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# Probabilistic analysis of the Two-Constraint Binary Knapsack Problem, case of small and moderate values of the constraints right-hand-sides coefficients

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## Abstract

The paper deals with the Two-Constraint Binary Knapsack Problem. It is assumed that some of the problem coefficients are realizations of mutually independent random variables. Asymptotical probabilistic properties of selected problem characteristics are investigated for the special cases of the Lagrange multipliers. Special attention is paid to the case of mixed, small and moderate, values of them

## 1 Introduction

Let us consider a Two-Constraint Binary Knapsack Problem in the following formulation:

$$\begin{aligned} z_{OPT}(n) &= \max \sum_{i=1}^n c_i \cdot x_i \\ \text{subject to} \quad & \sum_{i=1}^n a_{1i} \cdot x_i \leq b_1(n) \\ & \sum_{i=1}^n a_{2i} \cdot x_i \leq b_2(n) \\ \text{where } x_i &= 0 \text{ or } 1, i = 1, \dots, n \end{aligned} \tag{1}$$

It is assumed that:

$$c_i > 0, a_{ji} > 0, 0 < b_j(n) \leq \sum_{i=1}^n a_{ji}, i = 1, \dots, n, j = 1, 2.$$

Without restricting the generality of considerations it may be also assumed that:

$$b_1(n) \leq b_2(n).$$

The goal of the assumptions that  $c_i, a_{ji} > 0, 0 < b_j(n) \leq \sum_{i=1}^n a_{ji}$ ,  $i = 1, \dots, n, j = 1, 2$ , is to avoid the trivial and degenerated problems. More precisely interpretation of the  $a_{ji} = 0$  or  $c_i = 0$  is far not obvious. When  $b_j(n) > \sum_{i=1}^n a_{ji}$  then the corresponding constraint is always fulfilled and therefore it may be removed from the problem formulation, otherwise if  $b_j(n) = 0$  then (1) has only the trivial solution i.e.  $x_i = 0, i = 1, \dots, n$  and  $z_{OPT}(n) = 0$ .

Two-Constraint Binary Knapsack Problem is special case of the binary multiconstraint knapsack problem, also known as  $m$ -constraint knapsack problem, see Nemhauser and Wolsey [9] and Martello and Toth [7], where in general case there is arbitrary number  $m$  of constraints, i.e.  $b_j(n), j = 1, \dots, m$ . Another important special case of the multiconstraint knapsack problem is classical (single constraint) or, in other words, Binary Knapsack Problem, which have only one constraint, i.e.  $j = 1$  (see Martello and Toth [7]). In the Szkatula's papers see [12] and [13] probabilistic analysis results of the different cases of the binary multiconstraint knapsack problem were presented. Moreover full case of the classical (single constraint) Binary Knapsack Problem was considered in the paper [13].

The Multi-Constraint Knapsack Problem is well known to be  $\mathcal{NP}$  hard and moreover, when  $m \geq 2$ , it is  $\mathcal{NP}$  hard in the strong sense (see Garey and Johnson [3]). It does mean that Two-Constraint Binary Knapsack Problem (1) is also  $\mathcal{NP}$  hard in the strong sense. Classical (one-constraint) Binary Knapsack Problem is  $\mathcal{NP}$  hard combinatorial optimization problem, however not in the strong sense.

The papers by Frieze and Clarke [2], Mamer and Schilling [6], Schilling [10] and [11] investigate the asymptotic value of  $z_{OPT}(n)$  for the random model of Multi-Constraint Knapsack Problem, where  $b_j(n) = 1, j = 1, \dots, m$ . Papers by Szkatula [12] and [13] were dealing with the random model of the Multi-Constraint Knapsack Problem, where  $b_j(n)$  are not restricted to be equal to 1. Papers by Meanti, Rinnooy Kan, Stougie and Vercellis [8], Lee and Oh [4] consider more general random models of Multi-Constraint Knapsack Problem but only for  $j = 1, 2$  some partial analytical results describing the growth of  $z_{OPT}(n)$  were obtained.

The aim of the present paper is to analyze the growth of the asymptotic value of  $z_{OPT}(n)$  for the class of random Two-Constraint Binary Knapsack Problems (1) with possibly full spectrum of the constraints right-hand-sides values. Results of the probabilistic analysis of this important problem may allow to describe asymptotic behavior of the  $z_{OPT}(n)$  for practically all combinations of values of  $b_1(n)$  and  $b_2(n)$  as well as other problem coefficients (considered as realizations of the random variables). Those results may help to better understand number of the theoretical issues related to Two-Constraint Binary Knapsack Problems as well as enable construction of more efficient algorithms for solving the practical instances of the (1).

## 2 Definitions

The following definitions are necessary for the further presentation:

**Definition 1** We denote  $V_n \approx Y_n$ , where  $n \rightarrow \infty$ , if

$$Y_n \cdot (1 - o(1)) \leq V_n \leq Y_n \cdot (1 + o(1))$$

when  $V_n, Y_n$  are sequences of numbers, or

$$\lim_{n \rightarrow \infty} P\{Y_n \cdot (1 - o(1)) \leq V_n \leq Y_n \cdot (1 + o(1))\} = 1$$

when  $V_n$  is a sequence of random variables and  $Y_n$  is a sequence of numbers or random variables, where  $\lim_{n \rightarrow \infty} o(1) = 0$  as it is usually presumed.

**Definition 2** We denote  $V_n \preceq Y_n (V_n \succeq W_n)$  if

$$V_n \leq (1 + o(1)) \cdot Y_n \quad (V_n \geq (1 - o(1)) \cdot W_n)$$

when  $V_n, Y_n (W_n)$  are sequences of numbers, or

$$\lim_{n \rightarrow \infty} P\{V_n \leq (1 + o(1)) \cdot Y_n\} = 1 \quad (\lim_{n \rightarrow \infty} P\{V_n \geq (1 - o(1)) \cdot W_n\} = 1)$$

when  $V_n$  is a sequence of random variables and  $Y_n (W_n)$  is a sequence of numbers or random variables, where  $\lim_{n \rightarrow \infty} o(1) = 0$ .

**Definition 3** We denote  $V_n \cong Y_n$  if there exist constants  $c'' \geq c' > 0$  such that

$$c' \cdot Y_n \preceq V_n \preceq c'' \cdot Y_n$$

where  $Y_n, V_n$  are sequences of numbers or random variables.

The following random model of (1) will be considered in the paper:

- $n \rightarrow \infty, i = 1, \dots, n, j = 1, 2.$
- $c_i, a_{ji}$  are realizations of mutually independent random variables and moreover  $c_i, a_{ji}$  are uniformly distributed over  $(0, 1]$ .
- $0 < \delta \leq b_1(n) \leq b_2(n) \leq n/2, b_j(n) \leq b_j(n+1)$ , for every  $n \geq 1$  and all  $b_j(n), j = 1, 2$ , are deterministic, where  $\delta$  is a constant.

Under the assumptions made about  $c_i, a_{ji}$  and  $b_j(n)$  the following always hold

$$0 \leq z_{OPT}(n) \leq \sum_{i=1}^n c_i \leq n, \quad \delta \leq b_j(n) \leq \sum_{i=1}^n a_{ji} \leq n, \quad j = 1, 2. \quad (2)$$

Moreover, from the strong law of large numbers it follows that

$$\sum_{i=1}^n c_i \approx E(c_1) \cdot n = n/2, \quad \sum_{i=1}^n a_{ji} \approx E(a_{11}) \cdot n = n/2.$$

Therefore, it is justified to enhance formula (2) in the following way:

$$0 \leq z_{OPT}(n) \preceq n/2, \quad 0 < \delta \leq b_1(n) \leq b_2(n) \preceq n/2 \quad (3)$$

Formula (3) shows that random model of the Two-Constraint Binary Knapsack Problem (1) is complete in the sense that nearly all possible instances of the problem are considered. In this respect the model where  $b_1(n) = b_2(n) = 1$  is just a very special case. Taking into account that  $\sum_{i=1}^n a_{ji} \approx n/2$  assumption that  $b_j(n) \leq b_j(n+1), j = 1, 2$ , for all  $n \geq 1$ , is quite logical.

The growth of  $z_{OPT}(n)$  - value of the optimal solution of the problem (1) may be influenced by the problem coefficients, namely:

$$n, c_i, a_{ji}, b_1(n), b_2(n), \text{ where } i = 1, \dots, n.$$

It is assumed that  $c_i, a_{ji}$  are realizations of the random variables and therefore their impact on the  $z_{OPT}(n)$  growth is in this case indirect. Moreover, we have assumed that  $n \rightarrow \infty$ . The aim of the probabilistic analysis is to investigate asymptotic behavior of  $z_{OPT}(n)$  when  $n \rightarrow \infty$ . The impact of the right-hand-side values -  $b_1(n), b_2(n)$  - is well illustrated by the Lagrange function and the problem dual to (1), see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [8], Szkatuła [12] and [13]. Due to the very complicated formulas, impossible to handle efficiently in the general case, the papers by Szkatuła [12] and [13] investigate only two important special cases of values of constraints right hand sides in the case of Multi-Constraint Knapsack Problem.

### 3 Lagrange and dual estimations

When the general knapsack type problem, with one or many constraints, is considered then Lagrange function and the corresponding dual problems, see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [8], Szkatuła [12] and [13] are very useful tools to perform various kind of analyses of the original problem. In the specific case of the Two-Constraint Binary Knapsack Problem Lagrange function of the problem (1) may be formulated as follows:

$$\begin{aligned} L_n(x) &= \sum_{i=1}^n c_i \cdot x_i + \sum_{j=1}^2 \lambda_j \cdot \left( b_j(n) - \sum_{i=1}^n a_{ji} \cdot x_i \right) = \\ &= \lambda_1 \cdot b_1(n) + \lambda_2 \cdot b_2(n) + \sum_{i=1}^n (c_i - \lambda_1 \cdot a_{1i} - \lambda_2 \cdot a_{2i}) \cdot x_i \end{aligned}$$

where  $x = [x_1, \dots, x_n]$  and  $\Lambda = [\lambda_1, \lambda_2]$  - vector of Lagrange multipliers. Moreover, let for every  $\Lambda, \lambda_j \geq 0, j = 1, 2$  :

$$\phi_n(\Lambda) = \max_{x \in \{0,1\}^n} L_n(x, \Lambda) = \max_{x \in \{0,1\}^n} \left\{ \sum_{j=1}^2 \lambda_j \cdot b_j(n) + \sum_{i=1}^n \left( c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} \right) x_i \right\}.$$

Using the following notation:

$$\begin{aligned} x_i(\Lambda) &= \begin{cases} 1 & \text{if } c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} & (4) \\ c_i(\Lambda) &= \begin{cases} c_i & \text{if } c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \\ a_{ji}(\Lambda) &= \begin{cases} a_{ji} & \text{if } c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

we have for every  $\Lambda$ ,  $\lambda_j \geq 0$ ,  $j = 1, 2$ :

$$\begin{aligned}\phi_n(\Lambda) &= \sum_{j=1}^2 \lambda_j \cdot b_j(n) + \sum_{i=1}^n \left( c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} \right) \cdot x_i(\Lambda) = \\ &= \sum_{j=1}^2 \lambda_j \cdot b_j(n) + \sum_{i=1}^n \left( c_i(\Lambda) - \sum_{j=1}^2 \lambda_j \cdot a_{ji}(\Lambda) \right)\end{aligned}$$

Obviously for  $i = 1, \dots, n$ ,  $j = 1, 2$ ,

$$c_i(\Lambda) = c_i \cdot x_i(\Lambda), \quad a_{ji}(\Lambda) = a_{ji} \cdot x_i(\Lambda).$$

Dual problem to Two-Constraint Binary Knapsack Problem (1) maybe formulated as follows:

$$\Phi_n^* = \min_{\Lambda \geq 0} \phi_n(\Lambda). \quad (5)$$

For every  $\Lambda \geq 0$  the following holds:

$$z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \sum_{j=1}^2 \lambda_j (b_j(n) - s_j(\Lambda)). \quad (6)$$

Let us denote:

$$\begin{aligned}z_n(\Lambda) &= \sum_{i=1}^n c_i \cdot x_i(\Lambda) = \sum_{i=1}^n c_i(\Lambda), \quad s_j(\Lambda) = \sum_{i=1}^n a_{ji} \cdot x_i(\Lambda) = \sum_{i=1}^n a_{ji}(\Lambda), \\ S_n(\Lambda) &= \sum_{j=1}^2 \lambda_j \cdot s_j(\Lambda), \quad B(\Lambda) = \sum_{j=1}^2 \lambda_j \cdot b_j(n).\end{aligned}$$

By definition of  $c_i(\Lambda)$  and  $a_{ji}(\Lambda)$ , see (4), we have:

$$c_i(\Lambda) \geq \sum_{j=1}^2 \lambda_j \cdot a_{ji}(\Lambda), \quad i = 1, \dots, n,$$

and therefore

$$z_n(\Lambda) \geq S_n(\Lambda). \quad (7)$$

For certain  $\Lambda$ ,  $x_i(\Lambda)$  given by (4) may provide feasible solution of (1), i.e.:

$$s_1(\Lambda) \leq b_1(n) \quad \text{and} \quad s_2(\Lambda) \leq b_2(n). \quad (8)$$

If the above holds then:

$$z_n(\Lambda) \leq z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + B(\Lambda) - S_n(\Lambda). \quad (9)$$

So, if (8) holds, then the below inequality also holds:

$$B(\Lambda) - S_n(\Lambda) \geq 0.$$

From (7) we get:

$$\frac{\phi_n(\Lambda)}{z_n(\Lambda)} = \frac{z_n(\Lambda)}{z_n(\Lambda)} + \frac{B(\Lambda) - S_n(\Lambda)}{z_n(\Lambda)} \leq 1 + \frac{B(\Lambda) - S_n(\Lambda)}{S_n(\Lambda)}.$$

Therefore if (8) holds, then the following inequality also holds:

$$1 \leq \frac{z_{OPT}(n)}{z_n(\Lambda)} \leq \frac{\Phi_n^*}{z_n(\Lambda)} \leq \frac{\phi_n(\Lambda)}{z_n(\Lambda)} \leq \frac{B(\Lambda)}{S_n(\Lambda)}. \quad (10)$$

Formulas (8) and (10) may allow to provide the asymptotical approximation of the  $z_{OPT}(n)$  i.e. the optimal solution value of the (1) problem. Namely if there exists such a set of Lagrange multipliers  $\Lambda(n)$  asymptotically fulfilling the formulas (8) and (10) then the below conjecture holds:

$$\text{If } \lim_{n \rightarrow \infty} \frac{B(\Lambda(n))}{S_n(\Lambda(n))} = 1 \text{ and (8) holds then } \lim_{n \rightarrow \infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1 \quad (11)$$

Therefore if (11) holds then  $x_i(\Lambda(n))$ ,  $i = 1, \dots, n$ , given by (4), provides the asymptotically sub-optimal solution of the Two-Constraint Binary Knapsack Problem (1). Moreover the value of  $z_n(\Lambda(n))$  is an asymptotical approximation of the optimal solution value of the Two-Constraint Binary Knapsack Problem i.e.  $z_{OPT}(n)$ .

## 4 Probabilistic analysis

In the present section of the paper some probabilistic properties of the Two-Constraint Binary Knapsack Problem (1) will be investigated. It is assumed that  $c_i$ ,  $a_{ji}$   $i = 1, \dots, n$ ,  $j = 1, 2$  are realizations of mutually independent random variables and moreover  $c_i$ ,  $a_{ji}$  are uniformly distributed over  $(0, 1]$ . Moreover it is assumed that  $0 < \delta \leq b_1(n) \leq b_2(n) \leq n/2$ ,  $b_j(n) \leq b_j(n+1)$ . In addition it is assumed that Lagrange multipliers  $\lambda_1$  and  $\lambda_2$ ,  $\lambda_2 \leq \lambda_1$ ,  $\Lambda = (\lambda_1, \lambda_2)$  are also deterministic. Monotonicity of constraints right hand sides,  $b_1(n) \leq b_2(n)$ , is in this case determining monotonicity of the Lagrange multipliers, i.e.  $\lambda_2 \leq \lambda_1$ . This is often used in the literature probabilistic model of the general knapsack problems and it suits very well also to the Two-Constraint Binary Knapsack Problem (1).

Let us first observe that due to the assumptions made the following holds, for  $i = 1, \dots, n$ ,  $j = 1, 2$ :

$$P(a_{ji} < x) = \begin{cases} 0 & \text{when } x \leq 0 \\ x & \text{when } 0 < x \leq 1 \\ 1 & \text{when } x \geq 1 \end{cases}, \quad P(c_i < x) = \begin{cases} 0 & \text{when } x \leq 0 \\ x & \text{when } 0 < x \leq 1 \\ 1 & \text{when } x \geq 1 \end{cases}. \quad (12)$$

In order to proceed with probabilistic analysis of the Two-Constraint Binary Knapsack Problem (1) it is necessary to consider probabilistic distribution of the following random variables

$$\sum_{j=1}^k \lambda_j \cdot a_{ji}, \quad k = 1 \text{ or } 2$$



Let  $(x)_+ = \frac{|x| + x}{2} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ ,  $j^* = \begin{cases} 2 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases}$ , Then for or  $i = 1, \dots, n$ ,  $j = 1, 2$ , the following holds:

$$F_1(x, \lambda_j) = P\{\lambda_j \cdot a_{ji} < x\} = \frac{1}{\lambda_j}((x)_+ - (x - \lambda_j)_+), \quad j = 1, 2,$$

$$F_2(x, \Lambda) = P\{\lambda_1 \cdot a_{1i} + \lambda_2 \cdot a_{2i} < x\} = \frac{1}{\lambda_j} \int_0^1 F_1(x - \lambda_j \cdot t, \lambda_j) dt = \quad (13)$$

$$= \frac{1}{\lambda_1 \cdot \lambda_2} ((x)_+^2 - (x - \lambda_1)_+^2 - (x - \lambda_2)_+^2 + (x - \lambda_1 - \lambda_2)_+^2)$$

The distribution functions of the random variables  $a_{ji}(\Lambda)$ ,  $c_i(\Lambda)$ ,  $i = 1, \dots, n$ ,  $j = 1, 2$  are:

$$G_{ji}(x, \Lambda) = P\{a_{ji}(\Lambda) < x\} =$$

$$= P\left\{a_{ji} < x \cup a_{ji} \geq x \cap \sum_{k=1}^2 \lambda_k \cdot a_{ik} \geq c_i\right\} = \quad (14)$$

$$= 1 - \int_x^1 \int_0^1 F_1(r - \lambda_j \cdot t, \lambda_j) dr dt$$

$$H_i(x, \Lambda) = P\{c_i(\Lambda) < x\} =$$

$$= P\left\{c_i < x \cup c_i \geq x \cap \sum_{k=1}^2 \lambda_k \cdot a_{ik} \geq c_i\right\} = \quad (15)$$

$$= 1 - \int_x^1 F_2(t, \Lambda) dt,$$

Using above formulas (14) and (15) expectations of the  $a_{ji}(\Lambda)$ ,  $c_i(\Lambda)$  could be expressed as follows:

$$E(a_{ji}(\Lambda)) = \int_0^1 x dG_{ji}(x, \Lambda) = \int_0^1 x \int_0^1 F_1(r - \lambda_j \cdot x, \lambda_j) dr dx = \quad (16)$$

$$= \frac{1}{\lambda_j} \left( \int_0^1 x \int_0^1 ((r - x \cdot \lambda_j)_+ - (r - x \cdot \lambda_j - \lambda_j)_+) dr dx \right)$$

$$\begin{aligned}
E(c_i(\Lambda)) &= \int_0^1 x dH_i(x, \Lambda) = \int_0^1 x \cdot F_2(x, \Lambda) dx = & (17) \\
&= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \int_0^1 x \cdot ((x)_+ - (x - \lambda_1)_+^2 - (x - \lambda_2)_+^2 + (x - \lambda_1 - \lambda_2)_+^2) dx = \\
&= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \left( \frac{1}{4} - \int_0^1 x \cdot ((x - \lambda_1)_+^2 + (x - \lambda_2)_+^2 - (x - \lambda_1 - \lambda_2)_+^2) dx \right).
\end{aligned}$$

It is easy to observe that above formulas (16) and (17) may take different formulations, depending on the mutual relations between  $\lambda_1, \lambda_2$  and  $x, r$  since several items of the above formulas may remain strongly positive or become 0, due to the function  $(\cdot)_+$  properties. In general 4 specific cases could be distinguished for  $i = 1, \dots, n, j = 1, 2$ :

1. Case of "large" values of the Lagrange multipliers  $1 \leq \lambda_2 \leq \lambda_1$ . In this case:

$$\begin{aligned}
E(a_{ji}(\Lambda)) &= \frac{1}{\lambda_j} \int_0^{1/\lambda_j} x \int_{x \cdot \lambda_j}^1 (r - x \cdot \lambda_j) dr dx = \frac{1}{24 \cdot \lambda_j^2 \cdot \lambda_j} & (18) \\
E(c_i(\Lambda)) &= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \int_0^1 x^3 dx = \frac{1}{8 \cdot \lambda_1 \cdot \lambda_2}.
\end{aligned}$$

2. Case of "mixed" values of the Lagrange multipliers  $\lambda_2 \leq 1 \leq \lambda_1$ . In this case:

$$\begin{aligned}
E(a_{1i}(\Lambda)) &= \frac{1}{\lambda_2} \left( \int_0^{1/\lambda_1} x \int_{x \cdot \lambda_1}^1 (r - x \cdot \lambda_1) dr dx - & (19) \\
&\quad - \int_0^{\frac{1-\lambda_2}{\lambda_1}} x \int_{(x \cdot \lambda_1 + \lambda_2)}^1 (r - x \cdot \lambda_1 - \lambda_2) dr dx \right) = \\
&= \frac{1 - \lambda_2^3 + 4\lambda_2^2 - 6\lambda_2 + 4}{24 \lambda_1^2}, \\
E(a_{2i}(\Lambda)) &= \frac{1}{\lambda_1} \left( \int_0^1 x \int_{x \cdot \lambda_2}^1 (r - x \cdot \lambda_2) dr dx \right) = \frac{1}{24} \frac{3\lambda_2^2 - 8\lambda_2 + 6}{\lambda_1}, \\
E(c_i(\Lambda)) &= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \left( \frac{1}{4} - \int_{\lambda_2}^1 x \cdot (x - \lambda_2)^2 dx \right) = \\
&= \frac{1}{24\lambda_1} (\lambda_2^3 - 6\lambda_2 + 8).
\end{aligned}$$

3. Case of "moderate" values of the Lagrange multipliers  $\lambda_2 \leq \lambda_1 \leq 1$ ,  $\lambda_2 + \lambda_1 \geq 1$ . In this case:

$$\begin{aligned}
 E(a_{ji}(\Lambda)) &= \frac{1}{\lambda_{j^*}} \left( \int_0^1 x \int_{x \cdot \lambda_j}^1 (r - x \cdot \lambda_j) dr dx - \right. \\
 &\quad \left. - \int_0^{(1-\lambda_{j^*})/\lambda_j} x \int_{(x \cdot \lambda_j + \lambda_{j^*})}^1 (r - x \cdot \lambda_j - \lambda_{j^*}) dr dx \right) = \\
 &= \frac{1}{24} \frac{3\lambda_j^4 - 8\lambda_j^3 + 6\lambda_j^2 - 6\lambda_{j^*}^2 + 4\lambda_{j^*} - \lambda_j^4 + 4\lambda_{j^*}^3 - 1}{\lambda_j^2 \lambda_{j^*}}, \\
 E(c_i(\Lambda)) &= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \left( \frac{1}{4} - \int_{\lambda_1}^1 x \cdot (x - \lambda_1)^2 dx - \int_{\lambda_2}^1 x \cdot (x - \lambda_2)^2 dx \right) = \\
 &= \frac{1}{24\lambda_1\lambda_2} (\lambda_1^4 - 6\lambda_1^2 + 8\lambda_1 + \lambda_2^4 - 6\lambda_2^2 + 8\lambda_2 - 3).
 \end{aligned} \tag{20}$$

4. Case of "small" values of the Lagrange multipliers  $\lambda_2 \leq \lambda_1 \leq 1$ ,  $\lambda_2 + \lambda_1 \leq 1$ . In this case:

$$\begin{aligned}
 E(a_{ji}(\Lambda)) &= \frac{1}{\lambda_{j^*}} \left( \int_0^1 x \int_{x \cdot \lambda_j}^1 (r - x \cdot \lambda_j) dr dx - \right. \\
 &\quad \left. - \int_0^1 x \int_{(x \cdot \lambda_j + \lambda_{j^*})}^1 (r - x \cdot \lambda_j - \lambda_{j^*}) dr dx \right) = \\
 &= \frac{1}{2} - \frac{1}{3} \lambda_j - \frac{1}{4} \lambda_{j^*}, \\
 E(c_i(\Lambda)) &= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \left( \frac{1}{4} - \int_{\lambda_1}^1 x \cdot (x - \lambda_1)^2 dx - \int_{\lambda_2}^1 x \cdot (x - \lambda_2)^2 dx + \right. \\
 &\quad \left. + \int_{\lambda_1 + \lambda_2}^1 x \cdot (x - \lambda_1 - \lambda_2)^2 dx \right) = \\
 &= \frac{1}{2} - \frac{1}{6} \lambda_1^2 - \frac{1}{4} \lambda_1 \lambda_2 - \frac{1}{6} \lambda_2^2.
 \end{aligned} \tag{21}$$

Probabilistic, or in other words average case, analysis consists in determining such Lagrange multipliers  $\lambda_1(n)$ ,  $\lambda_2(n)$  that when  $n \rightarrow \infty$ ,  $x_i(\Lambda(n))$ ,  $i = 1, \dots, n$ , defined by (4) will provide solutions of the Two-Constraint Binary Knapsack Problem (1) which are, in the sense of convergence in probability, see Loeve [5], providing solutions which are asymptotically feasible, i.e.  $s_j(\Lambda(n))$  is satisfying (8) and moreover if  $S_n(\Lambda(n))$  is fulfilling (11) then, due to (10),  $\lim_{n \rightarrow \infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1$  and  $z_n(\Lambda(n))$  is suboptimal solution of the (1) and moreover

$$z_{OPT}(n) \approx z_n(\Lambda(n)) \approx E(z_n(\Lambda(n))).$$

The above goal may be achieved by determining  $\Lambda(n)$  as the solution of the following system of equations:

$$E(s_1(\Lambda(n))) = b'_1(n), \quad E(s_2(\Lambda(n))) = b'_2(n), \quad (22)$$

where  $b'_j(n) = b_j(n) - \epsilon_j(n)$ ,  $\epsilon_j(n) = o(b_j(n))$  and  $s_j(\Lambda(n)) \approx b'_j(n) \approx b_j(n)$ ,  $s_j(\Lambda(n)) \leq b_j(n)$ ,  $j = 1, 2$  (in the sense of convergence in probability) and therefore  $\Lambda(n)$  is fulfilling both (8) and (11).

It may also happen that the system of equations (22) has no solutions, e.g. when difference between  $b_1(n)$  and  $b_2(n)$  (asymptotical divergence) is too large. In this case only  $\lambda_1(n) > 0$  and  $\lambda_2(n) = 0$  which means that second constraint in the Two-Constraint Binary Knapsack Problem (1) formulation is excessive and could be removed. It does mean that in this situation problem (1) reduces to the classical single constraint knapsack problem. In the Szkatuła paper [13] the following formulas summarizing behavior of the optimal solution value was presented:

$$z_{OPT}(n) \approx z_n(\Lambda(n)) \approx E(z_n(\Lambda(n)))$$

and

$$E(z_n(\Lambda(n))) = \begin{cases} \sqrt{\frac{2 \cdot n \cdot b'_1(n)}{3}} & \text{if } \delta \leq b'_1(n) \leq \frac{n}{6}, \\ \frac{1}{4} \cdot \left( \frac{n}{2} + 6 \cdot b'_1(n) \cdot \left( 1 - \frac{b'_1(n)}{n} \right) \right) & \text{if } \frac{n}{6} \leq b'_1(n) \leq \frac{n}{2}. \end{cases} \quad (23)$$

**Lemma 1** *If  $c_i, a_{ji}$   $i = 1, \dots, n$ ,  $j = 1, 2$ , are realizations of mutually independent random variables uniformly distributed over  $(0, 1)$ , and there does not exist  $0 < \lambda_2(n) \leq \lambda_1(n)$ , being solution of the (22) for all  $n \geq 1$ . Then second constraint is excessive in the sense that it is always fulfilled if the first one is true. In this case  $\lambda_2(n) = 0$  and problem is equivalent to the single constraint knapsack problem, and*

$$z_{OPT}(n) \approx \begin{cases} \sqrt{\frac{2 \cdot n \cdot b'_1(n)}{3}} & \text{if } \delta \leq b'_1(n) \leq \frac{n}{6}, \\ \frac{1}{4} \cdot \left( \frac{n}{2} + 6 \cdot b'_1(n) \cdot \left( 1 - \frac{b'_1(n)}{n} \right) \right) & \text{if } \frac{n}{6} \leq b'_1(n) \leq \frac{n}{2}. \end{cases} \quad (24)$$

**Proof.** In this case the system of equations (22) has no solution and therefore Two-Constraint Binary Knapsack Problem (1) is equivalent to the single constraint knapsack problem, refer to Szkatuła [13]; (24) follows immediately from (23). ■

Each of the 4 cases where  $\lambda_1(n), \lambda_2(n) > 0$  mentioned above should be considered separately. Let us observe that  $E(s_j(\Lambda(n))) = n \cdot E(a_{j1}(\Lambda(n)))$ ,  $E(z_n(\Lambda(n))) = n \cdot E(c_1(\Lambda(n)))$ .

**Lemma 2** *If  $c_i, a_{ji}$   $i = 1, \dots, n$ ,  $j = 1, 2$ , are realizations of mutually independent random variables uniformly distributed over  $(0, 1)$ , and if  $\delta \leq b'_1(n) \leq b'_2(n) \leq \sqrt{\frac{n \cdot b'_1(n)}{24}}$ , where  $\delta$  is a constant, then*

$$\lambda_1(n) = \frac{1}{b'_1(n)} \sqrt[3]{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}}, \quad \lambda_2(n) = \frac{1}{b'_2(n)} \sqrt[3]{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}} \quad (25)$$

is the solution of (22) and

$$E(z_n(\Lambda(n))) = 3 \cdot \sqrt[3]{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}}. \quad (26)$$

**Proof.** Above formulas follow immediately from the (18) and (22). From the Lemma 2 assumptions and from (25) it follows that  $1 \leq \lambda_2(n) \leq \lambda_1(n)$ . Moreover condition  $\delta \leq b'_1(n) \leq b'_2(n) \leq \sqrt{\frac{n \cdot b'_1(n)}{24}}$  holds only when  $b'_1(n) \leq b'_2(n) \leq \frac{n}{24}$  (not vice versa). ■

When  $\lambda_2(n) \leq 1 \leq \lambda_1(n)$ , defined above case of of mixed values of Lagrange multipliers, where values of  $\lambda_1(n)$ ,  $\lambda_2(n)$ ,  $\lambda_2(n) \leq 1 \leq \lambda_1(n)$ ,  $E(s_1(\Lambda(n)))$ ,  $E(s_2(\Lambda(n)))$  and  $E(z_n(\Lambda(n)))$  are provided by (19) problem of defining exact values of  $\lambda_1(n)$  and  $\lambda_2(n)$  fulfilling is (22) is complex due to the formulation provided by (19). Let us denote

$$\alpha = -\lambda_2^3(n) + 4\lambda_2^2(n) - 6\lambda_2(n) + 4; \quad \beta = 3\lambda_2^2(n) - 8\lambda_2(n) + 6$$

then

$$\gamma = \beta - \alpha = (\lambda_2(n) - 1)(\lambda_2^2(n) - 2)$$

taking into account that in this case  $0 \leq \lambda_2(n) \leq 1$  the following hold

$$0 \leq \gamma \leq 2 \tag{27}$$

Then (19) and (22) could be reformulated as follows:

$$\begin{aligned} \frac{1}{24} \frac{\alpha}{\lambda_1^2(n)} &= \frac{b'_1(n)}{n} \\ \frac{1}{24} \frac{\beta}{\lambda_1(n)} &= \frac{b'_2(n)}{n} \end{aligned} \tag{28}$$

From the above equations putting  $\alpha = 24 \cdot \lambda_1^2 \cdot \frac{b'_1(n)}{n}$  and  $\beta = \gamma + \alpha$  we will receive:

$$24 \cdot \lambda_1^2(n) \cdot \frac{b'_1(n)}{n} + \gamma = 24 \cdot \lambda_1(n) \cdot \frac{b'_2(n)}{n}$$

And therefore  $\lambda_1(n) \geq 1$ , when  $\gamma \cdot n \cdot b'_1(n) \leq 6 \cdot b_2^2(n)$ , is provided by the formula

$$\lambda_1(n) = \frac{1}{12 \cdot b'_1(n)} \left( 6 \cdot b'_2(n) + \sqrt{6 \cdot b_2^2(n) - \gamma \cdot n \cdot b'_1(n)} \right) \tag{29}$$

Due to (27) we will receive

$$\frac{b'_2(n)}{2b'_1(n)} \leq \lambda_1(n) \leq \frac{b'_2(n)}{b'_1(n)}$$

Above formulas were received under assumption that there exist  $\lambda_1(n)$  and  $\lambda_2(n)$  being solution of (22). Let us observe that due to (28)

$$\lambda_1(n) = \sqrt{\frac{\alpha \cdot n}{24 \cdot b'_1(n)}} \tag{30}$$

$$\beta - \alpha = \sqrt{\frac{\alpha}{24}} \left( \frac{b'_2(n)}{\sqrt{n \cdot b'_1(n)}} - \sqrt{\frac{b'_1(n)}{n}} \right) = \gamma$$

**Remark 1** By the definition  $0 \leq \lambda_2(n) \leq 1$  and  $\gamma = 0$  if  $\lambda_2(n) = 1$ . In this case due to (30) and (29) it holds

$$b'_1(n) = b'_2(n) = \frac{n}{24} \text{ and } \lambda_1(n) = 1$$

If  $0 < \lambda_2(n) < 1 < \lambda_1(n)$  then by definition

$$1 < \alpha < 4, \sqrt{\frac{1}{24}} < \sqrt{\frac{\alpha}{24}} < \sqrt{\frac{1}{6}}, 0 < \gamma < 2; \frac{b'_1(n)}{n} \leq \frac{1}{2}.$$

**Lemma 3** *If there exist constant  $\delta > 0$  such that  $\delta \leq b'_1(n) < \frac{n}{24} < b'_2(n)$  and  $\lim_{n \rightarrow \infty} \frac{b'_2(n)}{\sqrt{n \cdot b'_1(n)}} = \infty$  then (22) has no solution and due to the Lemma 1*

$$z_{OPT}(n) \approx \sqrt{\frac{2 \cdot n \cdot b'_1(n)}{3}}$$

*Proof.* Proof follows from above considerations, namely equation

$$\sqrt{\frac{\alpha}{24}} \left( \frac{b'_2(n)}{\sqrt{n \cdot b'_1(n)}} - \sqrt{\frac{b'_1(n)}{n}} \right) = \gamma$$

has no solution since its left-hand-side is tending to infinity while right-hand-side is limited,  $0 < \gamma < 2$ . ■

**Lemma 4** *If  $c_i, a_{ji}$   $i = 1, \dots, n$ ,  $j = 1, 2$ , are realizations of mutually independent random variables uniformly distributed over  $(0, 1)$ , and if  $b'_1(n) \geq \max\{\frac{n}{6}, \frac{4}{3} \cdot b'_2(n) - \frac{n}{6}\}$  then*

$$\begin{aligned} \lambda_1(n) &= \frac{1}{7} \cdot \left( 6 - 48 \cdot \frac{b'_1(n)}{n} + 36 \cdot \frac{b'_2(n)}{n} \right), \\ \lambda_2(n) &= \frac{1}{7} \cdot \left( 6 + 36 \cdot \frac{b'_1(n)}{n} - 48 \cdot \frac{b'_2(n)}{n} \right) \end{aligned} \quad (31)$$

is the solution of (22) and

$$\begin{aligned} E(z_n(\Lambda(n))) &= \frac{1}{7} \cdot \left( \frac{n}{2} + 6 \cdot b'_1(n) + 6 \cdot b'_2(n) + \right. \\ &\quad \left. + 36 \cdot \frac{b'_1(n) \cdot b'_2(n)}{n} - 24 \cdot \frac{(b'_1(n))^2}{n} - 24 \cdot \frac{(b'_2(n))^2}{n} \right) \end{aligned} \quad (32)$$

**Proof.** From the (21) it could be obtained that  $\lambda_1(n)$  and  $\lambda_2(n)$  given by formula (31) are solving the equation (22) and fulfilling the condition  $\lambda_2(n) + \lambda_1(n) \leq 1$ . ■

**Lemma 5** *If  $c_i, a_{ji}$   $i = 1, \dots, n$ ,  $j = 1, 2$ , are realizations of mutually independent random variables uniformly distributed over  $(0, 1)$ , and if  $\frac{n}{6} \leq b'_1(n) < \min\{b'_2(n), \frac{4}{3} \cdot b'_2(n) - \frac{n}{6}\}$  then*

$$\lambda_1(n) = 3 \cdot \left( \frac{1}{2} - \frac{b'_1(n)}{n} \right), \quad \lambda_2(n) = 0$$

is the optimal set of Lagrange multipliers and

$$E(z_n(\Lambda(n))) = \frac{1}{4} \cdot \left( \frac{n}{2} + 6 \cdot b'_1(n) \cdot \left( 1 - \frac{b'_1(n)}{n} \right) \right)$$

**Proof.** In this case the system of equations (22) has no solution and therefore Two-Constraint Binary Knapsack Problem (1) is equivalent to the single constraint knapsack problem, refer to Szkatuła [13]; (24) follows immediately from (23). Condition  $\lambda_2(n) + \lambda_1(n) \leq 1$  holds also in this case. ■

In the cases considered in Lemma 4 and Lemma 5 the condition  $\lambda_2(n) + \lambda_1(n) \leq 1$  and formula (3) are providing that following right-hand-sides of the constraints:

$$\frac{5 \cdot n}{12} \leq b'_1(n) + b'_2(n) \leq n,$$

are fulfilling assumptions of the Lemma 4.

## 5 Concluding remarks

In this paper results describing probabilistic properties of the Two-Constraint Binary Knapsack Problem (1) in the case of smaller values of constraints right-hand-sides (when  $1 \leq \lambda_2(n) \leq \lambda_1(n)$ , i.e. large ) as well as large values of constraints right-hand-sides (when  $\lambda_2(n) + \lambda_1(n) \leq 1$ , corresponding to small values of  $\lambda_1(n)$  and  $\lambda_2(n)$ ) are considered and exact values of all relevant problem coefficients like  $\lambda_1(n)$ ,  $\lambda_2(n)$ ,  $E(z_n(\Lambda(n)))$  etc. are provided. Moreover case of small and moderate values of the constraints right-hand-sides, i.e. when  $\lambda_2(n) \leq 1 \leq \lambda_1(n)$ , was considered. Case when second constraint is excessive, i.e. when two-constraint knapsack problem is equivalent to single constraint knapsack problem, was identified.

In the paper distribution functions of the various random variables representing important problems characteristics are presented.

The future research will be aimed at investigation of 2 remaining cases (mixed and moderate values) of the mutual relations between  $\lambda_1(n)$  and  $\lambda_2(n)$ , feasibility of the received solutions and estimations of the Two-Constraint Binary Knapsack Problem (1) optimal solution values  $z_{OPT}(n)$  growth, when  $n \rightarrow \infty$ .

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the 1990s, the number of people in the UK who are aged 65 and over has increased from 10.5 million to 13.5 million (1990-2000) (ONS 2001).

There is a growing awareness of the need to address the health care needs of the elderly population. The Department of Health (2000) has set out a strategy for the NHS to meet the needs of the elderly population. This strategy is based on the following principles: (1) to ensure that the elderly population has access to the services they need; (2) to ensure that the services are of high quality; (3) to ensure that the services are cost-effective; and (4) to ensure that the services are sustainable. The strategy is based on the following key areas: (1) primary care; (2) secondary care; (3) community care; and (4) long-term care.

The strategy for the NHS to meet the needs of the elderly population is based on the following key areas: (1) primary care; (2) secondary care; (3) community care; and (4) long-term care. The strategy is based on the following key areas: (1) primary care; (2) secondary care; (3) community care; and (4) long-term care. The strategy is based on the following key areas: (1) primary care; (2) secondary care; (3) community care; and (4) long-term care.

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