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The method of virtual power and a simple multipolar rod model

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It is shown that an application of the method of virtual power to a simple multipolar rod model, with the force-system space representing force and couple distributions, leads to the correct equations of motions and, in the case of elastic rods, to a torsion-dependent strain-energy function — results that had eluded previous attempts to represent three-dimensionally deforming rods by simple body models.

Wykazano, że zastosowanie metody mocy wirtualnych do prostego modelu pręta mikropolarnego prowadzi do poprawnych równań ruchu. W przypadku prętów sprężystych energia odkształcenia jest funkcją skręcania, w przeciwieństwie do wcześniejszych wyników opartych na zastosowaniu klasycznego modelu ośrodka do trójwymiarowego stanu odkształcenia pręta.

Показано, что применения метода виртуальных мощностей к простой модели микрополярного стержня приводит к правильным уравнениям движения. В случае упругих стержней энергия деформирования является функцией кручения, в отличие от ранее полученных результатов, основанных на применении классической модели среды к трехмерному состоянию деформации стержня.

1. Introduction

In 1958 Ericksen and Truesdell [1] inaugurated the modern theory of rods by reintroducing the idea of E. and F. Cosserat of treating a rod as a one-dimensional continuum whose deformation is determined not only by the positions of its particles but by additional vector variables which they called *directors*. Continua with directors were subsequently also used as models for liquid crystals [2], shells [3], and other bodies.

In general, if $\mathscr{B}$ is a manifold with boundary of dimension at most three and if $\mathscr{L}$ is a set of mappings λ of $\mathscr{B}$ into some finite-dimensional manifold $\mathscr{M}$, then the pair $(\mathscr{B}, \mathscr{L})$ will be said to represent a *simple* body model if $\mathscr{M} = \mathscr{E}$, the three-dimensional Euclidean affine space (with associated vector space E), and a *generalized* body model otherwise. More particularly, $\mathscr{M}$ may be of the form $\mathscr{E}\times\mathscr{F}$, where $\mathscr{F}$ is an additional manifold whose dimension is the number of additional local degrees of freedom accounted for in the model. If a point in $\mathscr{F}$ may be represented by an ordered n-tuple $(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ of vectors (not necessarily linearly independent) in E, then these vectors are called directors and the body model is called *directed* or *oriented*, or, if $\mathscr{B}$ is a continuum, a *Cosserat continuum*.

In the continuum model originally treated by the Cosserats, the directors are three in number and form an orthonormal triad, so that $\mathscr{F}$ is a three-dimensional manifold (sometimes denoted F_3); points in $\mathscr{F}$ may, consequently, be represented by three angles (such as the Euler angles), and, if one prefers to work with directors, one needs only use two, say $\mathbf{d}_1$ and $\mathbf{d}_2$, since $\mathbf{d}_3 = \mathbf{d}_1\times\mathbf{d}_2$ if the triad is right-handed. Clearly, this model is the simplest one that, in the case of a rod, would account for torsion and shear. It was pointed out by ANTMAN [4] that virtually every deformational phenomenon that has a name will be accounted for by a model with two directors that are orthogonal but not restricted to be of unit length. However, rod theories with an unrestricted number of independent directors have been proposed [5] to account for displacement fields in the three-dimensional rod whose dependence on the transverse coordinates is of a degree higher than the first. The "micropolar" rod model discussed by KAFADAR [6] is essentially equivalent to the Cosserat rod; here $\mathscr{F}$ is the group of proper orthogonal tensors.

The configuration space of a simple body model, as defined above, will be denoted $\mathscr{K}$; a configuration $\varkappa \in \mathscr{K}$ is therefore a mapping $\varkappa: \mathscr{B} \to \mathscr{E}$, and a spatial virtual velocity field $\mathbf{v}$ is a mapping $\mathbf{v} \cdot \varkappa(\mathscr{B}) \to E$, that is, a vector field on $\varkappa(\mathscr{B})$ (not necessarily defined everywhere); a material virtual velocity field is given by $\mathbf{v} \circ \varkappa$, and will be denoted η. The space of material virtual velocity fields in a configuration $\varkappa$ is, in some sense, the tangent space to $\mathscr{K}$ at $\varkappa$; as a topological vector space with an appropriate topology, it will be denoted $V_\varkappa$. A *force system* may then be regarded as a continuous linear functional on $V_\varkappa$; the value of a force system $\mathbf{f}$ at a virtual velocity field η will be denoted $\langle \mathbf{f}, \eta \rangle$ and is called the *virtual power* of $\mathbf{f}$ on η. Most often in mechanics force systems are assumed to be given by measures on some σ-algebra of subsets of $\overline{\varkappa(\mathscr{B})}$ (the closure of $\varkappa(\mathscr{B})$); they are necessarily so if $\mathscr{B}$ is discrete. For example, $V_\varkappa$ may be the Banach space of continuous E-valued functions, or an L^p space with an appropriate trace space. If the E-valued measure representing a force system $\mathbf{f}$ is also denoted $\mathbf{f}$, then

$$\langle \mathbf{f}, \mathbf{v} \circ \varkappa \rangle = \int_{\overline{\varkappa(\mathscr{B})}} \mathbf{v}(x) \cdot \mathbf{f}(dx).$$

If $\mathscr{B}$ is a differentiable manifold (possibly with boundary) and $\varkappa$ is a diffeomorphism, then force systems may more generally be represented by distributions on $\overline{\varkappa(\mathscr{B})}$(1). If, for example, the distribution is of order one, then a force system $\mathbf{f}$ may equivalently be represented by a pair of measures $(\mathbf{f}^0, \mathbf{f}^1)$ on $\overline{\varkappa(\mathscr{B})}$, the former with values in E and the latter with values in the space of second rank tensors on E, such that

$$\langle \mathbf{f}, \mathbf{v} \circ \varkappa \rangle = \int_{\overline{\varkappa(\mathscr{B})}} [\mathbf{v}(x) \cdot \mathbf{f}^0(dx) + \nabla \mathbf{v}(x) : \mathbf{f}^1(dx)].$$

Such a body model is called a *simple dipolar continuum* [7]. The generalization to simple *multipolar* continua, where force systems are distributions of order higher than one, is obvious.

In 1965 TADJBAKHSH [8] showed that a rod undergoing planar extension and flexure may be modeled as a one-dimensional simple dipolar continuum. A later attempt by LAWS [9] to extend this result to three-dimensional deformation, however, proved unsatisfactory, in that the torsional moment was not involved in the equations of motion and consequently, for the elastic rod, the strain energy could not depend on the torsion. Moreover, no improvement was obtained by assuming the model to be multipolar of a higher grade, for example, by assuming the force system to be given by a distribution of order two. This negative

(1) If $\mathscr{M}$ is a manifold with boundary, then by a distribution on $\overline{\mathscr{M}}$ we mean an ordered pair consisting of a distribution on $\mathring{\mathscr{M}}$ and a distribution on $\partial\mathscr{M}$ (as a manifold).

result shows the difficulties that arise when the velocity-field space is assumed to have a mathematical structure without regard to the physically necessary force systems.

The situation is different when the *method of virtual power* [10, 11] is used; in this method a space of force systems, based on the physics of the problem, is assumed *a priori* and placed in duality with the velocity-field space. Physically significant conclusions can then be drawn by using some formal results of duality theory [12–13]. It will be shown here that the correct equations of motion for a rod, as well as a torsion-dependent strain energy, are derived when the force systems are assumed to take the form of very specific distributions of order two, representing distributed (and possibly concentrated) forces and couples.

2. The method of virtual power for simple body models

Forces systems must, of course, be algebraically dual to velocity fields; in other words, since $\mathscr{K}$ (being a space of mappings of $\mathscr{B}$ into the affine space $\mathscr{E}$) is itself an affine space with an associated vector space K, the space $F_\varkappa$ of force systems in a configuration $\varkappa$ is a subspace of K^*, the algebraic dual of K. The velocity-field space $V_\varkappa$ may now be defined as the *weak* tangent space to $\mathscr{K}$ at $\varkappa$ with respect to the weak duality topology $\sigma(F_\varkappa, V_\varkappa)$. That is, if χ, defined as $t \to \chi_t \in \mathscr{K}$, is a motion, then there exist at every time t the limits $\dot{\chi}_{t\pm}$ (one-sided velocity fields) defined by

$$\langle \mathbf{f}, \dot{\chi}_{t\pm}\rangle = \pm \lim_{h\to 0+} \frac{1}{h}\langle \mathbf{f}, \chi_{t\pm h} - \chi_t\rangle$$

for every $\mathbf{f} \in F_{\chi t}$. The velocity-field space $V_\varkappa$ is generated by the one-sided velocity fields in every motion such that $\chi_t = \varkappa$; it is a topological vector space whose dual is the quotient space $F_\varkappa / V_\varkappa \overset{\text{def}}{=} V'_\varkappa$, the space of equivalence classes of force systems that differ from one another by a force system whose virtual power on any virtual velocity field in $V_\varkappa$ is zero (a *reactive* or *workless* force system); see [14] for more details.

In order to formulate the equations of motion within the context of the method of virtual power, it is convenient to assume that all rigid-body displacements are *a priori* possible, so that every velocity-field space $V_\varkappa$ contains the subspace $V_\varkappa^\perp$ whose elements are the *rigid-body velocity fields* (also called *distributors*), given by

$$\boldsymbol{\eta}(X) = \boldsymbol{\alpha} + \boldsymbol{\beta} \times \big(\varkappa(X) - x_0\big),$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are arbitrary vectors in E and x_0 is an arbitrary point in $\mathscr{E}$. If $\mathbf{f}_t^e$ denotes the external force system acting on the body at time t as a result of its interaction with other bodies (governed by physical laws, such as gravity or friction) and if $\mathbf{f}_t^i$ denotes the inertial force system of the body at time t (derived from its inertia [14]), then the laws of motion are equivalent to what GERMAIN [10] calls the *fundamental axiom of the method of virtual power*, namely

$$\mathbf{f}_t^e + \mathbf{f}_t^i \in V_\varkappa^{r\perp}, \tag{1}$$

where $V_\varkappa^{r\perp}$ is the orthogonal or annihilator of $V_\varkappa^r$, that is, the space $\{\mathbf{f} \in V'_\varkappa | \langle \mathbf{f}, \eta\rangle = 0 \forall \eta \in V_\varkappa^r\}$. It can be shown [14] that Eq. (1) is equivalent to Euler's equations of motion.

A *deformation* of a simple body model may be identified with an equivalence class of configurations that differ by a rigid-body displacement. If this equivalence relation is denoted $\mathcal{R}$, then the space of deformations may be identified with the quotient manifold $\mathcal{K}/\mathcal{R}$, and its tangent space (defined with respect to the appropriate duality topology) at a representative configuration $\varkappa$ — the *deformation-rate space* — is isomorphic to the quotient vector space $V_\varkappa/V_\varkappa^r$. It follows [14] that the space of equilibrated force systems, i.e., those whose torsors vanish, is isomorphic to the dual of the deformation-rate space.

Let the deformation space be represented by a manifold $\mathcal{G}$; specifically, assume a bijection k from $\mathcal{K}/\mathcal{R}$ onto $\mathcal{G}$ and define $l = k \circ j$, where j is the canonical injection from $\mathcal{K}$ into $\mathcal{K}/\mathcal{R}$, so that a deformation process corresponding to a motion χ is given by $l \circ \chi$. The deformation-rate space at $l(\varkappa)$ will be denoted $D_{l(\varkappa)}$, and there must exist in some sense a derivative of l at $\varkappa$, denoted $l'(\varkappa)$ (a linear mapping from $V_\varkappa$ into $D_{l(\varkappa)}$), such that $\frac{d}{dt}l(\chi_t) = l'(\chi_t)\cdot\dot{\chi}_t$. Clearly, $l'(\varkappa)\cdot\eta = 0$ if and only if $\eta \in V_\varkappa^r$; 0 consequently $l'(\varkappa)$ defines an isomorphism, say $\lambda_{l(\varkappa)}$, of $V_\varkappa/V_\varkappa^T$ onto $D_{l(\varkappa)}$, such that, if $\mathbf{f} \in V_\varkappa^{T\perp}$ (that is, if $\mathbf{f}$ is equilibrated), then

$$\langle \mathbf{f}, \boldsymbol{\eta}\rangle = \langle \mathbf{f}, \lambda_{l(\varkappa)}^{-1}\cdot l'(\varkappa)\cdot\boldsymbol{\eta}\rangle = \langle \lambda_{l(\varkappa)}^{T-1}\cdot\mathbf{f}, l'(\varkappa)\cdot\boldsymbol{\eta}\rangle,$$

where we use the same symbol to denote $\mathbf{f}$ as an element of $V_\varkappa^{r\perp}$ ($\subset V_\varkappa'$) and of $(V_\varkappa/V_\varkappa^r)'$, the two being isomorphic, and where $\lambda_{l(\varkappa)}^{T-1}$ denotes the transpose of $\lambda_{l(\varkappa)}^{-1}\cdot\lambda_{l(\varkappa)}^{T-1}\cdot\mathbf{f} \in D_{l(\varkappa)}'$ is the *internal force system* corresponding to the equilibrated force system $\mathbf{f}$.

In an actual motion χ, let the deformation process be denoted $\xi \overset{\text{def}}{=} l\circ\chi$, and let the internal force system at time t be denoted $\mathbf{s}_t \overset{\text{def}}{=} \lambda_{\xi_t}^{T-1}\cdot(\mathbf{f}_t^e + \mathbf{f}_t^i)$; then

$$\langle \mathbf{f}_t^e + \mathbf{f}_t^i, \boldsymbol{\eta}\rangle = \langle \mathbf{s}_t, l'(\chi_t)\cdot\boldsymbol{\eta}\rangle. \tag{2}$$

Eq. (2) embodies the *principle of virtual work* for deformable bodies. When $\boldsymbol{\eta}$ equals the actual velocity field $\dot{\chi}_t$, then

$$\langle \mathbf{f}_t^e + \mathbf{f}_t^i, \dot{\chi}_t\rangle = \langle \mathbf{s}_t, \dot{\xi}_t\rangle \overset{\text{def}}{=} P_t^d,$$

P_t^d being known as the *deformation power* at time t. A body is called *elastic* if there exists a function $\Phi: \mathcal{G} \to R$, called the *strain-energy functional*, such that

$$P_t^d = \frac{d}{dt}\Phi(\xi_t). \tag{3}$$

3. Kinematics of a simple rod model

In applying the method of virtual power to the simple rod model, we recognize at the outset that a physically meaningful force system on a rod consists of a force distribution and a couple distribution, together with end forces and end couples. Now the power of a force distribution is on a linear velocity field, and the power of a couple distribution is on an angular velocity field. In a Cosserat rod model, the appropriate angular velocity is that of the director triad. In the absence of directors, it would appear that the appropriate angular velocity should be that of the Frenet vectors. We shall see that with this choice the correct equations of motion are deduced, and the strain energy of an elastic rod does indeed depend on the torsion. In deriving these results, we shall have recourse to a *principle of localization*, namely, the postulate that *there exists a sufficiently large family of subbodies having the same structure as the body under consideration.* In the case of a rod model, for which $\mathscr{B} = [0, L] \subset R$ (with material points given by $S \in [0, L]$), the assumption is that a subbody given by $[S_1, S_2]$, with $0 \leqslant S_1 < S_2 \leqslant L$, likewise defines a simple rod model.

For a rod undergoing three-dimensional deformations which is modelled as an extensible curve, the deformation is specified by the triple $(\lambda, \overline{\varphi}, \overline{\tau})$, where λ is the stretch and $\overline{\varphi}$ is the Lagrangian curvature, and $\overline{\tau}$ is the Lagrangian *torsion or tortuosity* (not to be confused with torsion in the mechanical sense, whose treatment requires a generalized body model), which is the rate (with respect to S) of rotation of the plane of curvature (*the osculating plane*) of the rod. If we denote by **t**, **n** and **b** the unit tangent, normal and binormal vector fields on the curve occupied by the deformed rod, then $\overline{\varphi}$ and $\overline{\tau}$ may be defined intrinsically by the Lagnragian version of Frenet's formulas:

$$\frac{d}{dS}\mathbf{t} = \overline{\varphi}\mathbf{n}, \qquad \frac{d}{dS}\mathbf{n} = \overline{\tau}\mathbf{b} - \overline{\varphi}\mathbf{t}, \qquad \frac{d}{dS}\mathbf{b} = -\overline{\tau}\mathbf{n}.$$

Alternatively, we may write these formulas as

$$\frac{d}{dS}\begin{Bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{Bmatrix} = (\overline{\varphi}\mathbf{b} + \overline{\tau}\mathbf{t}) \times \begin{Bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{Bmatrix}.$$

When the rod is moving, with the motion described by

$$\chi_t(S) = x_0 + \mathbf{r}_t(S), \qquad 0 \leqslant S \leqslant L,$$

we have the time-dependent triple $(\lambda_t, \overline{\varphi}_t, \overline{\tau}_t)$; in particular

$$\lambda_t = |\mathbf{r}_t'|, \tag{4}$$

where we define $()' = \partial()/\partial S$ for convenience, so that $\mathbf{r}_t' = \lambda_t \mathbf{t}_t$, $\mathbf{r}_t'' = \lambda_t' \mathbf{t}_t + \lambda_t \mathbf{t}_t' = \lambda_t' \mathbf{t}_t + {}$ $+ \lambda_t \overline{\varphi}_t \mathbf{n}_t$ (from the first of Frenet's formulas), and therefore

$$\lambda_t \overline{\varphi}_t = \mathbf{n}_t \cdot \mathbf{r}_t''. \tag{5}$$

Moreover,

$$\mathbf{t}_t'' = \left(\frac{1}{\lambda_t}\mathbf{r}_t'\right)'' = (\overline{\varphi}_t \mathbf{n}_t)' = \overline{\varphi}_t' \mathbf{n}_t + \overline{\varphi}_t(-\overline{\varphi}_t \mathbf{t}_t + \overline{\tau}_t \mathbf{b}_t),$$

so that

$$\overline{\tau}_t = \frac{1}{\overline{\varphi}_t}\mathbf{b}_t \cdot \left(\frac{1}{\lambda_t}\mathbf{r}_t'\right)''. \tag{6}$$

Eqs. (4)–(6) represent the equation $\xi_t = l(\chi_t)$ for the simple rod model.

To obtain the rates (material time derivatives) of $\lambda_t, \overline{\varphi}_t$ and $\overline{\tau}_t$ in terms of $\dot{\mathbf{r}}_t$, i.e., in order to determine $l'(\chi_t)$, we may proceed as follows. Since the triad $(\mathbf{t}_t, \mathbf{n}_t, \mathbf{b}_t)$ is orthonormal, there must be an angular velocity field $\boldsymbol{\omega}_t$ such that

$$\begin{Bmatrix}\dot{\mathbf{t}}_t\\ \dot{\mathbf{n}}_t\\ \dot{\mathbf{b}}_t\end{Bmatrix} = \boldsymbol{\omega}_t \times \begin{Bmatrix}\mathbf{t}_t\\ \mathbf{n}_t\\ \mathbf{b}_t\end{Bmatrix},$$

and therefore $\dot{\mathbf{r}}_t' = \dot{\lambda}_t \mathbf{t}_t + \lambda_t \omega_t \times \mathbf{t}_t$. Thus

(7) $$\dot{\lambda}_t = \mathbf{t}_t \cdot \dot{\mathbf{r}}_t',$$

while

$$\frac{1}{\lambda_t} \mathbf{t}_t \times \dot{\mathbf{r}}_t' = \mathbf{t}_t \times (\boldsymbol{\omega}_t \times \mathbf{t}_t) = \boldsymbol{\omega}_t - \mathbf{t}_t \boldsymbol{\omega}_t \cdot \mathbf{t}_t.$$

Now

$$\dot{\mathbf{t}}_t = -\frac{\dot{\lambda}_t}{\lambda_t^2} \mathbf{r}_t' + \frac{1}{\lambda_t} \dot{\mathbf{r}}_t' = -\mathbf{t}_t \frac{\dot{\lambda}_t}{\lambda_t} + \frac{1}{\lambda_t} \dot{\mathbf{r}}_t', \qquad \dot{\mathbf{t}}_t' = -\mathbf{n}_t \overline{\varphi}_t \frac{\dot{\lambda}_t}{\lambda_t} - \mathbf{t}_t \left(\frac{\dot{\lambda}_t}{\lambda_t}\right)' + \left(\frac{1}{\lambda_t} \dot{\mathbf{r}}_t'\right)',$$

so that $\boldsymbol{\omega}_t \cdot \mathbf{t}_t = \dfrac{1}{\overline{\varphi}_t} \left(\dfrac{1}{\lambda_t} \dot{\mathbf{r}}_t'\right)' \cdot \mathbf{b}_t$, and therefore

(8) $$\omega_t = \frac{1}{\lambda_t} \mathbf{t}_t \times \dot{\mathbf{r}}_t' + \frac{1}{\overline{\varphi}_t} \left(\frac{1}{\lambda_t} \dot{\mathbf{r}}_t'\right)' \cdot \mathbf{b}_t \mathbf{t}_t.$$

On the other hand, $\mathbf{t}_t' = \dot{\overline{\overline{\varphi}_t \mathbf{n}_t}} = (\boldsymbol{\omega}_t \times \mathbf{t}_t)'$, so that

$$0 = \dot{\overline{\overline{\varphi}_t \mathbf{n}_t}} - (\boldsymbol{\omega}_t \times \mathbf{t}_t)' = \dot{\overline{\varphi}}_t \mathbf{n}_t + \overline{\varphi}_t (\boldsymbol{\omega}_t \times \mathbf{n}_t) - \boldsymbol{\omega}_t' \times \mathbf{t}_t - \boldsymbol{\omega}_t \times (\overline{\varphi}_t \mathbf{n}_t),$$

and therefore $\boldsymbol{\omega}_t' \times \mathbf{t}_t = \dot{\overline{\varphi}}_t \mathbf{n}_t$, from which follow

(9) $$\dot{\overline{\varphi}}_t = \boldsymbol{\omega}_t' \times \mathbf{t}_t \cdot \mathbf{n}_t = \mathbf{b}_t \cdot \boldsymbol{\omega}_t',$$

and

$$\mathbf{n}_t \cdot \boldsymbol{\omega}_t' = 0.$$

Similarly, by noting that $\dot{\mathbf{b}}_t' = -\dot{\overline{\overline{\tau}_t \mathbf{n}_t}} = (\boldsymbol{\omega}_t \times \mathbf{b}_t)$, we obtain

(10) $$\dot{\overline{\tau}}_t = -\boldsymbol{\omega}_t' \times \mathbf{b}_t \cdot \mathbf{n}_t = \mathbf{t}_t \cdot \boldsymbol{\omega}_t',$$

and thus

(11) $$\boldsymbol{\omega}_t' = \mathbf{t}_t \dot{\overline{\tau}}_t + \mathbf{b}_t \dot{\overline{\varphi}}_t.$$

Eqs. (7)–(10) together define $l'(\chi_t)$.

A rigid-body velocity field is given by

$$\dot{\mathbf{r}}_t(S) = \boldsymbol{\alpha} + \boldsymbol{\beta} \times \mathbf{r}_t(S),$$

so that $\dot{\mathbf{r}}_t' = \lambda_t \boldsymbol{\beta} \times \mathbf{t}_t$ and therefore

$$\boldsymbol{\omega}_t = \mathbf{t}_t \times (\boldsymbol{\beta} \times \mathbf{t}_t) + \frac{1}{\overline{\varphi}_t} \mathbf{t}_t \boldsymbol{\beta} \times \mathbf{t}_t' \cdot \mathbf{b}_t = \mathbf{t}_t \times (\boldsymbol{\beta} \times \mathbf{t}_t) + \mathbf{t}_t \boldsymbol{\beta} \times \mathbf{n}_t \cdot \mathbf{b}_t = \boldsymbol{\beta}$$

as expected. It follows from Eqs. (7) and (11) than an actual velocity field is a rigid-body velocity field if and only if $\dot{\lambda}_t = 0$, $\dot{\overline{\tau}}_t = 0$, and $\dot{\overline{\varphi}}_t = 0$.

4. The equations of motion

In the method of virtual power, the equations of motion take the following form: if the external force system acting on the body at time t is denoted $\mathbf{f}_t^e$, and if the inertial force system of the body at time t is $\mathbf{f}_t^i$, then

$$\langle \mathbf{f}_t^e + \mathbf{f}_t^i, \boldsymbol{\eta} \rangle = 0 \forall \boldsymbol{\eta} \in V_{\chi_t}^T.$$

The inertial force system is derived by a canonical formula from the inertia functional [14]. Here we will take the inertia as classical, so that the inertial force system is given by

$$\langle \mathbf{f}_t^i, \boldsymbol{\eta} \rangle = -\int_0^L \bar{\rho}_0 \dot{\mathbf{v}}_t \cdot \boldsymbol{\eta} dS,$$

where $\bar{\rho}_0$ is mass density per unit reference length, and the external force system will be assumed (as discussed above) to take the form

$$\langle \mathbf{f}_t^e, \boldsymbol{\eta} \rangle = \int_0^L (\mathbf{p} \cdot \boldsymbol{\eta} + \mathbf{m} \cdot \boldsymbol{\omega}) dS + [\mathbf{P} \cdot \boldsymbol{\eta} + \mathbf{M} \cdot \boldsymbol{\omega}]_0^L,$$

where $\boldsymbol{\omega}$ is related to $\boldsymbol{\eta}$ by

$$\boldsymbol{\omega} = \frac{1}{\lambda_t} \mathbf{t}_t \times \boldsymbol{\eta}' + \frac{1}{\bar{\varphi}_t} \left(\frac{1}{\lambda_t} \boldsymbol{\eta}' \right)' \cdot \mathbf{b}_t \mathbf{t}_t .$$

$\mathbf{p}$ and $\mathbf{m}$ are the distributed force and couple (per unit reference length), respectively, and $\mathbf{P}$ and $\mathbf{M}$ are, respectively, the end force and end couple. But in accordance with the principle of localization, $\mathbf{P}$ and $\mathbf{M}$ are also defined on $(0, L)$, namely, $\mathbf{P}(S)$ ($\mathbf{M}(S)$) is the force (moment) exerted by the subbody $[S, L]$ on the subbody $[0, S]$, and if we assume sufficient differentiability we can obtain

$$\langle \mathbf{f}_t^e, \boldsymbol{\eta} \rangle = \int_0^L [\mathbf{p} \cdot \boldsymbol{\eta} + (\mathbf{P} \cdot \boldsymbol{\eta})' + \mathbf{m} \cdot \boldsymbol{\omega} + (\mathbf{M} \cdot \boldsymbol{\omega})'] dS.$$

Let the torsor of $\mathbf{f}_t^e + \mathbf{f}_t^i$, i.e., its restriction to $V_{\chi_t}^r$, be represented with respect to x_0 by $(\mathbf{F}_t^d, \mathbf{M}_t^{d[x_0]})$. As we noted above, if $\boldsymbol{\eta}(S) = \boldsymbol{\alpha} + \boldsymbol{\beta} \times \mathbf{r}_t(S)$, then $\boldsymbol{\eta}' = (1/\lambda_t) \boldsymbol{\beta} \times \mathbf{t}_t$ and $\boldsymbol{\omega}(S) = \boldsymbol{\beta}$. It follows, first off, that

$$\mathbf{F}_t^d = \int_0^L (\mathbf{p} - \bar{\rho}_0 \dot{\mathbf{v}} + \mathbf{P}') dS = 0,$$

and therefore, by localization,

$$\mathbf{p} + \mathbf{P}' = \bar{\rho}_0 \dot{\mathbf{v}}. \tag{12}$$

We can now simplify $\mathbf{f}_t^e + \mathbf{f}_t^i$ to obtain

$$\langle \mathbf{f}_t^e + \mathbf{f}_t^i, \boldsymbol{\eta} \rangle = \int_0^L [\mathbf{P} \cdot \boldsymbol{\eta}' + \mathbf{m} \cdot \boldsymbol{\omega} + (\mathbf{M} \cdot \boldsymbol{\omega})'] dS.$$

It then follows that

$$\mathbf{M}_t^{d[x_0]} = \int_0^L (\mathbf{r}_t \times \mathbf{P} + \mathbf{m} + \mathbf{M}') dS = 0,$$

and, again by localization,

$$\mathbf{r}_t \times \mathbf{P} + \mathbf{m} + \mathbf{M}' = 0. \tag{13}$$

Eqs. (12) and (13) are the standard equations of motion for rods with classical inertia.

5. Elastic rods

With the help of Eqs. (7), (11), (12) and (13), we can show that the deformation power reduces to

$$P_t^d = \int_0^L (N_t \dot{\lambda}_t + M_t \dot{\bar{\varphi}}_t + T_t \dot{\bar{\tau}}_t)\, dS, \tag{14}$$

where $N_t = \mathbf{P} \cdot \mathbf{t}_t$, $M_t = \mathbf{M} \cdot \mathbf{b}_t$, and $T = \mathbf{M} \cdot \mathbf{t}_t$ are, respectively, the axial force, the bending moment in the osculating plane and the torsional moment at time t. The triple $(N, M, T): [0, L] \to R^3$ thus constitutes the internal force system of the simple rod model. The shear forces $\mathbf{P} \cdot \mathbf{n}$ and $\mathbf{P} \cdot \mathbf{b}$, and the other bending moment $\mathbf{M} \cdot \mathbf{n}$, are not part of the internal force system but constitute *internal reactions*.

If the rod is elastic, then a reasonable assumption (consistent with the localization principle) for the strain-energy functional Φ is that $\Phi(\lambda(\cdot), \bar{\varphi}(\cdot), \bar{\tau}(\cdot))$ is the value at $(0, L)$ of a positive (for stability) Borel measure on $(0, L)$. By the Lebesgue decomposition theorem, such a measure can be written as the sum of a measure that is absolutely continuous with respect to Lebesgue measure and one which is singular. A strain energy represented by a singular measure that is concentrated at, say, S_1 may be interpreted physically as a spring located there. In the absence of springs, the Radon–Nikodym theorem implies the existence of a function $W(\lambda(\cdot), \bar{\varphi}(\cdot), \bar{\tau}(\cdot), S)$ such that

$$\Phi(\lambda(\cdot), \bar{\varphi}(\cdot), \bar{\tau}(\cdot)) = \int_0^L W(\lambda(\cdot), \bar{\varphi}(\cdot), \bar{\tau}(\cdot), S) dS. \tag{15}$$

By the principle of localization, Eq. (3) remains valid when the limits of integration in Eqs. (14) and (15) become (S_1, S_2) in place of $(0, L)$, provided the functions $\lambda(\cdot)$ etc. are interpreted as their restrictions to (S_1, S_2). Since the equations must be valid when the integrands are continuous, there result the local equations

$$N_t(S)\dot{\lambda}_t(S) + M_t(S)\dot{\bar{\varphi}}_t(S) + T_t(S)\dot{\bar{\tau}}_t(S) = \frac{d}{dt} W(\lambda_t(\cdot), \bar{\varphi}_t(\cdot), \bar{\tau}_t(\cdot), S), \tag{16}$$

where $\lambda(\cdot)$ etc. are now to be interpreted as the restrictions of the functions to an arbitrarily small neighborhood of S, but they cannot be replaced automatically by their local values $\lambda(S)$; for example, W may depend on $\lambda(\cdot)$ through $\lambda(S)$, $\lambda'(S)$, $\lambda''(S)$ and so on. If such is the case, then the dependence on the derivatives may be eliminated through the use of the chain rule: let $\mathbf{s}_t$ denote (N_t, M_t, T_t) and let ξ_t denote $(\lambda_t, \bar{\varphi}_t, \bar{\tau}_t)$; if $W_t(S) = W(\xi_t(S), \xi_t'(S), \ldots, \xi_t^{(n)}(S), S)$ then

$$\left(\mathbf{s}_t - \frac{\partial W}{\partial \xi_t(S)}\right)\dot{\xi}_t(S) - \sum_{k=1}^{k=n} \frac{\partial W}{\partial \xi_i^{(k)}(S)} \dot{\xi}_t^{(k)}(S) = 0.$$

Since a deformation process may, in general, be found such that $\dot{\xi}_t(\cdot), \dot{\xi}_t'(\cdot), \ldots, \dot{\xi}_t^{(n)}(\cdot)$ have arbitrary values at S, the coefficients of $\dot{\xi}_t^{(k)}(S)$ must vanish, so that $W_t(S) = W(\xi_t(S), S)$, and

$$\mathbf{s}_t(S) = \frac{\partial W}{\partial \xi_t(S)}. \tag{17}$$

If the dependence of W on $\xi_t(\cdot)$ is not through its derivatives, then a result identical to the preceding one may be derived through a method based on a proof due to GURTIN [15]. Consider a deformation process $t \to \xi_t(\cdot)$ during a time interval $[t_0, t_1]$ such that, at the given point $S \in (0, L)$, $\xi_t(S) = \xi$ (a constant) at all $t \in [t_0, t_1]$, while $\xi_{t_1}(\cdot) = \xi 1(\cdot)$, where $1(\cdot)$ is the unit function on $(0, L)$. In words, such a deformation process changes an arbitrary deformation state into one of uniform deformation (extension, curvature and torsion) while keeping the deformation constant at one point; it is not difficult to show that such a process is possible. Now, since $\dot{\xi}_t(S) = 0$, it follows from Eq. (16) that $\frac{d}{dt} W(\xi_t(\cdot), S) = 0$, so that $W(\xi_t(\cdot), S) = W(\xi 1(\cdot), S) \stackrel{\text{def}}{=} W(\xi, S)$. It follows that the dependence of W on $\xi(\cdot)$ is only through its local values, and Eq. (17) holds. Returning to the internal-force and deformation variables peculiar to the rod, we write this equation as

$$N = \frac{\partial W}{\partial \lambda}, \qquad M = \frac{\partial W}{\partial \bar{\varphi}}, \qquad T = \frac{\partial W}{\partial \bar{\tau}}. \tag{18}$$

The third of these equations shows explicity the torsion dependence of the strain energy that eluded LAWS.

It should be pointed out, in conclusion, that the purpose of this note is not to produce a general rod theory, but only to show the applicability of the method of virtual power. For one thing, the fact that the strain-energy function W depends on the total material curvature $\bar{\varphi}$, rather than on its components in two mutually perpendicular planes, means that a rod that is modeled by a simple rod model must be transversely isotropic: it must have equal stiffness in all possible planes of bending. Transverse anisotropy, along with torsion, can be handled by means of a simplified Cosserat rod model in which one of the directors remains tangential; this is essentially the Kirchhoff–St.Venant–Clebsch model [16].

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