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A GOAL PROGRAMMING MODEL OF THE REFERENCE POINT METHOD

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Abstract: Real-life decision problems are usually so complex they cannot be modeled with a single objective function thus creating a need for clear and efficient techniques of handling multiple criteria to support the decision process. The most widely used technique is Goal Programming (GP). It is clear and appealing but strongly criticized due to its noncompliance with the efficiency (Pareto-optimality) principle. In the paper we show how the GP model with relaxation of some traditional assumptions can be extended to an efficient decision support technique meeting the efficiency principle.

Key words: Multiple Criteria Programming, Goal Programming

1. Introduction

Consider a decision problem defined as an optimization problem with k objective functions. For simplification we assume, without loss of generality, that all the objective functions are to be minimized. The problem can be formulated as follows

minimize $\mathbf{F}(\mathbf{x})$ (1)

subject to $\mathbf{x} \in Q$ (2)

where

$\mathbf{F} = (F_1, \ldots, F_k)$ - k objective functions,

Q feasible set of the problem,

$\mathbf{x}$ vector of decision variables.

Consider further an achievement vector

$\mathbf{q} = \mathbf{F}(\mathbf{x})$

which measures achievement of several decisions $\mathbf{x}$ with respect to the specified set of k objectives $F_1,\ldots,F_k$. Let Y denote the set of all the attainable achievement vectors

$Y = \{ \mathbf{q}=\mathbf{F}(\mathbf{x}) : \mathbf{x} \in Q \}$

i.e., all the vectors $\mathbf{q}$ corresponding to feasible solutions. It is clear that an achievement vector is better than another if all of its individual achievements are better or at least one individual achievement is better whereas no other one is worse. Such a relation is called domination of achievement vectors and it is mathematically formalized as follows (in minimization problems as that under consideration):

if $\mathbf{q}' \neq \mathbf{q}''$ *and* $q_i' \leq q_i''$ *for all* $i=1,\ldots,k$

then $\mathbf{q}'$ *dominates* $\mathbf{q}''$ *and* $\mathbf{q}''$ *is dominated by* $\mathbf{q}'$

Unfortunately, there usually does not exist an achievement vector dominating all others with respect to all the criteria, i.e.,

there does not exist $\mathbf{y} \in Y$ *such that for any* $\mathbf{q} \in Y$ $y_i \leq q_i$ *for all* $i=1,\ldots,k$.

Thus in terms of strict mathematical relations we cannot distinguish the best achievement vector. The nondominated vectors are noncomparable on the basis of the specified set of objective functions.

The feasible solutions (decisions) that generate nondominated achievement vectors are called efficient or Pareto-optimal solutions to the multiobjective problem. That means each feasible decision for which one cannot improve any individual achievement without worsening another one is an efficient decision.

It seems clear that the solution of multiobjective optimization problems should simply depend on identification of the efficient solutions. However, even finite characteristic of the efficient set for a real-life problem is usually so large that it cannot be considered as a solution to the decision problem. So, there arises need for further analysis, or rather decision support, to help the DM select one efficient solution for future implementation. Of course, the original objective functions do not allow one to select any efficient solution as better than any other one. Therefore, this analysis depends on additional information about the DM's preferences.

The Goal Programming (GP) approach, originally proposed by Charnes and Cooper (1961) and further developed by others, requires one to transform objectives into goals by specification of an aspiration level for each objective. An optimal solution is then the one that minimizes the weighted deviations from the aspiration levels. The aspiration levels and weights can be considered as a set of parameters satisfying our implementation requirements.

Goal Programming, unfortunately, does not satisfy the efficiency (Pareto-optimality) principle. Simply, the GP approach does not suggest decisions that optimize the objective functions. It only yields decisions that have the closest outcomes to the specified aspiration levels. This weakness of Goal Programming has led to the development of the quasisatisficing approach. This approach deals with the so-called scalarizing achievement function which, when optimized, generates efficient decisions related to the specified aspiration levels. The best formalization of the quasisatisficing approach to multiobjective optimization was proposed and developed mainly by Wierzbicki (1982) as the reference point method. The reference point method was later extended to allow additional information from the DM and, eventually, led to efficient implementations with successful applications (see Lewandowski and Wierzbicki, 1989).

In this paper we show how the implementation techniques of Goal Programming can be used to model the reference point approach. Thereby we also show how Goal Programming with relaxation of some traditional assumptions can be extended to an efficient decision support technique meeting the efficiency principle and other standards of multiobjective optimization theory.

2. GP Model of the Reference Point Method

The reference point method is an interactive technique. The basic idea of the interactive scheme is as follows. The decision maker (DM) specifies requirements, as in GP, in terms of aspiration levels. Depending on the specified aspiration levels a special scalarizing achievement function is built which, while being minimized, generates an efficient solution to the problem. The computed efficient solution is presented to the DM as the current solution in a form that allows comparison with the previous ones and modification of the aspiration levels if necessary.

The scalarizing achievement function, obviously, not only guarantees efficiency of the solution but also reflects the DM's expectation specified via the aspiration levels. While building the function the following assumption regarding the DM's expectations is made:

A1. The DM prefers outcomes that satisfy all the aspiration levels to any outcome that does not.

One of the simplest scalarizing functions takes the following form (compare Steuer, 1986):

$$\max_{1 \le i \le k} \{s_i (F_i(\mathbf{x}) - a_i)\} + \varepsilon \sum_{i=1}^{k} s_i (F_i(\mathbf{x}) - a_i) \tag{3}$$

where: a_i denote aspiration levels, $s_i > 0$ are scaling factors, ε is an arbitrarily small positive number.

Minimization of the scalarizing achievement function (3) over the feasible set Q generates an efficient solution. The selection of the solution within the efficient set depends on two vector parameters: an aspiration vector $\mathbf{a}$ and scaling vector $\mathbf{s}$. In practical implementations the former is usually designated as a control tool to be used by the DM whereas the latter is automatically calculated from a predecision analysis (compare Grauer et al., 1984). Small scalar ε is introduced only to guarantee efficiency in case of a nonunique optimal solution.

The reference point method, though using the same main control parameters (aspiration levels), always generates an efficient solution to the multiobjective problem whereas GP does not. We will show, however, that the reference point method can be modeled via the GP methodology. Function (3) is built as a sum of the weighted Chebychev norm of the differences between individual achievements and the corresponding aspiration levels and a small regularization term (the sum of the differences). Usage of the Chebychev norm is important to generate efficient solutions for nonconvex problems (e.g., discrete ones) and it must always be accompanied by some regularization term.

Let us concentrate on the main term. The Chebychev norm is available in GP modeling as Fuzzy Goal Programming. The differences $F_i(\mathbf{x}) - a_i$ can be easily expressed in terms of goal deviations n_i and p_i defined according to the equations

$$F_i(\mathbf{x}) + n_i - p_i = a_i \quad \text{for} \quad i=1,\ldots,k$$

$$n_i \geq 0, \; p_i \geq 0 \; \text{ and } \; n_i p_i = 0$$

Thus nothing prohibits modeling the main term of the scalarizing achievement function via the GP methodology. We can form an equivalent GP achievement function:

$$g_1(\mathbf{n},\mathbf{p}) = \max_{1 \leq i \leq k} (-v_i n_i + w_i p_i) \tag{4}$$

where weights v_i and w_i associated with several goal deviations replace the scaling factors used in the scalarizing achievement function, e.g., for an exact model of the function (3) one needs to put $v_i = w_i = s_i$. However, there is one specificity in the function (4). Namely, there is a negative weight $-v_i$ associated with the negative deviation n_i. This is the reason why the reference point method attempts to reach an efficient solution even if the aspiration levels are attainable. This small change of the coefficient represents, however, a crucial change in the GP philosophy, where all the weights are assumed to be nonnegative. If we accept negative weights we can consider the function (4) as a specific case of GP achievement functions.

Adding a regularization term to the function (4) can destroy its GP form. However, using lexicographic optimization we can avoid the problem of choosing an arbitrarily small positive parameter ε (compare (3)) and introduce the regularization term as an additional priority level:

$$g_2(\mathbf{n},\mathbf{p}) = \sum_{i=1}^{k} (-v_i n_i + w_i p_i)$$

Finally, we can form the following lexicographic GP problem:

P1: *lexmin* $\mathbf{g}(\mathbf{n},\mathbf{p}) = [\, g_1(\mathbf{n},\mathbf{p}), g_2(\mathbf{n},\mathbf{p}) \,]$

subject to

$F_i(\mathbf{x}) + n_i - p_i = a_i \quad \text{for } i=1,\ldots,k$

$n_i \geq 0, \; p_i \geq 0 \; \textit{and} \; n_i p_i = 0$

$\mathbf{x} \in Q$

The above lexicographic GP problem generates always an efficient solution to the original multiobjective problem (Proposition 1) satisfying simultaneously rules of the reference point approach, i.e., assumption **A1** (Proposition 2).

Proposition 1.

For any aspiration levels a_i and any positive weights v_i and w_i if $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ is an optimal solution to the problem **P1** then $\bar{\mathbf{x}}$ is an efficient solution to the multiobjective problem (1)-(2).

Proof

Let $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ be an optimal solution to the problem **P1**. Suppose that $\bar{\mathbf{x}}$ is not efficient to the problem (1)-(2). That means there exist a vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leq F_i(\bar{\mathbf{x}}) \quad \text{for all } i=1,\ldots,k \tag{5}$$

and for some index j $(1 \leq j \leq k)$

$$F_j(\mathbf{x}) < F_j(\bar{\mathbf{x}})$$

or in other words

$$\sum_{i=1}^{k} F_i(\mathbf{x}) < \sum_{i=1}^{k} F_i(\bar{\mathbf{x}}) \tag{6}$$

The deviations $\bar{n}_i$ and $\bar{p}_i$ satisfy the following relations:

$\bar{p}_i = (\, F_i(\bar{\mathbf{x}}) - a_i \,)_+$

$\bar{n}_i = (\, a_i - F_i(\bar{\mathbf{x}}))_+$

where $(.)_+$ denotes the nonnegative part of a quantity.

Let us define similar deviations for the vector .

$p_i = (\, F_i(\mathbf{x}) - a_i \,)_+ \quad \text{for } i=1,\ldots,k$

$n_i = (\, a_i - F_i(\mathbf{x}))_+ \quad \text{for } i=1,\ldots,k$

$(\mathbf{x},\mathbf{n},\mathbf{p})$ is a feasible solution to the problem **P1** and due to (5) and (6) for any positive weights v_i and w_i the following inequalities are satisfied:

$$w_i p_i \leq w_i \bar{p}_i \quad \text{for } i=1,\ldots,k$$

$$-v_i n_i \leq -v_i \bar{n}_i \quad \text{for } i=1,\ldots,k$$

and

$$\sum_{i=1}^{k} (-v_i n_i + w_i p_i) < \sum_{i=1}^{k} (-v_i \bar{n}_i + w_i \bar{p}_i)$$

Hence we get

$$g_1(\mathbf{n},\mathbf{p}) \leq g_1(\bar{\mathbf{n}},\bar{\mathbf{p}}) \text{ and } g_2(\mathbf{n},\mathbf{p}) < g_2(\bar{\mathbf{n}},\bar{\mathbf{p}})$$

which contradicts optimality of $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ for the problem **P1**. Thus $\bar{\mathbf{x}}$ must be an efficient solution to the original multiobjective problem (1)-(2). ■

Proposition 2.

For any aspiration levels a_i and any positive weights v_i and w_i if $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ is an optimal solution to the problem **P1** then any deviation $\bar{p}_i$ is positive only if there does not exist any vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leq a_i \quad \text{for all } i=1,\ldots,k$$

Proof

Let $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ be an optimal solution to the problem **P1**. Suppose that for some j

$$\bar{p}_j > 0, \text{ i.e., } F_j(\bar{\mathbf{x}}) > a_j$$

and there exist a vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leq a_i \quad \text{for all } i=1,\ldots,k$$

Let us define deviations for the vector $\mathbf{x}$

$$p_i = (F_i(\mathbf{x}) - a_i)_+ = 0 \quad \text{for all } i=1,\ldots,k$$

$$n_i = (a_i - F_i(\mathbf{x}))_+ \geq 0 \quad \text{for all } i=1,\ldots,k$$

$(\mathbf{x},\mathbf{n},\mathbf{p})$ is a feasible solution to the problem **P1** and for any positive weights v_i and w_i the following inequality is satisfied:

$$\max_{1\leq i\leq k} (-v_i n_i + w_i p_i) \leq 0 < w_j \bar{p}_j \leq \max_{1\leq i\leq k} (-v_i \bar{n}_i + w_i \bar{p}_i)$$

Hence

$$g_1(\mathbf{n},\mathbf{p}) < g_1(\bar{\mathbf{n}},\bar{\mathbf{p}})$$

which contradicts optimality of $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ for the problem **P1**. Thus there does not exist any vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \le a_i \quad \text{for all } i=1,\dots,k$$

and thereby the assumption **A1** is satisfied. ■

Note that neither proposition assumes any specific relation between weights. It is not necessary because we directly put into the problem **P1** the requirements

$$n_i p_i = 0 \quad \text{for } i=1,\dots,k \tag{7}$$

to guarantee proper calculation of all the deviations. It turns out, however, that requirements (7) can be simply omitted in the constraints of the problem **P1** provided that the weights satisfy some relations natural for the reference point philosophy. This is made precise in Proposition 3.

Proposition 3.

For any aspiration levels a_i, if the weights satisfy relations

$$0 < v_i < w_i \quad \text{for } i=1,\dots,k \tag{8}$$

then any $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ optimal solution to the problem **P1** with omitted constraints (7) satisfy these requirements, i.e.,

$$\bar{n}_i \bar{p}_i = 0 \quad \text{for } i=1,\dots,k$$

Proof

Let **P1′** denote the problem **P1** with omitted constraints (7) and let $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ be an optimal solution to **P1′**. Suppose that for some j

$$\bar{n}_j \bar{p}_j > 0$$

Then we can decrease both $\bar{n}_j$ and $\bar{p}_j$ by the same small positive quantity. That means, for small enough positive δ the vector $(\bar{\mathbf{x}},\bar{\mathbf{n}}-\delta\mathbf{e}_j,\bar{\mathbf{p}}-\delta\mathbf{e}_j)$ is feasible to the problem **P1′**. Due to (8) the following inequality is valid

$$-v_j(\bar{n}_j-\delta) + w_j(\bar{p}_j-\delta) < -v_j\bar{n}_j + w_j\bar{p}_j$$

Hence we get

$$g_1(\bar{\mathbf{x}},\bar{\mathbf{n}}-\delta\mathbf{e}_j,\bar{\mathbf{p}}-\delta\mathbf{e}_j) \le g_1(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$$

$$g_2(\bar{\mathbf{x}},\bar{\mathbf{n}}-\delta\mathbf{e}_j,\bar{\mathbf{p}}-\delta\mathbf{e}_j) < g_2(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$$

which contradicts optimality of $(\bar{\mathbf{x}},\bar{\mathbf{n}},\bar{\mathbf{p}})$ for the problem **P1′**. Thus $\bar{n}_i \bar{p}_i = 0$ for $i=1,\dots,k$

■

3. Conclusions

In the paper we have shown that the implementation techniques of Goal Programming can be used to model the reference point method. Namely, we have shown that employing the lexicographic and Fuzzy GP with properly defined weights we receive a GP achievement function that satisfies all the requirements for the scalarizing achievement function used in the reference point approach. The properly defined weights mean, among others, usage of some negative weights. This is the reason why the scalarizing achievement function attempts to reach an efficient solution even if the aspiration levels are attainable. This small technical change represents, however, a crucial change in the GP philosophy, where all the weights are assumed to be nonnegative. We do not want to debate whether Goal Programming with negative weights is still Goal Programming or not. Instead of dealing with that scholastic problem we are interested in practical advantages of the relations proved in the paper.

From our point of view the most important is the possibility of using efficient GP implementation techniques to model the reference point approach. It allows one to simplify and demystify implementations of the reference point method and thereby extend applications of this powerful method. Moreover, it provides an opportunity to build unique decision support systems based on both approaches: GP and reference point.

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