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On the stationary motion of a granulated medium with constant density

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WE CONSIDER a model system of equations describing the stationary motion of a granulated medium with constant density. Our aim is to prove the existence of a weak solution of the first boundary-value problem for this system.

Rozważa się układ równań dla modelu stacjonarnego ruchu ośrodka sypkiego o stałej gęstości. Celem pracy jest przeprowadzenie dowodu istnienia słabego rozwiązania pierwszego zagadnienia brzegowego dla tego układu.

Рассматривается система уравнений для модели стационарного движения сыпучей среды с постоянной плотностью. Целью работы является провести доказательство существования слабого решения первой краевой задачи для этой системы.

0. Introduction and main results

THE PURPOSE of this paper is to prove the existence of a solution of the following boundary-value proble:

(0.1) $$-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f+\eta(\omega\times u) \quad \text{in } D,$$

(0.2) $$\operatorname{div} u=0 \quad \text{in } D,$$

(0.3) $$(u\cdot\nabla)\omega+F(p)\omega=\phi \quad \text{in } D,$$

(0.4) $$u=0 \quad \text{on } S,$$

where D is a bounded domain in R^3 with a smooth boundary S.

Equations (0.1)–(0.3) present conservation laws: conservation of momentum, mass and moment of momentum, respectively, of a granulated medium with constant density [6] in the case of the stationary motion of the medium.

The functions $u(x)=(u_1(x), u_2(x), u_3(x))$, $\omega(x)=(\omega_1(x), \omega_2(x), \omega_3(x))$ and $p(x)$ denote the velocity vector, angular velocity vector of rotation of particles and pressure, respectively. The functions $f(x)=(f_1(x), f_2(x), f_3(x))$ and $\phi(x)=(\phi_1(x), \phi_2(x), \phi_3(x))$ denote the exterior mass forces and the density of moments of these forces, respectively; $\eta, \nu=\text{const}>0$ are the Magnus and viscosity coefficients. The function $F=F(p)$ characterizes the friction between the particles. By $\omega\times u$ we mean the vector product of vectors ω and u; ∇, Δ and div are the usual gradient, Laplacian and divergence operators, so that Δu, $(u\cdot\nabla)u$, $(u\cdot\nabla)\omega$, ∇p are vectors with components Δu_i, $u_j(\partial/\partial x_j)u_i$, $u_j(\partial/\partial x_j)\omega_i$ and $(\partial/\partial x_i)p$, $i=1, 2, 3$, respectively (repeated indices are summed) $\operatorname{div} u=(\partial/\partial x_j)u_j$.

Assuming that f, ϕ and F are given, we shall prove that there exist functions u, ω and p satisfying Eqs. (0.1)–(0.3) in D and the boundary data (0.4) on S.

The time-dependent motion of granulated media with constant density has been studied by several authors [1, 2], [6, 7]. The existence of weak or strong solutions of a few initial and initial boundary-value problems for the time-dependent version of the system (0.1)–(0.3) has been established. The considerations of this paper concern the stationary case and seem to be new.

Before formulating the main result (Theorem 0.1) we define the basic function spaces that we shall use.

$H_0^1(D)$ = closure of $C_0^\infty(D; R^3)$ in the norm

$$||u||_{H_0^1(D)} = \Big(\int_D |\nabla u|^2\Big)^{1/2},$$

$\mathscr{V} = \{u \in C_0^\infty(D, R^3) : \operatorname{div} u = 0\}$, V — closure of $\mathscr{V}$ in $H_0^1(D)$, $L^q(D)$ — the set of classes of functions $f\colon D \to R^k$, L^q — integrable in D, with the norm

$$||f||_{L^q(D)} = \Big(\int_R |f|^q\Big)^{1/q}$$

($k = 1$ or 3, $q \geqslant 1$). $W_q^m(D)$ closure of $C^\infty(\bar{D}, R^k)$, $k = 1$ or 3, in the norm

$$||f||_{W_q^m(D)} = \Big(\sum_{|\alpha| \leqslant m} ||D^\alpha f||^q_{L^q(D)}\Big)^{1/q} \qquad (q \geqslant 1).$$

By $(\cdot\,,\,\cdot)$ we denote the scalar product on $L^2(D)$.

THEOREM 0.1. *Suppose that*

(0.5) $$f \in L^{3/2}(D), \quad \phi \in L^2(D),$$

(0.6) *F is a continuous operator from $L^q(D)$ to $L^2(D)$ for some $q \in (1, 6/5]$,*

(0.7) *$F(p)(x) \geqslant m > 0$ for each $p \in W^1_{3/2}(D)$ with $\int_D p(x)dx = 0$, and almost all $x \in D$.*

Then, there exist functions

(0.8) $$u \in V \cap W^2_{3/2}(D),$$

(0.9) $$p \in W^1_{3/2}(D), \qquad \int_D p(x)\,dx = 0,$$

(0.10) $$\omega \in L^2(D)$$

satisfying Eqs. (0.1)–(0.3) *in the sense of distribution on D.*

The boundary condition (0.4) is contained in the function (0.8).

The assumption (0.7) manifests the positiveness of the function F characterizing the friction between the particles of the medium and being determined experimentally for individual flows. For more information about F see [1].

The plan of the remaining sections of this paper is as follows. Section 1 is devoted to study an auxiliary elliptic problem related to Eq. (0.3). In Sect. 2 we prove the solvability of the linearized system (0.1), (0.2), (0.4) in u and p. Section 3 presents the proof of Theorem 0.1.

Several absolute constants in this paper are denoted by the letter C without bothering to distinguish them with subscripts.

1. Solvability of an auxiliary elliptic problem

In this section we shall prove the existence of a solution $\omega = \omega_\varepsilon$ of the problem (1.1) below. This problem is related to Eq. (0.3) as follows. Let $\{\varepsilon(n)\}$, $n = 1, 2, \ldots$ be any sequence of positive reals tending to zero and let $\omega_{\varepsilon(n)}$ be a solution of the problem (1.1) with $\varepsilon = \varepsilon(n)$. Then, a subsequence of the sequence $\{\omega_{\varepsilon(n)}\}$ converges, in a sense that will be determined in Sect. 3, to a solution of Eq. (0.3).

LEMMA 1.1. Suppose $u \in V$, $F(p) \in L^2(D)$, $F(p)(x) \geqslant m > 0$ for a.e. $x \in D$, $\phi \in L^2(D)$. Then, the problem

$$-\varepsilon\Delta\omega + (u\cdot\nabla)\omega + F(p)\omega = \phi \qquad (\varepsilon > 0), \tag{1.1}$$
$$\omega|_S = 0$$

has a unique solution ω in $H_0^1(D)$ and the following inequality holds:

$$\varepsilon\|\nabla\omega\|^2_{L^2(D)} + \frac{m}{2}\|\omega\|^2_{L^2(D)} \leqslant \frac{2}{m}\|\phi\|^2_{L^2(D)}. \tag{1.2}$$

Proof. To prove the existence of a solution of the above problem we shall use the Galerkin method. Suppose that $\omega \in H_0^1(D)$ is a solution of Eq. (1.1). Then, as one can easily see,

$$\varepsilon(\nabla\omega, \nabla v) + b(u, \omega, v) + \big((F(p)\omega, v\big) = (\phi, v) \tag{1.3}$$

for all $v \in H_0^1(D)$, where

$$b(u, \omega, v) = \int_D (u\cdot\nabla)\omega\cdot v.$$

Conversely, any $\omega \in H_0^1(D)$ satisfying Eq. (1.3) for all $v \in H_0^1(D)$ is a weak solution of Eq. (1.1). We shall prove the existence of such ω. Let a_i, $i = 1, 2, \ldots$ be a basis of $H_0^1(D)$, $a_i \in C_0^\infty(D, R^3)$. We define an approximate solution ω_m of Eq. (1.3) by

$$\omega_m = \sum_{i=1}^{m} r_{i,m} a_i, \qquad r_{i,m} \in R, \tag{1.4}$$
$$\varepsilon(\nabla\omega_m, \nabla a_k) + b(u, \omega_m, a_k) + \big(F(p)\omega_m, a_k\big) = (\phi, a_k), \qquad k = 1, 2, \ldots, m.$$

To prove the existence of such ω_m we use the following well-known [10, Ch. 2, §1]

LEMMA 1.2. Let X be a finite-dimensional Hilbert space with a scalar product $[\cdot, \cdot]$ and norm $[\cdot]$ and let P be a continuous mapping from X into itself such that

$$[P(x), x] > 0 \quad \text{for} \quad [x] = k > 0.$$

Then there exists $x \in X$, $[x] \leqslant k$ such that $P(x) = 0$.

As X we take the space spanned by the vectors a_k, $k = 1, 2, \ldots, m$, with the norm of $H_0^1(D)$, and as P — a mapping from X into itself defined by

$$\big(\nabla P(\omega), \nabla v\big) = \varepsilon(\nabla\omega, \nabla v) + b(u, \omega, v) + \big(F(p)\omega, v\big) - (\phi, v) \qquad \text{for all} \quad v \in X.$$

For any fixed ω in X we can treat the right-hand side of this identity as a continuous functional in v on X. In view of the Riesz–Frechet theorem $P(\omega)$ is uniquely determined. Since $b(u, \omega, \omega) = 0$, then we have

$$(\nabla P(\omega), \nabla\omega) = \varepsilon||\omega||^2_{H^1_0(D)} + \big(F(p)\omega, \omega\big) - (\phi, \omega)$$
$$\geqslant \varepsilon||\omega||^2_{H^1_0(D)} + ||\omega||_{L^2(D)}(m||\omega||_{L^2(D)} - |\phi||_{L^2(D)}).$$

This and the Poincaré lemma yield the existence of $k > 0$ such that

$$\big(\nabla P(\omega), \nabla\omega\big) > 0 \quad \text{for} \quad ||\omega||_{H^1_0(D)} = k.$$

The proof of the continuity of P presents no difficulties. Thus we have established the existence of an approximate solution ω_m of Eq. (1.3) for any integer m. From lemma 1.2 it follows also that the sequence $\{\omega_m\}$, $m = 1, 2, \ldots$ is jointly bounded in $H^1_0(D)$. We can choose then a subsequence $\{\omega_{m'}\}$ of $\{\omega_m\}$ converging to a function ω weakly in $H^1_0(D)$ and strongly in $L^2(D)$. Tending to infinity with m' in Eq. (1.4) and noticing that $b(u, v, \omega) = -b(u, \omega, v)$ for all $u \in V$, $v, \omega \in H^1_0(D)$, we conclude that ω satisfies Eq. (1.3) with $v = a_k$, $k = 1, 2, \ldots$. Since $\{a_k\}$, $k = 1, 2, \ldots$ is a basis of $H^1_0(D)$, then ω satisfies Eq. (1.3) for all $v \in H^1_0(D)$. We have thus proved the existence of a weak solution of Eq. (1.1). Setting $v = \omega$ in Eq. (1.3), we get

$$\varepsilon||\nabla\omega||^2_{L^2(D)} + m||\omega||^2_{L^2(D)} \leqslant \varepsilon(\nabla\omega, \nabla\omega) + \big(F(p)\omega, \omega\big) = (\phi, \omega)$$
$$\leqslant \frac{2}{m}||\phi||^2_{L^2(D)} + \frac{m}{2}||\omega||^2_{L^2(D)},$$

hence Eq. (1.2). It is clear that the solution is unique. The proof of the lemma is complete.

2. Solvability in u and p of the linearized problem (0.1), (0.2), (0.4)

In this Section we assume ω to be a given function in $L^2(D)$ and prove that the linearized problem (0.1), (0.2), (0.4) has a solution in u and p. We start with some useful inequalities.

LEMMA 2.1. (i) If $h, u \in H^1_0(D)$, then $(h \cdot \nabla)u \in L^{3/2}(D)$ and

$$||(h \cdot \nabla)u||_{L^{3/2}(D)} \leqslant C||\nabla h||_{L^2(D)}||\nabla u||_{L^2(D)}.$$

(ii) If $u \in H^1_0(D)$ and $\omega \in L^2(D)$ then $\omega \times u \in L^{3/2}(D)$ and

$$||\omega \times u||_{L^{3/2}(D)} \leqslant C||\omega||_{L^2(D)} \cdot ||\nabla u||_{L^2(D)}.$$

(iii) If $f \in L^{3/2}(D)$ then $f \in V'$ (the dual of V) and

$$||f||_{V'} \leqslant C||f||_{L^{3/2}(D)}.$$

Proof. To prove (i) and (ii) we use the inequality

$$(2.1) \qquad ||h||_{L^6(D)} \leqslant C||\nabla h||_{L^2(D)} \quad \text{for all} \quad h \in H^1_0(D).$$

Its elementary proof can be found in Chapter I, § 1 of [5].

By the inequality (2.1) and Hölder's inequality

$$\int_D |(h \cdot \nabla)u|^{3/2} \leqslant C \int_D |h|^{3/2} \cdot |\nabla u|^{3/2} \leqslant C\Big(\int_D |h|^6\Big)^{1/4}\Big(\int_D |\nabla u|^2\Big)^{3/4}$$
$$\leqslant C\Big(\int_D |\nabla h|^2\Big)^{3/4}\Big(\int_D |u\nabla|^2\Big)^{3/4},$$

hence (i). In a similar way we obtain (ii). To get (iii), observe that for any function v in V

$$\left|\int_D f\cdot v\right| \leqslant ||f||_{L^{3/2}(D)}\cdot ||v||_{L^3(D)} \leqslant c||f||_{L^{3/2}(D)}\cdot ||v||_{L^6(D)} \leqslant C||f||_{L^{3/2}(D)}\cdot ||\nabla u||_{L^2(D)}.$$

LEMMA 2.2. Suppose that $f \in L^{3/2}(D)$, $\omega \in L^2(D)$ and $h \in V$. Then there exist uniquely determined functions $u \in V \cap W^2_{3/2}(D)$ and $p \in W^1_{3/2}(D)$ with $\int_D p = 0$ satisfying

$$-\nu\Delta u+(h\cdot\nabla)u+\nabla p = f+\eta(\omega\times u)$$

in the sense of distribution on D. Moreover,

$$||\nabla u||_{L^2(D)} < C||f||_{L^{3/2}(D)}, \tag{2.2}$$

$$||p||_{W^1_{3/2}(D)}+||u||_{W^2_{3/2}(D)} \leqslant C||f||_{L^{3/2}(D)}(1+||\nabla h||_{L^2(D)}+||\omega||_{L^2(D)}). \tag{2.3}$$

Proof. We prove the existence of the relevant u and p in two steps. At first we establish the existence of a function u in V such that

$$\nu(\nabla u, \nabla v)+b(h, u, v) = (f, v)+\eta(\omega\times u, v) \tag{2.4}$$

for all $v \in V$. The proof is based on lemma 1.2 and the Galerkin method.

We omit the details. Notice that by setting $v = u$ in Eq. (2.4) we immediately get the inequality (2.2).

From Proposition 1.1 in Chapter I of [10] we conclude the existence of a distribution p such that

$$-\nu\Delta u+(h\cdot\nabla)u-f-\eta(\omega\times u) = \nabla p$$

in the sense of distribution on D. Clearly p is not uniquely determined. We shall now prove that there exist a relevant p satisfying the inequality (2.3). We fix $u \in V$ satisfying Eq. (2.4) for all $v \in V$ and consider the linear problem

$$\begin{aligned} -\nu\Delta\tilde{u}+\nabla p &= g \quad \text{in } D,\\ \operatorname{div}\tilde{u} &= 0 \quad \text{in } D,\\ \tilde{u} &= 0 \quad \text{on } S, \end{aligned} \tag{2.5}$$

where $g = f+\eta(\omega\times u)-(h\cdot\nabla)u$. By lemma 2.1 along with the inequality (2.2) $g \in L^{3/2}(D) \subset$ $\subset V'$ and

$$||g||_{L^{3/2}(D)} \leqslant C||f||_{L^{3/2}(D)}(1+||\nabla h||_{L^2(D)}+||\omega||_{L^2(D)}). \tag{2.6}$$

From the uniqueness in V of a solution u of the problem (2.5) we conclude that $\tilde{u} = u$. Now, from the results of [3, 8, 9] we infer the existence of a unique $p \in W^1_{3/2}(D)$ satisfying $\int_D p = 0$, the problem (2.5) and along with u, the inequality

$$||p||_{W^1_{3/2}(D)}+||u||_{W^2_{3/2}(D)} \leqslant C||g||_{L^{3/2}(D)}.$$

This inequality and the inequality (2.6) imply the inequality (2.3). The lemma is proved.

3. The proof of theorem 0.1

We prove Theorem 0.1 in two steps. At first we show that the nonlinear problem

$$
\begin{aligned}
&-\nu\Delta u+(u\cdot\nabla)u+\nabla p = f+\eta(\omega\times u),\\
&\operatorname{div} u = 0,\\
&(-1/N)\Delta\omega+(u\cdot\nabla)\omega+F(p)\omega = \phi \qquad (N \text{ — a positive integer}),\\
&u|_S = 0,\\
&\omega|_S = 0
\end{aligned}
\tag{3.1}
$$

has a solution $(u, p, \omega) = (u^N, p^N, \omega^N)$. Then we find a solution of the problem (0.1)–(0.4) as a limit of a subsequence of (u^N, p^N, ω^N), $N = 1, 2, 3, \ldots$.

LEMMA 3.1. Suppose (0.5), (0.6) and (0.7). Then there exists a solution

$$(u^N, p^N, \omega^N),\ u^N \in V\cap W^2_{3/2}(D), \quad p^N \in W^1_{3/2}(D), \quad \int_D p^N = 0, \quad \omega^N \in H^1_0(D)$$

of the problem (3.1).

P r o o f. We use Schauder's principle. Let

$$K = \{(u, p, \omega) \in V\times W^1_{3/2}(D)\times H^1_0(D): \|\nabla u\|_{L^2(D)} \leqslant C\|f\|_{L^{3/2}(D)},$$

$$\|p\|_{W^1_{3/2}(D)} \leqslant C\|f\|_{L^{3/2}(D)}\left(1+\frac{2}{m}\|\phi\|_{L^2(D)}+C\|f\|_{L^{3/2}(D)}\right),$$

$$\int_D p(x)\,dx = 0,$$

$$\|\omega\|_{L^2(D)} \leqslant \frac{2}{m}\|\phi\|_{L^2(D)}, \qquad \|\nabla\omega\|_{L^2(D)} \leqslant (2N/m)^{1/2}\cdot\|\phi\|_{L^2(D)}\},$$

where C is the constant from Lemma 2.1.

We define an operator Φ in $V\times W^1_{3/2}(D)\times H_0(D)$ by the formula $\Phi(\bar{u}, \bar{p}, \bar{\omega}) = (u, p, \omega)$, where (u, p) and ω are the unique solutions (see Lemmas 1.1 and 2.2) of the following problems:

$$
\begin{aligned}
&-\nu\Delta u+(\bar{u}\cdot\nabla)u+\nabla p = f+\eta(\bar{\omega}\times u) && \text{in } D,\\
&\operatorname{div} u = 0 && \text{in } D,\\
&u|_S = 0 &&\\
&(-1/N)\Delta\omega+(\bar{u}\cdot\nabla)\omega+F(\bar{p})\omega = \phi && \text{in } D,\\
&\omega|_S = 0. &&
\end{aligned}
$$

From the inequalities (1.2), (2.2) and (2.3) it follows that $\Phi(K) \subset K$. Since K is a compact, convex subset of $Y_q = L^2(D)\times L^q(D)\times L^2(D)$, for $q \in [1, 3)$, [4] then to prove Lemma 3.1 it suffices to show that Φ is continuous in the topology of Y_q for some $q \in [1, 3)$.

Let

$$(\bar{u}_i, \bar{p}_i, \bar{\omega}_i) \in K \quad \text{for} \quad i = 1, 2, 3, \ldots, \quad (\bar{u}, \bar{p}, \bar{\omega}) \in K \quad \text{and}$$

$$\Phi(\bar{u}_i, \bar{p}_i, \bar{\omega}_i) = (u_i, p_i, \omega_i), \quad \Phi(\bar{u}, \bar{p}, \bar{\omega}) = (u, p, \omega).$$

We have then

$$-\nu\Delta(u-u_i)+\nabla(p-p_i) = -(\bar{u}\cdot\nabla)(u-u_i)-((\bar{u}-\bar{u}_i)\cdot\nabla)u_i + \eta\{\bar{\omega}\times(u-u_i)+(\bar{\omega}-\bar{\omega}_i)\times u_i\} \quad \text{in } D,$$

$$(3.2) \quad \operatorname{div}(u-u_i) = 0 \quad \text{in } D,$$

$$u-u_i = 0 \quad \text{on } S.$$

From Eq. (3.2), the embedding $W^2_{3/2}(D)\subset W^1_3(D)$ and the results of [3, 8, 9] are easily obtained:

$$||p-p_i||_{W^1_{6/5}(D)} \leqslant C\{||\nabla(u-u_i)||_{L^2(D)}+||u-u_i||_{L^2(D)}+||\bar{u}-\bar{u}_i||_{L^2(D)}+||\bar{\omega}-\bar{\omega}_i||_{L^2(D)}\}.$$

Again, from Eq. (3.2), Poincaré's inequality and by standard energy estimate

$$||u-u_i||^2_{L^2(D)}+||\nabla(u-u_i)||^2_{L^2(D)} \leqslant C\{||\bar{u}-\bar{u}_i||_{L^2(D)}+||\bar{\omega}-\bar{\omega}_i||_{L^2(D)}\}.$$

Combining the above inequalities we obtain

$$||u-u_i||_{L^2(D)}+||p-p_i||_{L^{6/5}(D)} \leqslant C\{||\bar{u}-\bar{u}_i||_{L^2(D)}+||\bar{\omega}-\bar{\omega}_i||_{L^2(D)} + (||\bar{u}-\bar{u}_i||_{L^2(D)}+||\bar{\omega}-\bar{\omega}_i||_{L^2(D)})^{1/2}\}.$$

Now, "multiplying" the equation

$$(-1/\mathrm{N})\Delta(\omega-\omega_i)+(\bar{u}\cdot\nabla)(\omega-\omega_i)+F(\bar{p})(\omega-\omega_i) = -\omega_i(F(\bar{p})-F(\bar{p}_i))+((\bar{u}-\bar{u}_i)\cdot\nabla)\omega_i$$

by a function $\psi\in C^\infty_0(D)$ and "integrating by parts" in D, we get

$$(-1/N)\int\limits_D(\omega-\omega_i)\Delta\psi-\int\limits_D(\omega-\omega_i)(\bar{u}\cdot\nabla)\psi+\int\limits_D F(\bar{p})(\omega-\omega_i)\psi = -\int\limits_D\omega_i(F(\bar{p})-F(\bar{p}_i))\psi+\int\limits_D((\bar{u}-\bar{u}_i)\cdot\nabla)\omega_i\psi.$$

Let

$$\bar{u}_i\to\bar{u} \quad \text{in} \quad L^2(D), \quad \bar{p}_i\to\bar{p} \text{ in } L^{6/5}(D), \quad \bar{\omega}_i\to\bar{\omega} \quad \text{in} \quad L^2(D).$$

From (0.6) we have also $F(\bar{p}_i)\to F(\bar{p})$ in $L^2(D)$. The sequence $\{\omega_i\}$ is bounded in $H^1_0(D)$. From its arbitrary subsequence $\{\omega_{i'}\}$ we can choose a subsequence $\{\omega_{i''}\}$ converging in $L^2(D)$ to a function $\tilde{\omega}$. Taking the limit $i''\to\infty$ in the above integral identity we get

$$(-1/N)\int\limits_D(\omega-\tilde{\omega})\Delta\psi+\int\limits_D(\omega-\tilde{\omega})(\bar{u}\cdot\nabla)\psi+\int\limits_D F(\bar{p})(\omega-\tilde{\omega})\psi = 0.$$

In view of Lemma 1.1 $\tilde{\omega}=\omega$. Hence $\omega_i\to\omega$ in $L^2(D)$. This completes the proof of Lemma 3.1.

Taking into account the estimates (1.2), (2.2) and (2.3) we conclude that

(3.3) $\{u^N\}$ stays bounded in V,

(3.4) $\{p^N\}$ stays bounded in $W^1_{3/2}(D)$,

(3.5) $\{\omega^N\}$ stays bounded in $L^2(D)$.

By the compactness of embeddings $V\subset L^2(D)$, $W^1_{3/2}(D)\subset L^q(D)$, $q\in[1, 3)$ we also have

(3.6) $\{u^N\}$ stays in a compact subset of $L^2(D)$.

(3.7) $\{p^N\}$ stays in a compact subset of $L^q(D)$ for any $q\in[1, 3)$.

From (3.3)–(3.7) we conclude the existence of subsequences (again denoted by $\{u^N\}$, $\{p^N\}$, $\{\omega^N\}$ for simplicity) converging to some u, p, ω as $N \to \infty$

$$(3.8) \qquad u^N \to u \quad \begin{cases} \text{weakly in } V, \\ \text{strongly in } L^2(D), \end{cases}$$

$$(3.9) \qquad p^N \to p \quad \begin{cases} \text{weakly in } W^1_{3/2}(D), \\ \text{strongly in } L^{6/5}(D), \end{cases}$$

$$(3.10) \qquad \omega^N \to \omega \quad \text{weakly in } L^2(D).$$

From (0.6) and (3.9) we conclude also

$$(3.11) \qquad F(p^N) \to F(p) \quad \text{strongly in } L^2(D).$$

We shall now prove that the limit functions u, p and ω are a distributional solution of the problem (0.1)–(0.3). Let a and b be any functions from $C_0^\infty(D, R^3)$ and $C_0^\infty(D, R)$ respectively. Making use of the problem (3.1) we can write the following equalities:

$$(3.12) \qquad \int_D \left(-u^N(u^N \cdot \nabla)a + \nu \nabla u^N \cdot \nabla a + \nabla p^N \cdot a\right) = \int_D \left(\eta(\omega^N \times u^N)a + fa\right),$$

$$(3.13) \qquad \int_D u^N \cdot \nabla b = 0,$$

$$(3.14) \qquad \int_D \left((-1/N)\omega^N \cdot \Delta a - \omega^N(u^N \cdot \nabla)a + F(p^N)\omega^N a\right) = \int_D \phi \cdot a.$$

Tending to infinity with N in Eqs. (3.12)–(3.14) and taking into account (3.8)–(3.11), we easily get Eq. (3.12) with $u^N = u$, $P^N = p$ and $\omega^N = \omega$, respectively, Eq. (3.13) with $u^N = u$ and

$$\int_D \left(-\omega(u \cdot \nabla)a + F(p)\omega a\right) = \int_D \phi \cdot a$$

in place of Eq. (3.14). The limit integral identities we have obtained are equivalent to the definition of a distributional solution (u, p, ω) of the problem (0.1)–(0.4) provided that (0.5), (0.8)–(0.10) hold. This completes the proof of Theorem 0.1.

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