SUPPORT SYSTEMS FOR DECISION AND NEGOTIATION PROCESSES
Preprints, IFAC/IFORS/IIASA/TIMS Workshop, June 24-26, 1992, Warsaw, Poland

A NEW APPROACH TO SOLVING MULTICRITERIA NONCONVEX OPTIMIZATION PROBLEMS

Matthew L. TenHuisen
Malgorzata M. Wiecek

Department of Mathematical Sciences
Clemson University
Clemson, SC 29634

Abstract: We develop a new approach to generate efficient solutions of the mulitcriteria optimization problem. The approach, specially designed for nonconvex problems, relates the generalized Lagrangian duality theory with multicriteria optimization. Theoretical results illustrated by an example are presented.

Keywords: multicriteria nonconvex optimization, generalized Lagrangian duality

1. Introduction

Many methods for generating and examining efficient solutions of mulitcriteria optimization problems have been developed over the last two decades. Some of the methods are designed specifically for linear problems and still others work well only on problems which have convex objective functions and a convex feasible region. We are interested in consistently solving multicriteria optimization problems which are not requried to satisfy any convexity assumptions.

The weighting method (Geoffrion, 1968), perhaps the most commonly applied, identifies an efficient solution by utilizing a supporting hyperplane in the objective space. It can be easily shown that this method may fail to generate efficient solutions of a nonconvex problem. The ε-constraint method developed by Haimes (Haimes and Chankong, 1983) is more capable of generating efficient solutions of nonconvex problems than the weighting method. This method may, however, be unable to generate every efficient solution if the efficient set is disconnected, a property naturally resulting from nonconvexity. The placement of the goal and weights given by the decision maker in the goal-attainment method (Gembicki, 1973) can affect the method's ability to generate efficient solutions. Similarly, in the Tchebyshev norm method (Zeleny, 1973) and Benson's method (Benson, 1978), the location of the reference point is crucial. While applying these methods to a nonconvex problem, many reference points may generate the same efficient solution. Kostreva et al. (1991) developed a method for problems with polynomial objective and constraint functions which generates both locally and globally efficient solutions. Bernau (1990) applied exact penalty functions to determine efficient solutions of nonlinear multicriteria optimization problems. Khanh (1991) used penalty functions to formulate a dual problem for a multicriteria nonconvex problem and obtained results relating duality to efficiency.

The field of engineering provides examples of real life applications of noncovex multicriteria optimization problems. Tabek et al. (1979) model aircraft control systems design with such problems. Osyczka (1984) uses a nonconvex muticriteria model in the area of machine gear design.

In this paper we introduce a new approach to solving multicriteria nonconvex optimization problems. The approach applies generalized Lagrangian duality theory developed by Roode (1968), Nakayama et al. (1975), and others.

In the next section we formulate the multicriteria optimization problem and show how to relate the generalized Lagrangian dual problem to it. In section 3 we present new theoretical foundations for generating weakly efficient solutions. Section 4 includes an example, and the paper is concluded in section 5.

2. Problem Formulation

Consider the multicriteria optimization problem (MOP) formulated as

$$\text{MOP:} \qquad \min \quad \{f_1(x), f_2(x), \ldots, f_m(x)\}$$
$$\text{s.t.} \quad x \in X$$

where each $f_j(x)$, $j = 1, 2, \ldots, m$ is a real-valued function defined on $X \subseteq R^n$. A point $x^0 \in X$ is called an efficient solution of MOP if there is no other point $x \in X$ such that $f_i(x) \le f_i(x^0)$, $i = 1, 2, \ldots, m$, with strict inequality holding for at least one i. A point $x^0 \in X$ is called a weakly efficient solution of MOP if there is no other point $x \in X$ such that $f_i(x) < f_i(x^0)$, $i = 1, 2, \ldots, m$. The image of any (weakly) efficient solution under the vector-valued mapping $(f_1(x), f_2(x), \ldots, f_m(x))$ is called a (weakly) nondominated solution.

Corresponding to MOP is the following ε-constraint problem (Haimes and Chankong, 1983)

$$P_k(\varepsilon): \qquad \min \quad f_k(x)$$
$$\text{s.t.} \quad f_j(x) \le \varepsilon_j \qquad j = 1, 2, \ldots, m; \quad j \ne k$$
$$x \in X$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1}, \ldots, \varepsilon_m)$. Let the function $f_j(x) - \varepsilon_j$ be denoted by $g_j(x)$ and $g(x) := (g_1(x), \ldots, g_{k-1}(x), g_{k+1}(x), \ldots, g_m(x))$. Let $w(y) := \inf\{f_k(x) : g(x) \le y, x \in X\}$ and $B := \{y : \text{there exists an } x \in X \text{ such that } g(x) \le y\}$. Then the epigraph of $w(y)$ is defined as $\text{epi } w(y) := \{(y, z) : z \ge w(y), y \in B\}$.

The generalized Lagrangian function (Nakayama et al., 1975) corresponding to $P_k(\varepsilon)$ is

$$\lambda) = f_k(x) + G[g(x), \lambda]$$

where $\lambda \in R^{m-1}$, $\lambda \geq 0$, and $G[g(x), \lambda]$ is a mapping from $R^n \times R^{m-1}$ to R^1, which is finite on X for each $\lambda \geq 0$ and satisfies the following properties:

i.) $G[0, \lambda] = 0$ for all $\lambda \geq 0$, (1)

ii.) $G[g(x), 0] = 0$ for all $x \in X$ such that $g(x) \leq 0$, (2)

iii.) if $g(x) > 0$, there exists a sequence $\{\lambda^k\}$, $\|\lambda^k\| \to \infty$, such that $\lim_{\lambda \to \infty} \sup G[g(x), \lambda] = \infty$, (3)

iv.) $G[g(x), \lambda]$ is nondecreasing with respect to $g(x)$. (4)

The generalized Lagrangian dual (GLD) problem is given as

$$\text{GLD:} \quad \max \; h(\lambda) \quad \text{s.t.} \; \lambda \in \Lambda$$

where $h(\lambda) = \inf_{x \in X} \mathcal{L}(x, \lambda)$, and $\Lambda = \{\lambda : \lambda \geq 0, \lambda \in R^{m-1}\}$. Associated with the generalized Lagrangian function $\mathcal{L}(x, \lambda)$ is the hypersurface of the form $z + G[y, \lambda] = b(\hat{y}, \hat{z})$, where $b(\hat{y}, \hat{z})$ is the z-intercept of the hypersurface which passes through the point $(\hat{y}, \hat{z})$. Directly related to GLD problem is the following lemma (Nakayama et al, 1975).

Lemma 1: $\hat{x}$ minimizes $\mathcal{L}(x, \lambda)$ on X if and only if the hypersurface of the form $z + G[y, \lambda] = b(g(\hat{x}), f_k(\hat{x}))$ supports epi $w(y)$ at $(g(\hat{x}), f_k(\hat{x}))$.

This lemma relates the generalized Lagrangian function to a supporting hypersurface of the feasible set in the image space.

3. Applying Generalized Lagrangian Duality

In this section we show how to generate weakly efficient solutions of MOP by applying generalized Lagrangian duality theory.

Theorem 1: $\hat{x}$ minimizes $\mathcal{L}(x, \lambda)$ if and only if $\hat{x}$ is a weakly efficient solution of MOP.

Proof: ($\Rightarrow$) If $\hat{x}$ minimizes $\mathcal{L}(x, \lambda)$, then by Lemma 1 there exists a hypersurface of the form $z + G[y, \lambda] = b(\hat{y}, \hat{z})$ which supports epi $w(y)$ at $(\hat{y}, \hat{z}) := (g(\hat{x}), f_k(\hat{x}))$.

If there does not exist a point $\bar{x} \in X$ such that $g(\bar{x}) < g(\hat{x})$, then $\hat{x}$ is weakly efficient.

Now, assume that there exists an $\bar{x} \in X$ such that

$$\bar{y} := g(\bar{x}) < \hat{y} := g(\hat{x}). \tag{5}$$

Since $\bar{x} \in X$, $(\bar{y}, \bar{z}) := (g(\bar{x}), f_k(\bar{x})) \in Y := \{(g(x), f_k(x)) : x \in X\} \subseteq$ epi $w(y)$. Let $(\bar{y}, \tilde{z})$ be a point in the hypersurface $z + G[y, \lambda] = b(\hat{y}, \hat{z})$. Since $z + G[y, \lambda] = b(\hat{y}, \hat{z})$ supports epi $w(y)$,

$$\bar{z} \geq \tilde{z}. \tag{6}$$

From (5) and the property of $G[y, \lambda]$ being nondecreasing with respect to y, we get

$$\tilde{z} = -G[\tilde{y}, \lambda] + b(\hat{y}, \hat{z}) \geq \hat{z} = -G[\hat{y}, \lambda] + b(\hat{y}, \hat{z}). \tag{7}$$

From (5), (6), and (7) there is $\tilde{z} \geq \hat{z}$ if $g(\tilde{x}) < g(\hat{x})$. Hence $f_k(\tilde{x}) \geq f_k(\hat{x})$ if $g(\tilde{x}) < g(\hat{x})$. Therefore, $(\hat{y}, \hat{z})$ is weakly nondominated, and $\hat{x}$ is weakly efficient.

($\Leftarrow$) Since $\hat{x} \in X$ is weakly efficient, $(\hat{y}, \hat{z}) := (g(\hat{x}), f_k(\hat{x}))$ is weakly nondominated. Therefore, if there exists an $\tilde{x} \in X$ such that $f_k(\tilde{x}) < f_k(\hat{x})$ then $g(\tilde{x}) \geq g(\hat{x})$. Similarly, if there exists an $\tilde{x} \in X$ such that $g(\tilde{x}) < g(\hat{x})$ then $f_k(\tilde{x}) \geq f_k(\hat{x})$. Hence, the hypersurface defined by

$$z = \begin{cases} f_k(\hat{x}) & \text{if } g(x) \leq g(\hat{x}) \\ -\infty & \text{if } g(x) > g(\hat{x}) \end{cases} \tag{8}$$

supports Y at $(\hat{y}, \hat{z})$. Using $z = -G[y, \lambda] + b(\hat{y}, \hat{z})$, $\lambda \geq 0$, we get $z = -G[g(x), \lambda] + b(g(\hat{x}), f_k(\hat{x}))$, where $b(g(\hat{x}), f_k(\hat{x})) = f_k(\hat{x})$ and

$$G[g(x), \lambda] = \begin{cases} 0 & \text{if } g(x) \leq g(\hat{x}) \\ +\infty & \text{if } g(x) > g(\hat{x}) \end{cases} \tag{9}$$

Note that $G[g(\hat{x}), \lambda]$ is nondecreasing with respect to g(x) and satisfies properties (1) - (4). Therefore, the hypersurface of the form $z + G[y, \lambda] = b(\hat{y}, \hat{z})$ given by (8) and (9) supports epi w(y) at $(\hat{y}, \hat{z})$. From Lemma 1, then, $\hat{x}$ minimizes $\mathcal{L}(x, \lambda)$. ■

Having generated a weakly nondominated solution as above, any duality gap associated with it, which would be encountered with classical Lagrangian duality, is resolved with generalized Lagrangian duality theory.

Theorem 2: If $\hat{x}$ is a weakly efficient solution to MOP and $\hat{\lambda}$ is a solution of GLD, then $f_k(\hat{x}) = h(\hat{\lambda})$ for some $k = 1, 2, ..., m$.

Proof: Since $\hat{\lambda}$ solves GLD, $h(\hat{\lambda}) \geq h(\lambda)$ for all $\lambda \geq 0$. From the definitions of $h(\lambda)$ and $\mathcal{L}(x, \lambda)$, $h(\lambda) := \inf_{x \in X} \{\mathcal{L}(x, \lambda)\} \geq \mathcal{L}(\hat{x}, \lambda) := f_k(\hat{x}) + G[g(\hat{x}), \lambda]$ where the inequality follows from Theorem 1 and the fact that $\hat{x}$ is a weakly efficient solution of MOP. Therefore, $h(\hat{\lambda}) \geq f_k(\hat{x}) + G[g(\hat{x}), \lambda]$ for all $\lambda \geq 0$ with $g(\hat{x}) \leq 0$. In particular, $h(\hat{\lambda}) \geq f_k(\hat{x}) + G[g(\hat{x}), 0]$. From (2), $G[g(\hat{x}), 0] = 0$, and therefore

$$h(\hat{\lambda}) \geq f_k(\hat{x}). \tag{10}$$

On the other hand, $h(\hat{\lambda}) := \inf_{x \in X} \{\mathcal{L}(x, \hat{\lambda})\} := \inf_{x \in X} \{f_k(x) + G[g(x), \hat{\lambda}]\} \leq$ $\inf_{x \in X} \{f_k(x) + G[g(x), \hat{\lambda}] : g(x) \leq 0\} \leq \inf_{x \in X} \{f_k(x) : g(x) \leq 0\} := f_k(\hat{x})$, where the last inequality follows from the property that $G[g(x), \hat{\lambda}] \le 0$ for $g(x) \le 0$ because of properties (1) and (4). Therefore,

$$h(\hat{\lambda}) \le f_k(\hat{x}). \tag{11}$$

Inequalities (10) and (11) imply that $h(\hat{\lambda}) = f_k(\hat{x})$. ■

Theorems 1 and 2 guarantee that all weakly efficient solutions, particularly those in the duality gap, can be generated using the generalized Lagrangian duality approach.

4. Example

We now apply the results of the previous section to solving the following bi-criteria nonconvex optimization problem.

$$\text{MOP:} \quad \min \ \{ \sqrt[3]{x+1}, \ 1 - x \} \quad \text{s.t.} \quad 0 \le x \le 7.$$

The ε-constraint problem is formulated as

$$P_1(\varepsilon): \quad \min \ \sqrt[3]{x+1} \quad \text{s.t.} \quad 1 - x \le \varepsilon, \quad 0 \le x \le 7.$$

$P_1(\varepsilon)$ is feasible only for $\varepsilon \ge -6$. In such a case,

$$w(y) = \begin{cases} \sqrt[3]{2-y} & \text{if } y \le 1 \\ 1 & \text{if } y > 1 \end{cases}.$$

A duality gap exists for any value of ε such that $-6 \le \varepsilon \le 1$. For simplicity, assume $\varepsilon = 0$. The solution of $P_1(0)$ is $x^* = 1$, and the objective value is $\sqrt[3]{2}$. The optimal value of the classical Lagrangian dual problem is $\frac{8}{7}$. Clearly $\sqrt[3]{2} > \frac{8}{7}$, and a duality gap exists for this problem with $\varepsilon = 0$.

Now consider the function

$$G[(1-x), \lambda] = \begin{cases} \lambda(1-x)^2 + \lambda(1-x) & \text{if } (1-x) > 0 \\ 0 & \text{if } (1-x) \le 0 \end{cases}.$$

The corresponding generalized Lagrangian function is

$$\mathcal{L}(x, \lambda) = \sqrt[3]{x+1} + \begin{cases} \lambda(1-x)^2 + \lambda(1-x) & \text{if } x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}.$$

When $\lambda^* = 1/(3\sqrt[3]{4})$, the optimal value of the generalized Lagrangian dual problem is $\sqrt[3]{2}$ which is equal to the value of $P_1(0)$ at $x^* = 1$. The duality gap is resolved, and the weakly efficient solution

$x^* = 1$ is generated. In addition, the function

$$z = \begin{cases} -G[y, \lambda] + w(\varepsilon) = -\frac{1}{3\sqrt[3]{4}}y^2 - \frac{1}{3\sqrt[3]{4}}y + \sqrt[3]{2} & \text{if } y > 0 \\ w(\varepsilon)) & \text{if } y \leq 0 \end{cases}$$

is a supporting hypersurface of epi $w(y)$ at $(0, w(0)) = (0, \sqrt[3]{2})$.

5. Conclusions

This short paper presents for the first time a theoretical development for applying the generalized Lagrangian dual problem to solving a general class of nonlinear, possibly nonconvex, multicriteria optimization problems. The relationship between weak efficiency and generalized Lagrangian duality is shown. The illustrative example demonstrates how to generate an efficient solution located in the duality gap.

The authors intend to continue further study in this area in order to develop new algorithmic procedures for generating efficient solutions.

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