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A Markov model of a complex technical system with multiple operation modes

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1. Introduction

In this paper, a complex system built of independent repairable components with constant failure and repair rates is considered. The system can operate in a finite number of operation modes, and its functioning is modeled by a Markov chain on the state space {0, 1,..., z}. The individual modes are represented by the states {1,...,z}, the failed system is represented by the state 0. The paper's aim is to present analytical formulas for the system's key reliability parameters, derived by the author.

As an example, let us take a small power supply network (microgrid) whose reliability block diagram (RBD) is displayed in Fig. 1. The boxes denoted $e_1,...,e_8$ represent the network's components listed below.

e_1 – distribution company's network

e_2 – renewable source connected to e_5

e_3 – low voltage bus bar

e_4 – low voltage cut-off switch + low voltage cable line + low voltage cut-off switch

e_5 – load point (LP)

e_6 – transfer switch + low voltage cut-off switch

e_7 – low voltage cut-off switch + low voltage cable line + low voltage cut-off

e_8 – load point

Remark: + denotes the serial connection between elements of e_4, e_6, or e_7. The failure and repair rates of e_4, e_6, and e_7 can be found from (5) and (6), where S={0,1}, $0 <_d 1$.

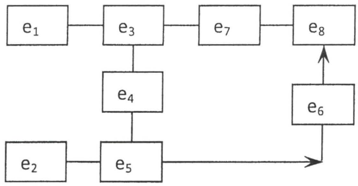

Fig. 1. RBD of a small power supply network. Arrows indicate the end fragment of an emergency supply path to e_8.

The exemplary microgrid is supplied from two power sources – the network of a distribution company, and a renewable energy source (e.g. a wind turbine). It also has two load points which supply power to the connected customers. Let us shortly analyze the network's functioning, with LP e_8 chosen as the point of reference. Two modes of operation are distinguished – normal and emergency ones. If all elements along the path (e_1, e_3, e_7, e_8) or (e_2, e_5, e_4, e_3, e_7, e_8) are operable, then e_8 is supplied in normal mode. When this is no longer possible, because, for example, e_3 or e_7 is failed, the supply of e_8 is switched to emergency mode by the transfer switch in e_6, provided that all elements along the path (e_2, e_5, e_6, e_8) or (e_1, e_3, e_4, e_5, e_6, e_8) are operable. The supply of e_8 is switched back into normal mode when this mode is restored. Obviously, if both normal and emergency modes are failed, then a power outage occurs at e_8. The functioning of e_8 can thus be modeled by a three-state stochastic process on the state space {0, 1, 2}.

We now give an outline of the paper. The paper is composed of five main sections numbered 3 through 6 and the 7-th concluding section. In section 3 the Markov property of the random process modeling the considered system's behavior is proved. The formulas for computing the system's key reliability parameters, i.e. the state probabilities and the inter-state transition intensities of the modeling process, are derived in sections 5 and 6 for which the theoretical background is presented in sections 3 and 4. The considerations of section 6 are based on the above given example, but can easily be extended to the general case. In section 7 it is shown how other parameters, characterizing the dynamics of switching between basic and emergency modes, can be obtained.

Readers interested in the reliability of power distribution systems (such a system serves as an example here) are referred to [2], while those needing an insight into the general theory of reliability – to [1] and [6].

2. Notation

The following notation will be used throughout the paper:

$\{e_i, 1 \leq i \leq n\}$ – the set of the system's components

$J = \{1,\ldots,n\}$ – the set of the components' indices

λ_i, μ_i – the failure and repair intensities of the component e_i

x_i – a binary variable representing the state of e_i; $x_i=1/x_i=0$ if e_i is operable/failed

x – the components' states vector, $x = [x_1,\ldots,x_n]$

$\{0,1\}^n$ – the set of binary vectors of length n

$[x, 1_i]$, $[x, 0_i]$ – the vector x with i-th coordinate set to 1 or 0 respectively

$d(x, y)$ – the number of coordinates in which the vectors x and y differ (the Hamming distance)

$X_i(t)$ – the state of the component e_i at time t

$p_i(t)$, $q_i(t)$ – the state probabilities of $X_i(t)$, i.e. $p_i(t) = Pr[X_i(t)=1]$, $q_i(t) = Pr[X_i(t)=0]$

$X(t)$ – the components' states vector at time t, i.e. $X(t) = [X_1(t),\ldots,X_n(t)]$

$\Phi(x)$ – the system's structure function expressing the system's state in relation to the components' states

S – the set of the system's states with the partial order transferred by Φ from the partial order in $\{0,1\}^n$, i.e.

$$(x, y \in \{0,1\}^n) \wedge (x < y) \Rightarrow \Phi(x) \leq \Phi(y) \tag{1}$$

where < and $\leq$ denote the strong and weak precedence relations in both $\{0,1\}^n$ and S. We adopt the usual partial order in $\{0,1\}^n$, i.e. x<y for $x,y \in \{0,1\}^n$ if d(x,y)>0 and $y_i - x_i = 1$ for each $x_i \neq y_i$.

$<_d$ – the direct precedence relation in both $\{0,1\}^n$ and S

Z(t) – the system's state at time t, i.e. Z(t) = Φ(X(t))

Z – the stochastic process (Z_t, t≥0), where Z_t = Z(t)

$P_a(t)$ – the state probability of Z at time t, i.e. $P_a(t)$ = Pr[Z(t)=a], $a \in S$

$\Lambda_{a \to b}(t)$ – the intensity with which Z changes its state from a to b at time t (a transition intensity), defined as follows:

$$\Lambda_{a \to b}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr[Z(t + \Delta t) = b \mid Z(t) = a] \tag{2}$$

$\Pi_{a \to b}^{crit}(i)$ – the set of binary vectors x such that x_i=1, Φ(x)=a, and $\Phi(x, 0_i)$=b

$\Theta_{a \to b}^{crit}(i)$ – the set of binary vectors x such that x_i=0, Φ(x)=a, and $\Phi(x, 1_i)$=b

$I_{a \leftrightarrow b}(i)$ – the importance of e_i to a transition between a and b, defined as follows:

$$I_{a \leftrightarrow b}(i) = \Pr\left[X \in \Pi_{a \to b}^{crit}(i) \mid X_i = 1\right] = \Pr\left[X \in \Theta_{b \to a}^{crit}(i) \mid X_i = 0\right] \tag{3}$$

i.e. $I_{a \leftrightarrow b}(i)$ is the probability that the failure/repair of e_i causes a transition from a to b, given that e_i is operable/failed.

$\vee$ – the "Boolean" sum of real numbers from the [0, 1] interval, defined as $r_1 \vee r_2 = r_1 + (1 - r_1)r_2$, where $r_1, r_2 \in [0, 1]$

Remark 1: It was shown in [1] that

$$p_i(t) = \frac{\mu_i}{\lambda_i+\mu_i} + \frac{\lambda_i}{\lambda_i+\mu_i}\exp[-(\lambda_i+\mu_i)t]\,, \quad q_i(t) = \frac{\lambda_i}{\lambda_i+\mu_i} - \frac{\lambda_i}{\lambda_i+\mu_i}\exp[-(\lambda_i+\mu_i)t] \tag{4}$$

Remark 2: If $x \in \Pi_{a\to b}^{crit}(i)$ then we say that x is critical to the system's transition from state a to b caused by e_i's failure. If, in turn, $x \in \Theta_{a\to b}^{crit}(i)$ then we say that x is critical to the system's transition from state a to b caused by e_i's renewal. These notions of criticality are generalizations of a path-vectors' or a cut-vectors' criticality for a two-state system. Clearly, if $x \in \Pi_{a\to b}^{crit}(i)$, then $[x, 0_i] \in \Theta_{b\to a}^{crit}(i)$.

Remark 3: $I_{a\leftrightarrow b}(i)$ is a generalization of the Birnbaum importance for a two-state system

Remark 4: $x <_d y$ if and only if $d(x,y)=1$ and $y_i - x_i = 1$ for $x_i \neq y_i$.

3. The theoretical background

We begin this section with three lemmas of auxiliary character that will be used to prove the main result – a theorem giving the formulas for the transition intensities of the process Z, i.e. intensities of transitions between different modes of the system's operation.

Lemma 1

Let $a,b \in S$. If the strong precedence relation between a and b holds, i.e. $a<b$, then $x<y$ for every $x \in \Phi^{-1}(a)$, $y \in \Phi^{-1}(b)$.

Proof: Let us assume that $a<b$ and $x \geq y$ for certain $x \in \Phi^{-1}(a)$, $y \in \Phi^{-1}(b)$. Then $\Phi(x) \geq \Phi(y)$, since Φ transfers the partial order from $\{0,1\}^n$ to S. As $\Phi(x)=a$ and $\Phi(y)=b$, the latter inequality contradicts the assumption. This ends the proof.

Lemma 2

Let $a,b \in S$, $a \neq b$. The direct precedence relation between a and b holds, i.e. $a <_d b$, if and only if $\exists\ x \in \Phi^{-1}(a), y \in \Phi^{-1}(b): x <_d y$.

Proof: Let us assume that $a <_d b$ and take arbitrary $v \in \Phi^{-1}(a)$, $w \in \Phi^{-1}(b)$. Then $v < w$ according to Lemma 1. If $\sim(v <_d w)$ then there exist $z_1,\ldots,z_k$ such that $v <_d z_1 <_d z_2 < \ldots <_d z_k <_d w$. Clearly, $a = \Phi(v) \leq \Phi(z_1) \leq \ldots \leq \Phi(z_k) \leq \Phi(w) = b$, as Φ transfers the partial order from $\{0,1\}^n$ to S. Since $a <_d b$, there exists $i \in \{0,1,\ldots,k\}$ such that $\Phi(v) = \ldots = \Phi(z_i) < \Phi(z_{i+1}) = \ldots = \Phi(w)$, where $z_0 = v$, $z_{k+1} = w$. Thus $x=z_i$ and $y=z_{i+1}$ fulfill the right-hand side condition. To prove the opposite implication, let us assume that the right-hand side condition holds and $\sim(a <_d b)$, i.e. $\sim(\Phi(x) <_d \Phi(y))$. Since $\Phi(x) < \Phi(y)$, there exists $c \in S$ such that $\Phi(x) < c < \Phi(y)$. Φ is a surjection from $\{0,1\}^n$ to S, hence there exists $z \in \{0,1\}^n$ such that $\Phi(x) < \Phi(z) < \Phi(y)$. According to Lemma 1, $x < z < y$ which contradicts the fact that $x <_d y$. The opposite implication has thus been proved by reductio ad absurdum, and the whole proof is completed.

Lemma 3

The following equalities hold:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr[X_i(t+\Delta t) = 0 | X_i(t) = 1] = \lambda_i$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr[X_i(t+\Delta t) = 1 | X_i(t) = 0] = \mu_i$$

$$\lim_{\Delta t \to 0} \Pr[X_i(t+\Delta t) = 1 | X_i(t) = 1] = \lim_{\Delta t \to 0} \Pr[X_i(t+\Delta t) = 0 | X_i(t) = 0] = 1 \qquad (5)$$

It should be noted that the probabilities in the last equality are not divided by Δt!

Proof: The first two equalities are simply definitions of λ_i and μ_i, $i \in J$. For the proof of the third one let us note that:

$$\Pr[X_i(t+\Delta t)=0|X_i(t)=1]=\lambda_i\Delta t+\varepsilon(\Delta t)$$

$$\Pr[X_i(t+\Delta t)=1|X_i(t)=0]=\mu_i\Delta t+\varepsilon(\Delta t) \tag{6}$$

where $\varepsilon(\Delta t)$ quickly converges to 0 as $\Delta t \to 0$, i.e.

$$\lim_{\Delta t\to 0}\frac{\varepsilon(\Delta t)}{\Delta t}=0 \tag{7}$$

We thus have:

$$\Pr[X_i(t+\Delta t)=1|X_i(t)=1]=1-\lambda_i\Delta t-\varepsilon(\Delta t)$$

$$\Pr[X_i(t+\Delta t)=0|X_i(t)=0]=1-\mu_i\Delta t-\varepsilon(\Delta t) \tag{8}$$

whence the third equality in (5) follows.

We can now formulate a theorem which gives the expressions for the transition intensities of Z.

Theorem 1

Let $a,b \in S$, $a \neq b$. Then $\Lambda_{a\to b} > 0$ only if $a <_d b$ or $a >_d b$, which means that direct transitions between a and b are only possible if one directly precedes the other. If $a <_d b$ then we have:

$$\Lambda_{a\to b}(t)=\frac{1}{\Pr[Z(t)=a]}\sum_{i\in J}\mu_i q_i(t) I_{a\leftrightarrow b}(i,t) \tag{9}$$

$$\Lambda_{b\to a}(t)=\frac{1}{\Pr[Z(t)=b]}\sum_{i\in J}\lambda_i p_i(t) I_{a\leftrightarrow b}(i,t) \tag{10}$$

Proof: The total probability formula and the independence of $X_j(t)$, $j \in J$ yield:

$$\Pr[Z(t+\Delta t)=b,\ Z(t)=a] =$$

$$= \sum_{x,y:\ \Phi(x)=a,\ \ \Phi(y)=b} \Pr[X(t+\Delta t)=y,\ X(t)=\ x] =$$

$$= \sum_{x,y:\ \Phi(x)=a,\ \ \Phi(y)=b,\ \ d(x,y)\geq 2} \Pr[X(t+\Delta t)=y,\ X(t)=\ x] +$$

$$+ \sum_{x,y:\ \Phi(x)=a,\ \ \Phi(y)=b,\ \ d(x,y)=1} \Pr[X(t+\Delta t)=y,\ X(t)=\ x] \qquad (11)$$

As the probability that two or more components change their states in an infinitesimal time interval Δt is almost equal to zero, i.e. this probability, divided by Δt, converges to zero as $\Delta t \to 0$, we have:

$$\Pr[Z(t+\Delta t)=b,\ Z(t)=a] =$$

$$= \varepsilon(\Delta t) + \sum_{x,y:\ \Phi(x)=a,\ \ \Phi(y)=b,\ \ d(x,y)=1} \Pr[X(t+\Delta t)=y,\ X(t)=\ x] \qquad (12)$$

where $\varepsilon(\Delta t)$ fulfills (7). In view of Lemma 2 and (2) it holds that $\Lambda_{a\to b}(t)>0$ only for $a <_d b$ or $a >_d b$. The first part of the theorem is this proved.

Let now $a <_d b$. Applying Lemma 1 we transform the sum in (10) as follows:

$$\sum_{x,y:\ \Phi(x)=a,\ \ \Phi(y)=b,\ \ d(x,y)=1} \Pr[X(t+\Delta t)=y,\ X(t)=\ x] =$$

$$= \sum_{x,y:\ \Phi(x)=a,\ \ \Phi(y)=b,\ \ x<y,\ \ d(x,y)=1} \Pr[X(t+\Delta t)=y,\ X(t)=\ x] =$$

$$= \sum_{i\in J} \sum_{x,y:\ \Phi(x)=a,\ \ \Phi(y)=b,\ \ y_i-x_i=1,\ \ \forall j\neq i\, x_j=y_j} \Pr[X(t+\Delta t)=y,\ X(t)=\ x] \qquad (13)$$

Due to the components' independence we have:

$$\Pr[X(t+\Delta t)=y,\ X(t)=\ x] = \prod_{j\in J} \Pr[X_j(t+\Delta t)=y_j,\ X_j(t)=\ x_j] \qquad (14)$$

The formulas (12) - (14) yield:

$$\Pr[Z(t+\Delta t)=b,\ Z(t)=a] =$$

$$= \varepsilon(\Delta t) + \sum_{i\in J}\sum_{x,y:\ \Phi(x)=a,\ \ \Phi(y)=b,\ \ y_i-x_i=1,\ \ \forall j\neq i\ x_j=y_j} \downarrow$$

$$\prod_{j\in J}\Pr\left[X_j(t+\Delta t)=y_j,\ X_j(t)=\ x_j\right] =$$

$$= \varepsilon(\Delta t) + \sum_{i\in J}\Pr[X_i(t+\Delta t)=1,\ X_i(t)=0] \times \downarrow$$

$$\sum_{x\in\Theta_{a\to b}^{crit}(i)}\prod_{j\in J\backslash\{i\}}\Pr\left[X_j(t+\Delta t)=x_j,\ X_j(t)=\ x_j\right] \qquad (15)$$

where the last equality follows from the definition of $\Theta_{a\to b}{}^{crit}(i)$. In an analogous way it is proved that for a $<_d$ b we have

$$\Pr[Z(t+\Delta t)=a,\ Z(t)=b] =$$

$$= \varepsilon(\Delta t) + \sum_{i\in J}\Pr[X_i(t+\Delta t)=0,\ X_i(t)=1] \times \downarrow$$

$$\sum_{x\in\Pi_{b\to a}^{crit}(i)}\prod_{j\in J\backslash\{i\}}\Pr\left[X_j(t+\Delta t)=x_j,\ X_j(t)=\ x_j\right] \qquad (16)$$

Let us note that for j∈J we have

$$\Pr\left[X_j(t+\Delta t)=x_j,\ X_j(t)=\ x_j\right] = \Pr\left[X_j(t+\Delta t)=x_j \mid X_j(t)=\ x_j\right]\Pr\left[X_j(t)=\ x_j\right] \qquad (17)$$

From (15) - (17), and Lemma 3 it follows that

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\Pr[Z(t+\Delta t)=b \mid Z(t)=a] =$$

$$= \frac{1}{\Pr[Z(t)=a]}\sum_{i\in J}\mu_i q_i(t)\sum_{x\in\Theta_{a\to b}^{crit}(i)}\prod_{j\in J\backslash\{i\}}\Pr\left[X_j(t)=\ x_j\right] =$$

$$= \frac{1}{\Pr[Z(t)=a]}\sum_{i\in J}\mu_i q_i(t)\sum_{x\in\Theta_{a\to b}^{crit}(i)}\frac{\Pr[X_i(t)=\ 0]}{\Pr[X_i(t)=\ 0]}\prod_{j\in J\backslash\{i\}}\Pr\left[X_j(t)=\ x_j\right] =$$

$$= \frac{1}{\Pr[Z(t)=a]}\sum_{i\in J}\mu_i q_i(t)\sum_{x\in\Theta_{a\to b}^{crit}(i)}\Pr[X(t)=x|X_i(t)=0] =$$

$$= \frac{1}{\Pr[Z(t)=a]}\sum_{i\in J}\mu_i q_i(t)\Pr\left[X(t)\in\Theta_{a\to b}^{crit}(i)|X_i(t)=0\right] =$$

$$= \frac{1}{\Pr[Z(t)=a]}\sum_{i\in J}\mu_i q_i(t) I_{a\leftrightarrow b}(i,t) \qquad (18)$$

and

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr[Z(t+\Delta t) = a \mid Z(t) = b] =$$

$$= \frac{1}{\Pr[Z(t)=b]} \sum_{i \in J} \lambda_i p_i(t) \sum_{x \in \Pi_{b \to a}^{crit}(i)} \prod_{j \in J \setminus \{i\}} \Pr[X_j(t) = x_j] =$$

$$= \frac{1}{\Pr[Z(t)=b]} \sum_{i \in J} \lambda_i p_i(t) \sum_{x \in \Pi_{b \to a}^{crit}(i)} \frac{\Pr[X_i(t)=1]}{\Pr[X_i(t)=1]} \prod_{j \in J \setminus \{i\}} \Pr[X_j(t) = x_j] =$$

$$= \frac{1}{\Pr[Z(t)=b]} \sum_{i \in J} \lambda_i p_i(t) \sum_{x \in \Pi_{b \to a}^{crit}(i)} \Pr[X(t) = x | X_i(t) = 1] =$$

$$= \frac{1}{\Pr[Z(t)=b]} \sum_{i \in J} \lambda_i p_i(t) \Pr[X(t) \in \Pi_{b \to a}^{crit}(i) | X_i(t) = 1] =$$

$$= \frac{1}{\Pr[Z(t)=b]} \sum_{i \in J} \lambda_i p_i(t) I_{a \leftrightarrow b}(i,t) \qquad (19)$$

The proof is thus completed.

An important conclusion can be drawn from Theorem 1. If Φ is a structure function, and the partial order in S is transferred by Φ from $\{0,1\}^n$, then Z is a (non-homogenous) Markov process with the transition intensities given by (9) and (10). Indeed, it can be easily shown that Pr[Z(t)=a], Pr[Z(t)=b], and $I_{a\leftrightarrow b}(i)$, $i \in J$ are functions of $p_i(t)$, hence these intensities are functions of t, and do not depend on the history of Z before time t. Also, they converge to constant values as $t \to \infty$, because $p_i(t)$ converges to $\mu_i/(\lambda_i+\mu_i)$ as $t \to \infty$. Z is thus asymptotically homogenous.

4. A Markov model of a two-mode object.

A key assumption about the considered system is that its components are independent, i.e. the TTF and TTR of any component do not depend on any other component's TTF or TTR. Clearly, this assumption may seem doubtful, because a component has to wait for repair if all maintenance teams are busy repairing other failed components, in which case the dependence of the component's TTR on the TTRs of other components takes place. However, if the components' failure rates are very small compared to their repair rates, i.e. $\lambda_i << \mu_i$, $i \in I$, which is often the case in practice, then the probability that a component fails when another component undergoes repair is close to zero. Moreover, if there are at least two maintenance teams, then repairs of two (or more) components can be performed simultaneously, if (notwithstanding the small failure rates) a component fails while another one is under repair. In consequence, the system's functioning can well be described by n independent two–state Markov chains, where each chain models the failure-repair process of the respective component. Such approach directly leads to the construction of a Markov chain with 2^n states. Nevertheless, it occurs that in order to model the system's functioning, as perceived by a user, the number of states can be greatly reduced. Such models will be constructed in this section. In the next section formulas for the transition intensities of the respective Markov chains will be derived.

We begin our considerations regarding multimode systems by analyzing a simple case of a two-mode technical system whose normal operation takes place in basic mode. When some of its components fail, it can be switched to emergency mode in which its operation continues until the failed components are repaired and the system is switched back to basic mode, or more component failures cause the system to stop its operation. A detailed model of such a system is presented in Fig. x+1 in the form of an inter-state transitions diagram. The meanings of individual states are given below the figure. This model takes into consideration each situation related to the system's bimodality.

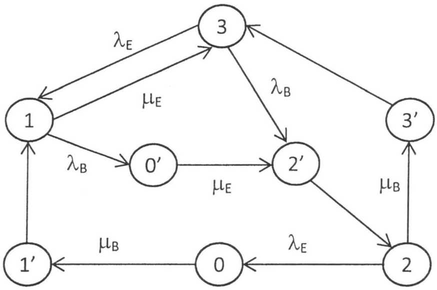

Fig. x+1. The detailed model of a two-mode system's operation – one maintenance team.

3 – both modes are operable, BM is active (EM has been switched to BM)

3′ – both modes are operable, EM is active

1 – BM is active, EM is under repair

1′ – BM is operable and inactive, EM is under repair

2 – BM is under repair, EM is active

2′ – BM is under repair, EM is operable and inactive

0 – both modes are failed, BM is under repair, EM is awaiting repair

0′ – both modes are failed, EM is under repair, BM is awaiting repair

Clearly, the stochastic process illustrated in Fig. x+1 has Markov property if the sojourn time in each state is exponentially distributed. However, Theorem 1 cannot be applied here, because the process's state space is not an image of $\{0,1\}^n$ for any function defined on $\{0,1\}^n$. In particular, the transition from 1′ to 1, 2′ to 2, or 3′ to 3 is not an effect of a component's state change, but that of switching between basic and emergency modes.

When there are two or more maintenance teams, then a repair of either mode can start immediately after its failure, and the states 0′ and 0 can be merged into one state – 0.

The resulting diagram is presented in Fig. x+2, and the remark regarding the application of Theorem 1 still holds.

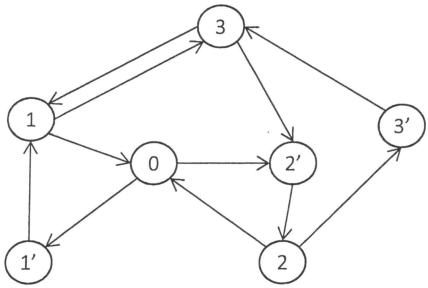

Fig. x+2. The model assuming no waiting times for repairs

Let us note that the transition chain 3⟶2'⟶2 is perceived by a system's user as the direct transition 3⟶2, i.e. from basic to emergency mode. Similarly, the transition chain 2⟶3'⟶3 is perceived as the direct transition 2⟶3, i.e. from emergency to basic mode. Further, the transition chains 0⟶1'⟶1 and 0⟶2'⟶2 are perceived as the direct transitions 0⟶1 and 0⟶2 respectively. In consequence, from a user's viewpoint, each of the states 1', 2', and 3' can be merged with the state 1, 2, or 3, respectively. The resulting diagram is shown in Fig. 3.

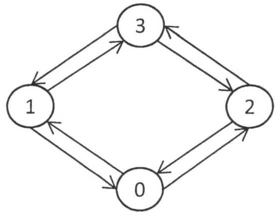

Fig. x+3. The four-state model

As each transition involves a failure or repair of at least one component, the state space of the process illustrated in Fig. x+3 is given by $S = \Phi(\{0,1\}^n) = \{0, 1, 2, 3\}$, where Φ is the respective structure function. The partial order in S, transferred from that in $\{0,1\}^n$, is given by the following relations: $0 <_d 1$, $0 <_d 2$, $1 <_d 3$, $2 <_d 3$. We can thus apply Theorem 1 in order to find the inter-state transition intensities.

Let us note that a user may not distinguish between the states 1 and 3, because in both cases the system operates in basic mode, and a transition between 1 and 3 does not cause a break in the system's operation, noticeable to a user. Thus our model can be further simplified, by merging the state 3 with 1, to the model presented in Fig. x+4.

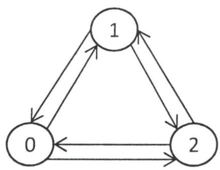

Fig. x+4. The three-state model

The state space of the process illustrated in Fig. x+4 is given by S = $\Phi(\{0,1\}^n) = \{0, 1, 2\}$, where Φ is the respective structure function. The partial order in S, transferred from that in $\{0,1\}^n$, is given by the following relations: $0 <_d 1$, $0 <_d 2$, $2 <_d 1$. Thus, as in the previous case, we can apply Theorem 1 in order to find the inter-state transition intensities, which will be done in the next section.

We conclude this chapter with a remark relevant to possible applications of the presented model. It can be assumed that switching between basic and emergency modes is done instantly, i.e. times of transitions 1'→1, 2'→2, and 3'→3 are much shorter than the remaining transition times. Thus, in the three-state model, the transitions 1→2 and 2→1 are associated with short breaks in the system operation. In turn, the sojourn in state 0 is perceived as a long break, because it involves restoring and activating basic or emergency mode.

5. Formulas for the transition intensities of a three-state system

In this section let Z be the three-state process illustrated in Fig. x+4. We will now derive formulas for the transition intensities of Z. This task will be made easier by using the intensities of transitions of two-state processes obtained from Z by aggregating the states 0+2 or 1+2. For simpler notation, we will omit the variable t where no confusion arises. These intensities are defined as follows:

$$\Lambda_{1\to 0+2}(t) = \lim_{u\to 0}\frac{1}{u}\Pr[Z(t+u) \in \{0,2\}|Z(t) = 1]$$

$$\Lambda_{0+2\to 1}(t) = \lim_{u\to 0}\frac{1}{u}\Pr[Z(t+u) = 1|Z(t) \in \{0,2\}]$$

$$\Lambda_{1+2\to 0}(t) = \lim_{u\to 0}\frac{1}{u}\Pr[Z(t+u) = 0|Z(t) \in \{0,2\}]$$

$$\Lambda_{0\to 1+2}(t) = \lim_{u\to 0}\frac{1}{u}\Pr[Z(t+u) \in \{1,2\}|Z(t) = 0] \tag{20}$$

where + is the aggregation operator. The definition of a component's importance to an inter-state transition yields:

$$I_{1\leftrightarrow 0+2}(i) = P_1|_{p_i=1} - P_1|_{p_i=0}$$

$$I_{1+2\leftrightarrow 0}(i) = P_{1+2}|_{p_i=1} - P_{1+2}|_{p_i=0} \qquad (21)$$

where $i \in J$, and $P_{1+2}(t) = Pr[Z(t) \in \{1,2\}] = P_1(t)+P_2(t)$. From (9) and (10) we obtain:

$$\Lambda_{1\to 0+2} = \frac{1}{P_1}\sum_{i\in J}\lambda_i p_i I_{1\leftrightarrow 0+2}(i)$$

$$\Lambda_{0+2\to 1} = \frac{1}{1-P_1}\sum_{i\in J}\mu_i q_i I_{1\leftrightarrow 0+2}(i)$$

$$\Lambda_{1+2\to 0} = \frac{1}{P_{1+2}}\sum_{i\in J}\lambda_i p_i I_{1+2\leftrightarrow 0}(i)$$

$$\Lambda_{0\to 1+2} = \frac{1}{1-P_{1+2}}\sum_{i\in J}\mu_i q_i I_{1+2\leftrightarrow 0}(i) \qquad (22)$$

The transition intensities of Z will be expressed using those defined by (22), and the importances $I_{1\leftrightarrow 2}(i)$, $i \in J$. However, the latter are not given by formulas as simple as (21). This is due to the fact that Z is not a two-state process, and the event $\{\Phi(x_i, i \in J)=1$ provided that $x_i=1\}$ may not include the event $\{\Phi(x_i, i \in I)=1$ provided that $x_i=0\}$. A method to compute $I_{1\leftrightarrow 2}(i)$, $i \in J$ will be presented in the next section.

As $0 <_d 1$, from (9) and (10) we get:

$$\Lambda_{1\to 2} = \frac{1}{P_1}\sum_{i\in J}\lambda_i p_i I_{1\leftrightarrow 2}(i)\,,\quad \Lambda_{2\to 1} = \frac{1}{P_2}\sum_{i\in J}\mu_i q_i I_{1\leftrightarrow 2}(i) \qquad (23)$$

From (20) and the law of total probability it follows that:

$$\Lambda_{1\to 0+2} = \Lambda_{1\to 0} + \Lambda_{1\to 2}\,,\ \Lambda_{0+2\to 1} = \frac{\Lambda_{0\to 1}P_0 + \Lambda_{2\to 1}P_2}{P_0+P_2} \qquad (24)$$

The equalities (24) yield:

$$\Lambda_{1\to 0} = \Lambda_{1\to 0+2} - \Lambda_{1\to 2} \qquad (25)$$

and

$$\Lambda_{0\to1} = \frac{\Lambda_{0+2\to1}(P_0+P_2)-\Lambda_{2\to1}P_2}{P_0} \quad (26)$$

It now remains to compute $\Lambda_{2\to0}$ and $\Lambda_{0\to2}$. By the same argument as just used we obtain:

$$\Lambda_{0\to1+2} = \Lambda_{0\to1} + \Lambda_{0\to2}\,, \quad \Lambda_{1+2\to0} = \frac{\Lambda_{1\to0}P_1+\Lambda_{2\to0}P_2}{P_1+P_2} \quad (27)$$

The equalities (27) yield:

$$\Lambda_{0\to2} = \Lambda_{0\to1+2} - \Lambda_{0\to1} \quad (28)$$

and

$$\Lambda_{2\to0} = \frac{\Lambda_{1+2\to0}(P_1+P_2)-\Lambda_{1\to0}P_1}{P_2} \quad (29)$$

As can be seen, all transition intensities of Z are expressed by $P_1(t)$, $P_2(t)$, and $I_{1\leftrightarrow2}(i,t)$, $i\in J$, which, in turn, are functions of $p_i(t)$, $i\in J$. Z is thus a Markov chain with time-dependent transition intensities. As follows from (4), each $p_i(t)$: $i\in J$ converges to $\mu_i/(\lambda_i+\mu_i)$ as $t\to\infty$, hence each $\Lambda_{a\to b}(t)$: $a,b\in\{0, 1, 2\}$ converges to a constant value. In consequence, Z is asymptotically homogenous.

A method to find $P_1(t)$, $P_2(t)$, and $I_{1\leftrightarrow0}(i,t)$, $i\in J$, which have to be known in order to use the formulas (23), (25) - (26), and (28) - (29), is presented in Section 6.

6. Computing P_1, P_2, and $I_{1\leftrightarrow0}(i)$ for an exemplary system

As can be expected, the method to compute the state probabilities P_1 and P_2, and the importances $I_{1\leftrightarrow0}(i)$, $i\in J$ is based on analyzing the system's RBD. This method will be illustrated on the exemplary system whose RBD is shown in Fig. 1.

We will first find the formulas for P_1 and P_2. Let us note that the events $\{x_7=0\}$, $\{x_7=1, x_3=0\}$, $\{x_7=1, x_3=1, x_4=0\}$, $\{x_7=1, x_3=1, x_4=1, x_5=0\}$, $\{x_7=1 ,..., x_5=1, x_6=0\}$, and $\{x_7=1 ,..., x_5=1, x_6=1\}$ are disjoint and exhaustive. From the law of total probability and the rules for computing the reliabilities of series-parallel systems, we obtain:

$$P_{1+2} = [\, q_7(p_1p_3p_4 \vee p_2)p_5p_6 + p_7q_3(p_2p_5p_6) +$$

$$+ p_7p_3q_4(p_1 \vee p_2p_5p_6) + p_7p_3p_4q_5(p_1) +$$

$$+ p_7p_3p_4p_5q_6(p_1 \vee p_2) + p_7p_3p_4p_5p_6(p_1 \vee p_2)\,] \cdot p_8 \qquad (30)$$

Reversing the order of components in the first "Boolean" sum, and transforming all the "Boolean" sums using the definition of the operator $\vee$, we obtain:

$$P_{1+2} = [\, q_7(p_2 + q_2p_1p_3p_4)p_5p_6 + p_7q_3p_2p_5p_6 +$$

$$+ p_7p_3q_4(p_1 + q_1p_2p_5p_6) + p_7p_3p_4q_5p_1 +$$

$$+ p_7p_3p_4p_5(p_1 + q_1p_2)\,] \cdot p_8 \qquad (31)$$

From (31) and the RBD in Fig. 1 it follows that:

$$P_1 = [\, p_7p_3q_4(p_1) + p_7p_3p_4q_5p_1 +$$

$$+ p_7p_3p_4p_5(p_1 + q_1p_2)\,] \cdot p_8 \qquad (32)$$

$$P_2 = [\, q_7(p_2 + q_2p_1p_3p_4)p_5p_6 + p_7q_3p_2p_5p_6 +$$

$$+ p_7p_3q_4q_1p_2p_5p_6\,] \cdot p_8 \qquad (33)$$

We now pass to the computation of $I_{1\leftrightarrow 2}(i)$. As $2 <_d 1$, transitions from 2 to 1 are triggered by components' repairs. From the RBD in Fig. 1 it can be seen that (1,3,7,8) and (2,5,4,3,7,8) are the basic minimal path-sets, and (2,5,6,8) and (1,3,4,5,6,8) are the emergency minimal path-sets. Let us note that if $i \in \{5,6,8\}$ then e_i belongs to the both emergency path-sets, which means that $\Phi(x) \neq 2$ if $x_i=0$. In consequence $\Theta_{2\to 1}^{crit}(i) = \emptyset$, and

$$I_{1\leftrightarrow 2}(i) = 0;\ i = 5,6,8 \qquad (34)$$

For $i \in \{1,2,3,4,7\}$ the formula for $I_{1\leftrightarrow 2}(i)$ is obtained by first selecting those components in the expression for P_2, which contain the variable q_i. Each vector x such that

x_i=0 (e_i is failed) and Φ(x)=2 (the system's state is 2) corresponds to one such component. Then for each selected component it is checked if the repair of e_i "opens" at least one basic path, all of which are "closed" before e_i's repair. If so, the component (after the removal of the variable q_i, and a possible further modification) is added to the expression for $I_{1\leftrightarrow 2}(i)$. Clearly, in order to obtain this expression, the variable q_i has to be deleted from each selected component of P_2, as $I_{1\leftrightarrow 2}(i)$ is a conditional probability provided that x_i=0. For i=1 the selected component is $p_7p_3q_4q_1p_2p_5p_6p_8$. As it contains the variables p_3, p_7, and p_8, the repair of e_1 opens the basic path {1,3,7,8}, hence

$$I_{1\leftrightarrow 2}(1) = p_7p_3q_4p_2p_5p_6p_8 \tag{35}$$

For i=2 the selected component is $q_7q_2p_1p_3p_4p_5p_6p_8$. However, the repair of e_2 does not open any basic path due to the presence of q_7 in the analyzed component, hence

$$I_{1\to 2}(2) = 0 \tag{36}$$

For i=3 the selected component is $p_7q_3p_2p_5p_6p_8$. The repair of e_3 opens (1,3,7,8) provided that e_1 is operable, or (2,5,4,3,7,8) provided that e_4 is operable. We thus have

$$I_{1\leftrightarrow 2}(3) = p_7p_2p_5p_6p_8(p_1 \vee p_4) = p_7p_2p_5p_6p_8(p_1 + q_1p_4) \tag{37}$$

For i=4 the selected component is $p_7p_3q_4q_1p_2p_5p_6p_8$. The repair of e_4 opens (2,5,4,3,7,8) (note that it cannot open (1,3,7,8) due to the presence of q_1), hence

$$I_{1\leftrightarrow 2}(4) = p_7p_3q_1p_2p_5p_6p_8 \tag{38}$$

For i=7 the selected components are is $q_7p_2p_5p_6p_8$ and $q_7q_2p_1p_3p_4p_5p_6p_8$. In case of the first component, the repair of e_7 opens (1,3,7,8) if e_1 and e_3 are operable, or (2,5,4,3,7,8) if e_3 and e_4 are operable. In case of the second component the repair of e_7 only opens (1,3,7,8) (due to the presence of q_2). In consequence

$$I_{1\leftrightarrow 2}(7) = p_2p_5p_6p_8(p_1 + q_1p_4)p_3 + q_2p_1p_3p_4p_5p_6p_8 \quad (39)$$

7. Conclusion

A method to compute the state probabilities and inter-state transition intensities for a complex system operating according to a three-state Markov model has been presented. This method, appropriately modified, can be applied to a broad spectrum of Markov-modeled multi-state systems. In particular, the systems that fulfill the assumptions of Theorem 1 are eligible. The computation of state probabilities and transition importances was presented in section 6 for the exemplary system, but it can easily be generalized to any system whose each path-set (obtained from its RBD) corresponds to one of the system's modes of operations (or states).

The key reliability parameters, i.e. the inter-state transition intensities can be used to obtain other characteristics of the system's behavior. This will now be shown for a three-state system considered in Sections 5 and 6. Let us adopt the following definitions:

L_a – the mean sojourn time in the state a, $a \in \{0,1,2\}$

$N_{a\rightarrow b}(u)$ – the mean number of times the system changes its state from a to b in a time interval of length u.

$N^{long}(u)$ – the average number of long brakes in the system's operation, resulting from failures of the both modes

$N^{short}(u)$ – the average number of short breaks in the system's operation, caused by switching between the both modes

By the argument similar to that used in [7] it can be shown that

$$L_a = \left(\sum_{b\in\{0,1,2\}, b\neq a} \Lambda_{a\rightarrow b}\right)^{-1} \quad (40)$$

$$N_{a\to b}(u) = uP_a\Lambda_{a\to b} \quad (41)$$

$$N^{long}(u) = N_{1\to 0}(u) + N_{2\to 0}(u) \quad (42)$$

$$N^{short}(u) = N_{1\to 2}(u) + N_{2\to 1}(u) \quad (43)$$

The above defined characteristics are particularly important for the reliability analysis of power distribution networks.

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