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Formulation of the brittle fracture criterion for three-dimensional problems

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THE CRACK propagation in three dimensions is considered. The fracture criterion takes into account the energy balance and the crack geometry at the same time. For the preferred fracture process, the ratio of the corresponding energy release rate and the magnitude of a newly-created crack surface reaches the maximum.

Rozważana jest propagacja szczeliny w trzech wymiarach. Kryterium pękania uwzględnia jednocześnie bilans energii i geometrię szczeliny. Dla preferowanego procesu pękania stosunek uwolnionej energii sprężystej do wielkości nowopowstałej powierzchni szczeliny osiąga maximum.

Рассматривается распространение трещины в трех измерениях. Критерий разрушения учитывает одновременно баланс энергии и геометрию трещины. Для предпочительного процесса разрушения отношение освобожденной упругой энергии к величине нововозникающей поверхности трещины достигает максимума.

1. Introduction

WE CONSIDER the fracture process as the propagation of a macroscopic crack in the perfect, elastic body.

The fracture process is determined by the applied loads and by the geometry of the initial crack and the body. The trajectory along which a crack spreads is distinguished as the trajectory of maximal energy release rate.

However, the prediction towards the fracture geometry in the complex three-dimensional case is a very heavy task. Yet, with the help of contemporary numerical methods a lot can be done.

Taking this into account, we formulate the fracture criterion using the idea of the energy release rate referring to the subsequent states of the propagating crack instead of using quantities referring to the "initial" state of stresses around the crack.

2. The crack as a surface defect

We consider the ideal three-dimensional linearly-elastic medium, cracked along the surface S. The two faces of the crack S^+ and S^- lie close together when the medium is unloaded, and displace with respect to each other when stresses due to external loadings appear in the medium. They remain stress-free at the same time, as the free surfaces of the medium.

In the frames of the linear theory of elasticity, the crack of arbitrary shape can be treated as the surface of discontinuity of the displacement field. The surface of discontinuity is of course the crack surface S, whereas the discontinuity $\mathbf{U}$ of the displacement field $\mathbf{u}$ depends on the stresses due to external loadings (see [1, 2, 4, 5]).

The condition of discontinuity of the displacement field is written in the form

$$\mathbf{u}^+(\boldsymbol{\zeta})-\mathbf{u}^-(\boldsymbol{\zeta}) \equiv |[\mathbf{u}(\boldsymbol{\zeta})]| = \mathbf{U}(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in S. \tag{2.1}$$

$\mathbf{u}^+$ and $\mathbf{u}^-$ are the limits of the field $\mathbf{u}$ on S^+ and S^- appropriately (Fig. 1).

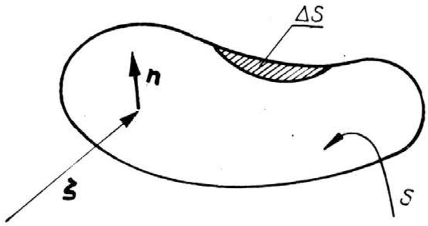

FIG. 1.

The growth of the surface S is the propagation of the crack.

To find the displacement and stress field of the crack, different methods are used (see [7, 8, 9, 10]). Only for the simplest geometries are exact solutions available.

From the theory of defects the method of singular boundary integral equations results.

The displacement and stress fields $\mathbf{u}$ and $\boldsymbol{\sigma}$ in the medium with a crack are represented in the form of a sum of the "external fields" $\mathbf{u}^e$, $\boldsymbol{\sigma}^e$, which would be produced by external loadings in the medium of the same shape but without a crack, and the "self-fields" $\mathbf{u}^s$, $\boldsymbol{\sigma}^s$ due to the crack itself.

The displacement field $\mathbf{u}^s$, due to the crack, is represented by the elastic potential of a double layer of the Lamé equation [1, 2, 4]:

$$u_i^s(\mathbf{x}) = -\int_S ds\, U_n n_b c_{nbrs} G_{ir,s}(\mathbf{x}, \boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in S, \tag{2.2}$$

$\mathbf{G}$ is the Green tensor of the Lamé equation.

The expression (2.2) has the following properties see [1]:

$$|[u_i^s]| = U_i, \tag{2.3}$$

$$n_k|[\sigma_{ik}^s]| = 0 \quad \text{on} \quad S, \quad \sigma_{ik}^s \equiv \sigma_{ik}^s(u_{i,k}^s). \tag{2.4}$$

The displacement discontinuity $\mathbf{U}$ for the crack is to be found from the condition of equilibrium of the crack surface:

$$n_k \sigma_{ik}^s(\mathbf{r}) + n_k \sigma_{ik}^e(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \in S. \tag{2.5}$$

The above condition leads to the strongly singular integral equation for the function $\mathbf{U}$.

The equation of equilibrium of the plane crack, opened by the normal stress σ applied to its surface, has the form

$$-\frac{\mu}{4\pi(1-\nu)}(\nabla_x^2+\nabla_y^2)\int_S ds'\, U(\mathbf{r}') \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sigma, \tag{2.6}$$

$\mathbf{r}, \mathbf{r}' \in S$, $U = 0$ at the boundary of S.

Only for the circular and elliptical crack does the exact solution exist; in other cases the equation has to be solved numerically.

As follows from the known exact solutions, the stress field of the crack in the linearly elastic medium has the singularity of the order $1/\sqrt{r}$ in the surroundings of the crack boundary, whereas the displacement discontinuity function behaves as $\sqrt{r}$ at the boundary of the surface S.

In two dimensions we represent the stress field of the plane crack (stressed in a given mode) at a distance r from the crack tip and at an angle θ with respect to the crack plane (or line) in the form

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta), \tag{2.7}$$

K is called the stress intensity-factor. For a crack of the length a under the constant stress σ, K is proportional to $\sigma\sqrt{a}$. The critical stress-intensity factor K_c, at which fracture occurs, is a material parameter and plays an essential role in fracture analysis.

In three dimensions the analogues of K vary, in general, along the crack boundary.

3. On the calculation of the elastic energy release rate

To be able to examine the energy balance for the fracture phenomenon, one must be able to estimate the elastic energy release rate ΔP corresponding to crack propagation.

The general expression for this quantity, independent of the boundary conditions on the external surface of the body with a crack, has the following form (see RICE in [12]):

$$\Delta P = \frac{1}{2} \int_{\Delta S} ds \mathbf{T} \Delta \mathbf{U}, \qquad \mathbf{T} = \mathbf{n}\boldsymbol{\sigma}, \tag{3.1}$$

where ΔS is the newly-created area of the crack surface S, T is the stress vector on ΔS before it became a part of the crack surface, and $\Delta \mathbf{U}$ is the displacement discontinuity on ΔS after the crack propagation. ΔP thus depends on the asymptotic values of stresses and displacements discontinuities at the crack boundary.

Only for simplest geometries ΔP can be calculated analytically.

For two-dimensional plane cracks under constant stress, the derivatives of P with respect to the crack length l are proportional to the squares of the stress-intensity factors (see [7, 9, 10,—11]). Assuming the crack propagates in its own direction we thus obtain

$$\Delta P \approx \frac{\partial P}{\partial l} \Delta l \sim K^2 \Delta l. \tag{3.2}$$

Also only for two-dimensional problems of semi-infinite plane cracks, the expression (3.1) is equivalent to the so-called Rice integral J, calculated over the arbitrary path around the crack tip.

For numerical calculations, another expression, suggested by the theory of defects, could be suitable.

To derive it, we represent the strain and stress fields as he sums of the "external" and "self" fields:

$$\mathbf{e} = \mathbf{e}^e + \mathbf{e}^s, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^e + \boldsymbol{\sigma}^s. \tag{3.3}$$

The body occupies the region V, the crack faces S^+ and S^- and the surface S^T where the external stresses $\mathbf{T}^e$ are applied belong to its boundary.

The potential energy of the body is equal

$$P = \frac{1}{2}\int\limits_V dv\boldsymbol{\sigma}\mathbf{e} - \int\limits_{S^T} ds\mathbf{T}^e\mathbf{u} = \frac{1}{2}\int\limits_V dv[\boldsymbol{\sigma}^e\mathbf{e}^e + 2\boldsymbol{\sigma}^e\mathbf{e}^s + \boldsymbol{\sigma}^s\mathbf{e}^s] - \int\limits_{S^T} ds\mathbf{T}^e[\mathbf{u}^e + \mathbf{u}^s]. \tag{3.4}$$

After integration by parts, taking into account the discontinuity condition (2.1), we obtain

$$P = P^e + \int\limits_S ds\mathbf{T}^e|[\mathbf{u}^s]| + \int\limits_{S^T} ds\mathbf{T}^e\mathbf{u}^s + \frac{1}{2}\int\limits_S ds\mathbf{T}^s|[\mathbf{u}^s]| - \int\limits_{S^T} ds\mathbf{T}^e\mathbf{u}^s, \tag{3.5}$$

where P^e depends only on the "external fields".

Taking into account that on S $\mathbf{T}^s = -\mathbf{T}^e$, we obtain for P the expression (see [2, 6])

$$P = P^e + \frac{1}{2}\int\limits_S ds\mathbf{T}^e\mathbf{U}. \tag{3.6}$$

For ΔP we obtain the expression:

$$\Delta P = \frac{1}{2}\int\limits_{S+\Delta S} ds\mathbf{T}^e\Delta\mathbf{U}. \tag{3.7}$$

To calculate this expression numerically, one has to find $\mathbf{U}$ first for the crack having the surface S, then for the crack having the surface $S+\Delta S$ and, subsequently, calculate the integral (3.7) over $S+\Delta S$.

We are now going to formulate the fracture criterion in terms of the quantity ΔP.

4. The three-dimensional fracture criterion

Having in mind the classical Griffith reasoning, we assume that in the brittle material a crack can propagate if the elastic energy released upon the crack growth is sufficient to provide all the energy that is required for the crack growth.

We assume that the energy ΔL necessary for the formation of the new surface area ΔS depends only on the magnitude of ΔS, not on its orientation and location on the crack boundary.

The fracture along the given surface ΔS can thus occur if for this surface

$$\Delta(P+L) \leqslant 0. \tag{4.1}$$

Denoting

$$\Delta L = 2\gamma\Delta S, \tag{4.2}$$

we obtain the fracture condition in the form

$$-\frac{\Delta P}{\Delta S} \geqslant 2\gamma. \tag{4.3}$$

The question arises now at which region of the boundary and at which angle the fracture can occur.

It is generally assumed that the fracture first occurs around those points of the boundary where the stress-intensities are the greatest. The fracture angle is indicated by the maximum circumferential normal stress criterion or by the minimum strain-energy-density criterion of G. C. SIH e.g. (see [9, 10, 11, 14]). Here the stresses, their intensities and strain-energy densities refer to the "initial" state of the crack, whereas the quantity ΔP refers to the "subsequent" states of the crack.

The fracture criterion, which is not only the energy criterion, but also gives directions towards fracture geometry, in terms of ΔP can thus be formulated as follows.

Considering the subject in three dimensions, we assume first that the crack propagation proceeds as the formation of the new relatively smooth surface areas, with smooth boundaries. This assumption is motivated by the fact that the stress intensity factor at the curved crack boundary decreases together with the curvature of the boundary (as follows from the known solutions for the circular and elliptical crack).

We then state that for the preferred process of crack propagation, the ratio $\Delta P/\Delta S$ will be comparatively the greatest; however, the necessary condition of crack propagation is Eq. (4.3).

The estimation of the expression $\Delta P/\Delta S$ for the postulated crack growth process in most cases has to be done numerically (see e.g. [13]).

5. The example of the elliptical crack

We consider now as an example the case of an elliptical crack in the field of the uniform normal stress σ.

The existence of the exact analytic solution (see [7, 11]), enables us to avoid numerical calculations. The crack is characterized by two parameters a and b, which are the major and minor semiaxes of the ellipse, constituting its surface.

We are going to compare two kinds of crack propagation processes, preserving the

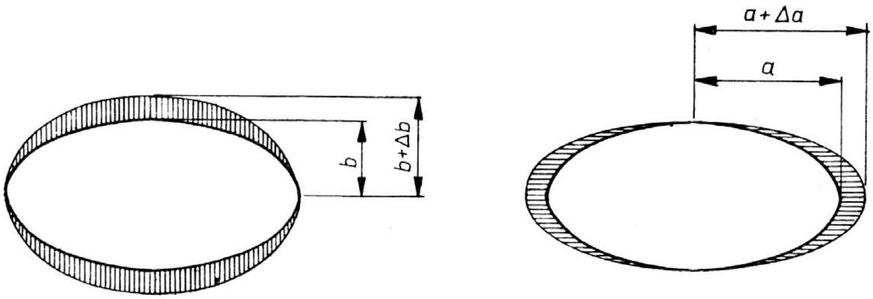

FIG. 2.

elliptical form of the crack — the propagation along the major and minor axis of the ellipse (Fig. 2).

The surface S of the ellipse and its derivatives are equal:

$$S = \pi ab, \tag{5.1}$$

$$\frac{\partial S}{\partial a} = \pi b, \quad \frac{\partial S}{\partial b} = \pi a. \tag{5.2}$$

The displacement discontinuity function U is equal:

$$U = \frac{1-\nu}{\mu} \frac{2b\sigma}{E\left(\sqrt{1-\frac{b^2}{a^2}}\right)} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}, \tag{5.3}$$

where E is the elliptical integral of the second kind. The elastic energy connected with the presence of the crack in the medium, according to Eq. (3.4) is equal:

$$P-P^e = -\frac{1}{2}\int_S ds\,\boldsymbol{\sigma}\mathbf{U} = -\frac{1-\nu}{\mu}\sigma^2 \frac{1}{E\left(\sqrt{1-\frac{b^2}{a^2}}\right)} \frac{2\pi}{3} ab^2. \tag{5.4}$$

Comparing the ratios of the derivatives of the elastic energy P and the surface S with respect to ellipse parameters, we obtain the following expression:

$$\frac{\frac{\partial P}{\partial b}}{\frac{\partial S}{\partial b}} = \alpha \frac{\frac{\partial P}{\partial a}}{\frac{\partial S}{\partial a}}, \tag{5.5}$$

where

$$\alpha = \frac{2-\frac{1}{E}\frac{\partial E}{\partial b}b}{1-\frac{1}{E}\frac{\partial E}{\partial a}a} = \frac{E(k)\,(k^2+1)-K(k)\,(1-k^2)}{E(k)\,(2k^2-1)+K(k)\,(1-k^2)}, \tag{5.6}$$

$k = \sqrt{1-\frac{b^2}{a^2}}$, $K(k)$ is the elliptical integral of the first kind.

The variation of the quantity α with the ratio $\frac{b}{a}$ is illustrated in Fig. 3. Since for all values $\frac{b}{a} < 1$ we have $\alpha > 1$, we come to the conclusion that the elliptical crack has the tendency to propagate in the direction of the minor axis and to reach the shape of a circle (see also [11]).

Analysing the formula giving the values of the stress-intensity factor along the crack boundary, we would come to the same qualitative conclusion; however, we would estimate the tendency of the crack to propagate in the direction of the minor axis as being much greater.

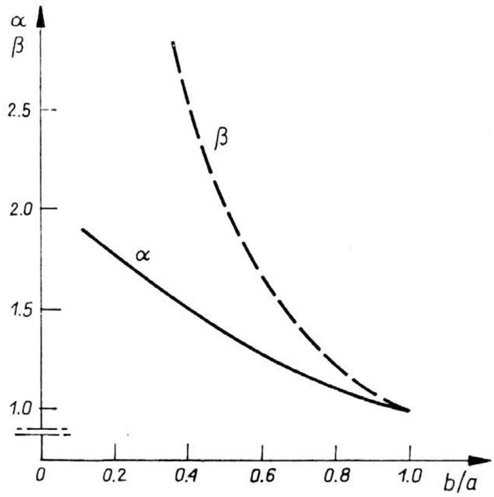

FIG. 3.

The stress-intensity factor given as the function of the polar angle φ, has the form

$$K_I = \frac{\sigma\sqrt{\pi b}}{E}\left[\sin^2\varphi + \frac{b^2}{a^2}\cos^2\varphi\right]^{1/4}. \tag{5.7}$$

The measure of the tendency of fracture in the direction of the minor axis, basing on the stress-intensity factor, could be the quantity

$$\beta = \frac{K_I^2\left(\frac{\pi}{2}\right)}{K_I^2(0)} = \frac{a}{b}. \tag{5.8}$$

We notice that the quantity β is essentially greater than the quantity α, especially for small b/a (see Fig. 3). For $b/a = 0.1$ $\alpha \approx 1.9$, whereas $\beta = 10$. We claim, that the quantity α takes into account better the fracture geometry than the quantity β.

6. Conclusions

The criterion formulated above of three-dimensional fracture takes into account the complete fracture geometry.

The calculation of the energy release rate can be carried out solving the integral equation for the displacement discontinuity function — for the "initial" crack and for the extended crack.

Predictions towards the fracture geometry must point out not only the regions of the crack boundary where the fracture first occurs and the inclination of the newly-created area of the crack surface, but also its shape, taking into account that its boundary should be a comparatively smooth curve.

The simple example of the elliptical crack indicates that predictions towards fracture geometry basing on the idea of the energy release rate, may differ from those basing on the idea of the stress-intensity factor in some aspects.

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