$

where $\varphi = \bar{\chi}$ or $\varphi = \bar{\theta}$ and $\varepsilon \in]0, \frac{1}{2}[$ is the number from Lemma 2.

If we choose $M_0 > 0$ such that (²)

$$(7.2)\qquad (C_1 N + C_2 N K_3)(0) \leqslant \frac{1}{2} M_0, \qquad [1 + C_2 N (K_1 + K_2)](0) \leqslant \frac{1}{6} M_0$$

(²) The symbols $C_i, K_i, i = 1, 2, 3$ are introduced in Theorem 2 and Lemma 2.

and $T_0 \in]0, \overline{T}]$ such that [3]

$$
\begin{aligned}
&[C_1 N + C_2 N K_3](t) \leqslant 2[C_1 N + C_2 N K_3](0),\\
(7.3)\qquad &[1 + C_2 N(K_1 + K_2)](t) \leqslant 2[1 + C_2 N(K_1 + K_2)](0) \quad \text{for} \quad t \in [0, T_0]\\
&[1 + (1 + T_0 C_3)\exp(T_0 C_3)](t) \leqslant 3,
\end{aligned}
$$

then by using the estimates (6.10), (6.12) and (7.2), (7.3) we obtain

$$
(7.4)\qquad \operatorname*{ess\,sup}_{t\in[0,T]} \Big(\sum_{k=0}^{m} ||\overset{(k)}{\chi}(t)||^2_{m-k} + \sum_{k=0}^{m-2} ||\overset{(k)}{\theta}(t)||^2_{m-k} + ||\overset{(m-1)}{\theta}(t)||^2_0 \Big) + \int_0^T ||\overset{(m-1)}{\theta}(t)||^2_1 dt
$$

$$
\leqslant 3\big(M_0 + C_2 N(K_1 + K_2) M_0\big) \leqslant M_0^2,
$$

i.e. $\mathscr{T}$ maps $Z(M_0, T)$ into itself for $T \in]0, T_0]$.

To prove that $\mathscr{T}$ is contractive if T is sufficiently small, we assume that $T \in]0, T_0]$ and $\{\overline{\chi}, \overline{\theta}\}$, $\{\overline{\overline{\chi}}, \overline{\overline{\theta}}\} \in Z(M_0, T)$ are given. Let us put

$$
\begin{aligned}
&\overline{\psi} = \overline{\chi} - \overline{\overline{\chi}}, && \overline{\phi} = \overline{\theta} - \overline{\overline{\theta}},\\
(7.5)\qquad &\{\tilde{\chi}, \tilde{\theta}\} = \mathscr{T}(\{\overline{\chi}, \overline{\theta}\}), && \{\tilde{\tilde{\chi}}, \tilde{\tilde{\theta}}\} = \mathscr{T}(\{\overline{\overline{\chi}}, \overline{\overline{\theta}}\}),\\
&\psi = \tilde{\chi} - \tilde{\tilde{\chi}}, && \phi = \tilde{\theta} - \tilde{\tilde{\theta}}.
\end{aligned}
$$

If we introduce the new abbreviate symbols by putting

$$
\begin{aligned}
A(\hat{\chi}, \hat{\theta})\{\check{\chi}, \check{\theta}\} &= A_1(\hat{\chi}, \hat{\theta})\check{\chi} + A_2(\hat{\chi}, \hat{\theta})\check{\theta},\\
B(\hat{\chi}, \hat{\theta})\{\check{\chi}, \check{\theta}\} &= B_1(\hat{\chi}, \hat{\theta})\check{\theta} + B_2(\hat{\chi}, \hat{\theta})\check{\chi} + B_3(\hat{\chi}, \hat{\theta})\dot{\check{\chi}},
\end{aligned}
$$

then the relations (7.5) will imply the equalities

$$
\begin{aligned}
&\ddot{\psi} + A(\overline{\chi}, \overline{\theta})\{\psi, \phi\} = \big(A(\overline{\overline{\chi}}, \overline{\overline{\theta}}) - A(\overline{\chi}, \overline{\theta})\big)\{\tilde{\tilde{\chi}}, \tilde{\tilde{\theta}}\} - \big(f(\overline{\overline{\chi}}, \overline{\overline{\theta}}) - f(\overline{\chi}, \overline{\theta})\big),\\
(7.6)\qquad &b(\overline{\chi}, \overline{\theta})\dot{\phi} + B(\overline{\chi}, \overline{\theta})\{\psi, \phi\} = \big(B(\overline{\overline{\chi}}, \overline{\overline{\theta}}) - B(\overline{\chi}, \overline{\theta})\big)\{\tilde{\tilde{\chi}}, \tilde{\tilde{\theta}}\}\\
&\qquad - \big(g(\overline{\overline{\chi}}, \overline{\overline{\theta}}) - g(\overline{\chi}, \overline{\theta})\big) + \big(b(\overline{\overline{\chi}}, \overline{\overline{\theta}}) - b(\overline{\chi}, \overline{\theta})\big)\dot{\tilde{\tilde{\theta}}},\\
&\psi(0) = \dot{\psi}(0) = 0, \quad \phi(0) = 0.
\end{aligned}
$$

Forming the duality products of Eq. $(7.6)_1$ with $2\dot{\psi}$ and Eq. $(7.6)_2$ with 2ϕ and integrating over $[0, t]$, we obtain respectively

$$
(7.7)\qquad ||\dot{\psi}||^2_0 + \langle A_1(\overline{\chi}, \overline{\theta})\psi, \psi\rangle = \int_0^t \{\langle \dot{A}_1(\overline{\chi}, \overline{\theta})\psi, \psi\rangle - 2\langle A_2(\overline{\chi}, \overline{\theta})\phi, \dot{\psi}\rangle\} d\sigma
$$

$$
+ 2\int_0^t \{\langle \big(A(\overline{\overline{\chi}}, \overline{\overline{\theta}}) - A(\overline{\chi}, \overline{\theta})\big)\{\tilde{\tilde{\chi}}, \tilde{\tilde{\theta}}\}, \dot{\psi}\rangle - \langle f(\overline{\overline{\chi}}, \overline{\overline{\theta}}) - f(\overline{\chi}, \overline{\theta}), \dot{\psi}\rangle\} d\sigma,
$$

and

$$
(7.8)\qquad ||\big(b(\overline{\chi}, \overline{\theta})\big)^{1/2}\phi||^2_0 + 2\int_0^t \langle B_1(\overline{\chi}, \overline{\theta})\phi, \phi\rangle d\sigma = \int_0^t \langle \dot{b}(\overline{\chi}, \overline{\theta})\phi, \phi\rangle d\sigma
$$

---

[3] We use the continuity properties of the functions C_i, K_i and the inequality (7.1).

(7.8) [cont.]

$$+2\int_0^t \{-\langle B_2(\bar{\chi},\bar{\theta})\psi,\phi\rangle-\langle B_3(\bar{\chi},\bar{\theta})\dot{\psi},\phi\rangle+\langle (b(\bar{\bar{\chi}},\bar{\bar{\theta}})-b(\bar{\chi},\bar{\theta}))\ddot{\tilde{\bar{\theta}}},\phi\rangle\}d\sigma$$

$$+2\int_0^t \{\langle (B(\bar{\bar{\chi}},\bar{\bar{\theta}})-B(\bar{\chi},\bar{\theta}))\{\tilde{\bar{\chi}},\tilde{\bar{\theta}}\},\phi\rangle-\langle g(\bar{\bar{\chi}},\bar{\bar{\theta}})-g(\bar{\chi},\bar{\theta}),\phi\rangle\}d\sigma.$$

If we use the estimates

$$\int_0^t \langle B_2(\bar{\chi},\bar{\theta})\psi,\phi\rangle d\sigma \leqslant \bar{C}_1(M)\int_0^t \|\psi\|_1\|\phi\|_1 d\sigma \leqslant \bar{C}_2(M)\int_0^t\left(\frac{1}{2\delta_1}\|\psi\|_1^2+\frac{\delta_1}{2}\|\phi\|_1^2\right)d\sigma,$$

$$(7.9)\qquad \int_0^t \langle (B(\bar{\bar{\chi}},\bar{\bar{\theta}})-B(\bar{\chi},\bar{\theta}))\{\tilde{\bar{\chi}},\tilde{\bar{\theta}}\},\phi\rangle d\sigma \leqslant \bar{C}_3(M)\int_0^t (\|\bar{\psi}\|_1+\|\bar{\phi}\|_1)(\|\tilde{\bar{\chi}}\|_m$$

$$+\|\tilde{\bar{\theta}}\|_m)\|\phi\|_0 d\sigma \leqslant \bar{C}_4(M)\int_0^t\left(\frac{1}{2}\|\bar{\psi}\|_1^2+\frac{\delta_2}{2}\|\bar{\phi}\|_1^2+\frac{1}{2}\left(1+\frac{1}{\delta_2}\right)\|\phi\|_0^2\right)d\sigma,$$

with arbitrary positive numbers δ_1, δ_2 and some positive constants $\bar{C}_i(M)$, $i = 1, 2, 3, 4$, and if we remark that similar estimates are true for other expressions in the right hand sides of Eqs. (7.7) and (7.8), we arrive at the inequality

$$(7.10)\qquad \|\dot{\psi}\|_0^2+\|\psi\|_1^2+\|\phi\|_0^2+\int_0^t\|\phi\|_1^2 d\sigma \leqslant \frac{\tilde{C}_1\delta_1}{2}\int_0^t\|\phi\|_1^2 d\sigma+\tilde{C}_2\int_0^t(\|\dot{\psi}\|_0^2+\|\psi\|_1^2$$

$$+\|\phi\|_0^2)d\sigma+\frac{\tilde{C}_3\delta_2}{2}\int_0^t\|\bar{\phi}\|_1^2 d\sigma+\tilde{C}_4\int_0^t(\|\dot{\bar{\psi}}\|_0^2+\|\bar{\psi}\|_1^2+\|\bar{\phi}\|_0^2)d\sigma,$$

with the constants $\tilde{C}_i$, $i = 1, 2, 3, 4$.

Putting $\tilde{C}_1\delta_1 \leqslant 1$ and using Gronwall's inequality, we get

$$(7.11)\qquad \|\dot{\psi}\|_0^2+\|\psi\|_1^2+\|\phi\|_0^2 \leqslant \tilde{C}_4 e^{\tilde{C}_2 t}\int_0^t(\|\dot{\bar{\psi}}\|_0^2+\|\bar{\psi}\|_1^2+\|\bar{\phi}\|_0^2)d\sigma+\frac{\tilde{C}_3\delta_2}{2}e^{\tilde{C}_2 t}\int_0^t\|\bar{\phi}\|_1^2 d\sigma$$

$$\leqslant T\tilde{C}_4 e^{\tilde{C}_2 T}\operatorname*{ess\,sup}_{t\in[0,T]}(\|\dot{\bar{\psi}}\|_0^2+\|\bar{\psi}\|_1^2+\|\bar{\phi}\|_0^2)+\frac{\tilde{C}_3\delta_2}{2}e^{\tilde{C}_2 T}\int_0^T\|\bar{\phi}\|_1^2 d\sigma.$$

Using again the inequality (7.10), we obtain

$$(7.12)\qquad \|\dot{\psi}\|_0^2+\|\psi\|_1^2+\|\phi\|_0^2+\int_0^t\|\phi\|_1^2 d\sigma \leqslant 2TC_4(T\tilde{C}_2 e^{\tilde{C}_2 T}+1)\operatorname*{ess\,sup}_{t\in[0,T]}(\|\dot{\bar{\psi}}\|_0^2+\|\bar{\psi}\|_1^2+\|\bar{\phi}\|_0^2)$$

$$+\tilde{C}_3\delta_2(\tilde{C}_2 Te^{\tilde{C}_2 T}+1)\int_0^T\|\bar{\phi}\|_1^2 dt.$$

Now it is clear that the operator $\mathcal{T}$ is contractive if the condition

$$\max\{2T\tilde{C}_4(T\tilde{C}_2 e^{\tilde{C}_2 T}+1), \quad \tilde{C}_3\delta_2(\tilde{C}_2 Te^{\tilde{C}_2 T}+1)\} \leqslant \alpha < 1$$

is satisfied and if the space $Z(M, T)$ is equipped with the metric

$$d(\{\bar{\chi}, \bar{\theta}\}, \{\bar{\bar{\chi}}, \bar{\bar{\theta}}\})$$

$$= \Big(\operatorname*{ess\,sup}_{t\in[0,T]} \big(||\dot{\bar{\chi}}(t)-\dot{\bar{\bar{\chi}}}(t)||_0^2+||\bar{\chi}(t)-\bar{\bar{\chi}}(t)||_1^2+||\bar{\theta}(t)-\bar{\bar{\theta}}(t)||_0^2\big)+\int_0^T ||\bar{\theta}(t)-\bar{\bar{\theta}}(t)||_1^2\,dt\Big)^{1/2}.$$

Completness of $Z(M, T)$ with respect to this metric follows from * weak precompactness of bounded sets and sequential * weak lower semicontinuity of the norm in the space $L^p([0, T], Z)$ where $1 \leqslant p \leqslant \infty$ and Z is Sobolev space.

By the contraction mapping principle the operator $\mathcal{T}$ has a unique fixed point $\{\chi, \theta\} \in Z(M, T)$ which is a desired solution of the problem (4.4)–(4.7).

Using Theorem 2 to obtain appropriate regularity of the solution, we complete the proof of Theorem 1.

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