Arch. Mech., 38, 1–2, pp. 97–102, 1986

On the regularity, uniqueness and continuous dependence for generalized solutions of some coupled problems in nonlinear theory of thermoelastic shells

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THE REGULARITY, uniqueness and continuous dependence results in Sobolev spaces for generalized solutions of two-dimensional nonlinear equations describing thermoelastic vibrations of shells coupled with three-dimensional nonlinear heat-conduction equation are formulated. The shell under consideration is partially clamped and partially supported, and there is no heat exchange with the surroundings. The nonhomogeneous initial conditions are imposed.

W pracy sformułowano twierdzenia o regularności, jednoznaczności i ciągłej zależności od danych dla uogólnionych rozwiązań dwuwymiarowych nieliniowych równań powłok, sprzężonych z trójwymiarowym, nieliniowym równaniem przewodnictwa cieplnego. Rozważana powłoka jest sztywnie utwierdzona na części brzegu i swobodnie podparta na jego pozostałej części. Założono brak wymiany ciepła z otoczeniem.

В работе сформулированы теоремы регулярности, единственности и напрерывной зависимости от данных для обобщенных решений двумерных, нелинейных уравнений оболочек, сопряженных с трехмерным, нелинейным уравнением теплопроводности. Рассматриваемая оболочка частично защемлена и частично шарнирно оперта по краям. Предположено отсутствие внешнего теплообмена.

1. Introduction

THE EXISTENCE of solutions for the two-dimensional nonlinear problem of vibrations of shells coupled with three-dimensional nonlinear heat-conduction equation was considered in [1]. In the present paper we deal with further mathematical questions concerning the same problem, i.e. with the regularity, uniqueness and continuous dependence on the data.

The equations under consideration are of the form (see [1], our modiffications of the notations of [1] allow for using the summation convention with $i, j = 1, 2$)

$$(1.1)\qquad \varrho h\ddot{w} - I\Delta\ddot{w} + D\Delta^2 w - k_{ij}\sigma^*_{ij} - (\sigma^*_{ji}w_{,i})_{,j} + b\Delta\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\theta dx_3 = q_1 \quad \text{in} \quad \Omega\times]0, T[,$$

$$(1.2)\qquad \varrho h\ddot{u}_i - \sigma^*_{ij,j} = p_i, \quad i = 1, 2 \quad \text{in} \quad \Omega\times]0, T[,$$

$$(1.3)\qquad c_0\dot{\theta} - \lambda_q\Delta_3\theta + bT_0(\dot{\varepsilon}_{ii} - x_3\Delta\dot{w}) = q_2 \quad \text{in} \quad Q\times]0, T[,$$

where

$$\sigma_{ij}^* = \sigma_{ij} - \delta_{ij} b \int_{-\frac{h}{2}}^{\frac{h}{2}} \theta\, dx_3, \qquad \sigma_{ij} = a_{ijkl}\varepsilon_{kl},$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + w_{,i}w_{,j}) - k_{ij}w,$$

Ω — the open domain in R^2 of variables x_1, x_2, $]0, T[$ — the open interval in R^1 of variable t, $\left]-\frac{h}{2}, +\frac{h}{2}\right[$ — the open interval in R^1 of variable x_3 and $Q = \Omega \times \left]-\frac{h}{2}, +\frac{h}{2}\right[$. Furthermore $\varphi_{,i} \equiv \partial\varphi/\partial x_i$, $\varphi_{,ij} \equiv \partial^2\varphi/\partial x_i \partial x_j$, $\dot{\varphi} \equiv \partial\varphi/\partial t$, $\ddot{\varphi} \equiv \partial^2\varphi/\partial t^2$, $\Delta\varphi \equiv \varphi_{,11} + \varphi_{,22}$, $\Delta_3\varphi \equiv \varphi_{,11} + \varphi_{,22} + \varphi_{,33}$. The constants in (1.1)–(1.3) have the following meaning; ϱ_3 — mass density, h — thickness of the shell, $I = \varrho h^3/12$, $D = Eh^3/12(1-\nu^2)$, ν — Poisson's ratio, E — Young modulus, $b = E\alpha_T/(1-\nu)$, α_T — coefficient of thermal expansion, T_0 — initial temperature, k_{11}, k_{22} — curvatures of the middle surface, $k_{12} = k_{21} = 0$, c_0 — specific heat at constant stress, λ_q — thermal conductivity;

(1.4)
$$\begin{gathered} a_{iiii} = Eh/(1-\nu^2); \quad a_{iijj} = \nu Eh/(1-\nu^2), \quad i \neq j; \\ a_{ijij} = Eh/(1+\nu), \quad i \neq j; \quad a_{ijkl} = 0 \quad \text{for other} \quad i, j, k, l = 1, 2 \end{gathered}$$

(in (1.4) there is no summation over repeated indices).

The following functions are assumed to be given:

$q_1 = q_1(x_1, x_2, t)$ normal load,

$p_i = p_i(x_1, x_2, t)$, $i = 1, 2$ load in the x_i-direction,

$q_2 = q_2(x_1, x_2, x_3, t)$ heat sources function.

The functions sought for are

$w = w(x_1, x_2, t)$ deflection of the shell,

$u_i = u_i(x_1, x_2, t)$, $i = 1, 2$ displacements in the x_i-direction,

$\theta = \theta(x_1, x_2, x_3, t)$ temperature difference.

Let Ω be the bounded domain with the cone property (see [10], p. 314), and let $\partial\Omega$ denote its boundary. Let us assume, that the shell under consideration is partially clamped and partially supported, i.e.,

(1.5)
$$\begin{gathered} w = 0 \\ w_{,n} \equiv \frac{\partial w}{\partial n} = 0 \qquad \text{on} \quad \partial_c\Omega \times]0, T[, \\ u_i = 0, \quad i = 1,2 \end{gathered}$$

and

(1.6)
$$\begin{gathered} w = 0 \\ M_n = 0 \qquad \text{on} \quad \partial_s\Omega \times]0, T[, \\ u_i = 0, \quad i = 1, 2, \end{gathered}$$

where $n = (n_1, n_2)$ is the unit outward normal to $\partial\Omega$,

$$M_n = -D\{\Delta w+(1-\nu)(2n_1 n_2 w_{,12}-n_2^2 w_{,11}-n_1^2 w_{,22})\},$$

$\partial_c\Omega$ — the clamped and $\partial_s\Omega$ — the supported part of $\partial\Omega$. $\partial_c\Omega$ or $\partial_s\Omega$ may be empty but $\partial_c\Omega\cup\partial_s\Omega = \partial\Omega$. In both the cases no heat exchange with the surroundings, is assumed, i.e.,

$$\theta = 0 \quad \text{on} \quad \partial Q\times]0, T[. \tag{1.7}$$

Finally impose the initial conditions (space variables x_i are omitted for the sake of brevity)

$$\begin{aligned} &w(0) = w^0, \quad u_i(0) = u_i^0 \\ &\dot{w}(0) = w^1, \quad u_i(0) = u_i^1 \quad i = 1, 2, \quad \theta(0) = \theta^0, \end{aligned} \tag{1.8}$$

where the right-hand sides are given functions.

2. Assumptions

Let us consider the following two types of domains

(2.1) $\partial\Omega$ is of class C^3,

(2.2) $\partial\Omega$ consists of finite number of closed straight-line segments Γ_k, $k = 1, 2, \ldots, N$.

We assume that in the case (2.1), $\partial_c\Omega = \phi$ (= null set) or $\partial_s\Omega = \phi$ (= null set) what means that the shell is either entirely supported, or entirely clamped, and in the case (2.2) the changes in the type of the boundary conditions (1.5), (1.6) are allowed only at the corner points $\{P_k\} = \Gamma_k\cap\Gamma_{k-1}$, $k = 1, 2, \ldots, N$, $\Gamma_0 \equiv \Gamma_N$. Let ω_k be the inner angle at the corner point P_k. We assume that

(2.3)
- $0 < \omega_k < 180^\circ$ for corner points P_k clamped along both sides;
- $0 < \omega_k < \overline{\omega}$ for the corner points in which the type of boundary conditions changes, where $\overline{\omega} \approx 126^\circ, 726699$;
- $0 < \omega_k < 90^\circ$ for corner points supported along both sides.

Let the function spaces $W^{m,p}(\mathrm{D})$, $H^m(\mathrm{D})$, $H_0^m(\mathrm{D})$, $L^p(0, T; X)$, $\mathrm{D} = \Omega$ or Q be defined as in [2], Chapt. I, and let

$$V(\Omega) = \{v \in H^2(\Omega)\colon v = v_{,n} = 0 \quad \text{on} \quad \partial_c\Omega$$

and $v = 0$ on $\partial_s\Omega$ in the trace sense}.

We suppose

$$\begin{aligned} &q_1 \in L^2(0, T; H^{-1}(\Omega)), \quad \dot{q}_1 \in L^2(0, T; H^{-1}(\Omega)), \\ &p_i \in L^2(0, T; L^2(\Omega)), \quad \dot{p}_i \in L^2(0, T; L^2(\Omega)), \quad i = 1, 2, \\ &q_2 \in L^\infty(0, T; L^2(Q)), \quad \dot{q}_2 \in L^2(0, T; L^2(Q)), \\ &w^0 \in H^3(\Omega), \quad w^0 \text{ satisfies Eqs. (1.5), (1.6) in the trace sense}, \\ &w^1 \in V(\Omega), \\ &\left.\begin{aligned} &u_i^0 \in H^2(\Omega)\cap H_0^1(\Omega), \\ &u_i^1 \in H_0^1(\Omega), \end{aligned}\right\} \quad i = 1, 2, \\ &\theta^0 \in H^2(Q). \end{aligned} \tag{2.4}$$

3. Results

The hypotheses (2.1)–(2.4) imply the following regularity result:

THEOREM 1. *There exist functions* w, u_1, u_2, θ *having the properties*

$$
\text{(3.1)}\qquad
\begin{aligned}
& w \in L^2\big(0, T; H^3(\Omega)\cap V(\Omega)\big)\cap C\big([0, T], V(\Omega)\big),\\
& \dot{w} \in L^\infty\big(0, T; V(\Omega)\big)\cap C([0, T], H_0^1(\Omega)),\\
& \ddot{w} \in L^\infty\big(0, T; H_0^1(\Omega)\big),\\
& \left.\begin{aligned}
& u_i \in L^2\big(0, T; H^2(\Omega)\cap H_0^1(\Omega)\big)\cap C\big([0, T], H_0^1(\Omega)\big),\\
& \dot{u}_i \in L^\infty\big(0, T; H_0^1(\Omega)\big)\cap C\big([0, T], L^2(\Omega)\big),\\
& \ddot{u}_i \in L^\infty\big(0, T; L^2(\Omega)\big),
\end{aligned}\right\} \quad i = 1, 2,\\
& \theta \in L^\infty\big(0, T; H^2(Q)\cap H_0^1(Q)\big)\cap C\big([0, T], H_0^1(Q)\big),\\
& \dot{\theta} \in L^\infty\big(0, T; L^2(Q)\big)\cap L^2\big(0, T; H_0^1(\Omega)\big),
\end{aligned}
$$

such that w, u_1, u_2, θ *satisfy the* Eq. (1.1) *in the distributional sense, the* Eqs. (1.2), (1.3) *almost everywhere on* $\Omega\times]0, T[$ *and* $Q\times]0, T[$, *respectively, the boundary conditions* (1.5) *in the classical sense, the boundary conditions* (1.6) *in the trace sense, and the initial conditions almost everywhere on* Ω *and* Q, *respectively.*

The continuous dependence and uniqueness results follow from the estimate, formulated in

THEOREM 2. *Let* $\bar{q}_i, \bar{p}_i, \bar{w}^0, \bar{w}^1, \bar{u}_i^0, \bar{u}_i^1, \bar{\theta}^0, i = 1, 2$ *and* $\tilde{q}_i, \tilde{p}_i, \tilde{w}^0, \tilde{w}^1, \tilde{u}_i^0, \tilde{u}_i^1, \tilde{\theta}^0, i = 1, 2$, *respectively, be the data satisfying conditions* (2.4). *Let* $\bar{w}$, $\bar{u}_i$, $\bar{\theta}$ *and* $\tilde{w}$, $\tilde{u}_i$, $\tilde{\theta}$, *respectively, be the corresponding solutions of the problem* (1.1)–(1.8). *The following inequality holds true*

$$
\text{(3.2)}\qquad
\begin{aligned}
&\|\dot{\bar{w}}(t)-\dot{\tilde{w}}(t)\|_{H^1(\Omega)}^2+\sum_{i=1}^{2}\|\dot{\bar{u}}_i(t)-\dot{\tilde{u}}_i(t)\|_{L^2(\Omega)}^2+\|\bar{w}(t)-\tilde{w}(t)\|_{H^2(\Omega)}^2+\sum_{i=1}^{2}\|\bar{u}_i(t)\\
&\quad-\tilde{u}_i(t)\|_{H^1(\Omega)}^2+\|\bar{\theta}(t)-\tilde{\theta}(t)\|_{H^1(\Omega)}^2+\int_0^T\|\bar{\theta}(t)-\tilde{\theta}(t)\|_{H^1(Q)}^2 \leqslant C\Big\{\|\bar{w}^1-\tilde{w}^1\|_{H^1(Q)}^2\\
&\quad+\sum_{i=1}^{2}\|\bar{u}^1-\hat{u}^1\|_{L^2(\Omega)}^2+\|\bar{w}^0-\tilde{w}^0\|_{H^2(\Omega)}^2+\sum_{i=1}^{2}\|\bar{u}_i^0-\tilde{u}_i^0\|_{H^1(\Omega)}^2+\|\bar{\theta}^0-\tilde{\theta}^0\|_{L^2(Q)}\\
&\quad+\int_0^T\Big[\|\bar{q}_1(t)-\tilde{q}_1(t)\|_{H^{-1}(\Omega)}^2+\sum_{i=1}^{2}\|\bar{p}_i(t)-\tilde{p}_i(t)\|_{L^2(\Omega)}^2+\|\bar{q}_2(t)-\tilde{q}_2(t)\|_{L^2(Q)}^2\Big]dt\Big\}
\end{aligned}
$$

with a positive constant C.

4. Sketch of proofs

The proof of the inequality (3.2) is similar to the proof of the Theorem 5.2 of the paper [3] and therefore we can restrict our considerations to the regularity problem.

As in [3] we can show the existence of the generalized solution of the problem (1.1)–(1.8) with the properties

$$
\begin{aligned}
& w,\, \dot{w} \in L^\infty\big(0,T; V(\Omega)\big), \qquad \ddot{w} \in L^\infty\big(0,T; H_0^1(\Omega)\big),\\
(4.1)\qquad & u_i,\, \dot{u}_i \in L^\infty\big(0,T; H_0^1(\Omega)\big), \quad \ddot{u}_i \in L^\infty\big(0,T; L^2(\Omega)\big), \quad i = 1,2,\\
& \theta,\, \dot{\theta} \in L^\infty\big(0,T; L^2(Q)\big) \cap L^2\big(0,T; H_0^1(Q)\big).
\end{aligned}
$$

Thus the functions w, u_1, u_2, θ satisfy in the distributional sense the following system of equations

$$
D\Delta^2 w = -\varrho h\ddot{w} + I\Delta\ddot{w} + k_{ij}\sigma_{ij}^* + (\sigma_{ij}^* w_{,i})_{,j} - b\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\theta\, dx_3 + q_1 \equiv \bar{Q}_1,
$$

$$
(4.2)\qquad -a_{ijkl}u_{k,lj} = \left(a_{ijkl}\left(\frac{1}{2}w_{,k}w_{,l} - k_{kl}w\right)\right)_{,j} - \varrho h\ddot{u}_i - \delta_y b\left(\int_{-\frac{h}{2}}^{\frac{h}{2}} \theta\, dx_3\right)_{,j} + p_i \equiv \bar{P}_i,
$$

$$
-\lambda\, \Delta_3\theta = -c_0\theta - bT_0(\varepsilon_{ii} - x_3\Delta\dot{w}) + q_2 \equiv \bar{Q}_2.
$$

Let $1 < r < 2$. The inequalities

$$
\int_\Omega |\sigma_{ij}|^r |w_i|^r dx \leqslant \Big(\int_\Omega |\sigma_{ij}|^2 dx\Big)^{\frac{r}{2}} \Big(\int_\Omega |w_{,i}|^{\frac{2r}{2-r}} dx\Big)^{\frac{2-r}{2}}
$$

$$
= ||\sigma_{ij}||^r_{L^2(\Omega)} ||w_{,i}||^r_{L^{\frac{2r}{2-r}}(\Omega)} \leqslant \text{const}\, ||\sigma_{ij}||^r_{L^2(\Omega)} ||w||^r_{H^2(\Omega)}
$$

imply

$$
\sigma_{ij}w_{,i} \in L^\infty\big(0,T; L^r(\Omega)\big)
$$

and

$$
(\sigma_{ij}w_{,i})_{,j} \in L^\infty\big(0,T; W^{-1,}\ (\Omega)\big).
$$

The embedding $H^{1+\varepsilon}(\Omega) \subset W^{1,q}(\Omega)$, $0 < \varepsilon < 1$, $\dfrac{1}{r} + \dfrac{1}{q} = 1$ (see [10] pp. 327–328, where $H_p^s(\Omega) = H_2^s(\Omega) \equiv H^{1+\varepsilon}(\Omega)$, $p = 2$, $s = 1+\varepsilon$, is the so-called interpolation space and $H_q^t(\Omega) \equiv W^{t,q}(\Omega)$, $t = 1$) yields $W^{-1,r}(\Omega) \subset H^{-1-\varepsilon}(\Omega)$ and, consequently,

$$
(\sigma_{ij}w_{,i})_{,j} \in L^\infty\big(0,T; H^{-1-\varepsilon}(\Omega)\big).
$$

It is easy to see that for the other terms of the sum $\bar{Q}_1$ similar inclusions hold and

$$
(4.3)\qquad \bar{Q}_1 \in L^2\big(0,T; H^{-1-\varepsilon}(\Omega)\big).
$$

Because $(\Delta^2)^{-1}$ maps $H^{-2}(\Omega)$ onto $H^2(\Omega)$ and $H^{-1}(\Omega)$ onto $H^3(\Omega)$ (see [5] in the smooth boundary case and [4] in the case of polygonal domain), then by interpolation (see [5, 10]) it maps $H^{-1-\varepsilon}(\Omega)$ onto $H^{3-\varepsilon}(\Omega)$. Thus, we have

$$
(4.4)\qquad w \in L^2\big(0,T; H^{3-\varepsilon}(\Omega)\big).
$$

But $w_{,i} \in L^2(0,T; H^{2-\varepsilon}(\Omega)) \subset L^2(0,T; L^\infty(\Omega))$ (see [7] § 4.1, Ch. 1) implies

$$
\sigma_{ij}w_{,i} \in L^2\big(0,T; L^2(\Omega)\big)
$$

and

$$(\sigma_{ij} w_{,i})_{,j} \in L^2(0, T; H^{-1}(\Omega)).$$

Using again the properties of $(\Delta^2)^{-1}$ we arrive at the inclusion

(4.5) $$w \in L^2(0, T; H^3(\Omega)).$$

It is easily verified that (4.5) yields $(w_{,k} w_{,l})_{,j} \in L^2(0, T; L^2(\Omega))$ and

(4.6) $$\bar{P}_i \in L^2(0, T; L^2(\Omega)).$$

Using the results of [9] and the method of continuity we may show as in [8], § 9, Chapt. III that (4.6) implies

(4.7) $$u_i \in L^2(0, T; H^2(\Omega)).$$

Similarly it may easily be proved that $\bar{Q}_2 \in L^\infty(0, T; L^2(Q))$. Using the results of [6], [9] we conclude that

(4.8) $$\theta \in L^\infty(0, T; H^2(Q)).$$

The assertions (3.1) follow then from (4.1), (4.5), (4.7), (4.8) and the Lemma 1.2, § 1.2 Chapt. I of [7].

The remaining part of the Theorem 1 is a consequence of (3.1) and of the inclusions $H^{2+\gamma}(\Omega) \subset C^\gamma(\bar{\Omega})$, $\gamma = 0,1$, $H^2(Q) \subset C^0(\bar{Q})$.

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Received February 21, 1985.