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Accuracy of a newly developed maximum likelihood estimators for the parameters of a Weibull process

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1. Introduction

The paper [1] presents a new method of estimating the scale (λ) and shape (α) parameters of the Weibull distribution. It is based on simulating m times-to-failure of each of n independent objects, where an object is subjected to m−1 minimal repairs, and considered unusable after the m-th failure. We thus obtain n i.i.d. samples of the random vector $[T_1,\ldots,T_m]$, where T_1 is the time of the first failure, and T_i – the time elapsed between the (i−1)-th minimal repair and the i-th failure, i=2,...,m. In other words, this procedure simulates n sequences of events of a non-homogenous Poisson process with Weilull intensity given by $r(t) = \alpha\lambda^\alpha t^{\alpha-1}$. From these n sequences the considered estimators of α and λ are obtained in the following way: firstly, the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\lambda}$ are constructed from a single sample:

$$\hat{\alpha} = \frac{m}{m\cdot\ln(T_1+\cdots+T_m)-\sum_{i=1}^{m}\ln(T_1+\cdots+T_m)}, \tag{1}$$

$$\hat{\lambda} = \frac{m^{1/\hat{\alpha}}}{T_1+\cdots+T_m}; \tag{2}$$

secondly, $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ defined as follows

$$\widehat{\Lambda} = \frac{\ln(\hat{\lambda}_1)+\cdots+\ln(\hat{\lambda}_n)}{n}, \ \widehat{\mathrm{A}} = \frac{1/\hat{\alpha}_1+\cdots+1/\hat{\alpha}_n}{n}, \tag{3}$$

are used as respective estimators of $\ln(\lambda)$ and $1/\alpha$, where $\hat{\lambda}_1,\ldots,\hat{\lambda}_n$ and $\hat{\alpha}_1,\ldots,\hat{\alpha}_n$ are i.i.d. instances of $\hat{\lambda}$ and $\hat{\alpha}$ respectively. Let us note that the estimators $\hat{\lambda}$ and $\hat{\alpha}$ are not obtained from an i.i.d. sample, but $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ are. For technical reasons, it is easier to estimate 1/α and ln(λ) rather than α and λ, because, as proved in [1], the biases of $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$ can be expressed in an analytical form as linear functions of $E(1/\hat{\alpha})$, i.e.

$$\frac{1}{\alpha} - E(1/\hat{\alpha}) = \frac{1}{m-1} E(1/\hat{\alpha}) \tag{4}$$

$$\ln(\lambda) - E[\ln(\hat{\lambda})] = \frac{m}{m-1} E(1/\hat{\alpha}) \left[\frac{\ln(m)}{m} - \frac{1}{m} + \Gamma'(1) - \ln(m) + \sum_{j=1}^{m} 1/j\right] \tag{5}$$

Let us note that the biases of $\hat{\Lambda}$ and $\hat{A}$ are equal to those of $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$, because, as it follows from (3)

$$E(\hat{\Lambda}) = E[\ln(\hat{\lambda})] \text{ and } E(\hat{A}) = E(1/\hat{\alpha}), \tag{6}$$

The formulas (4) and (5) are the main results presented in [1]. However, for an estimation technique to be complete it is also necessary to assess the estimation accuracy. This problem was left open in [1], but it will be addressed herein. Such accuracy is usually expressed in terms of confidence levels and confidence intervals. Let us recall some basics on this subject. If $X_1, \dots, X_n$ is an i.i.d. random sample from a random variable X such that $E(X) = \mu < \infty$ and $Var(X) = \sigma^2 < \infty$, and α is a small positive number, then for sufficiently large n it holds that

$$\Pr\left(\hat{\mu}_n - z_{1-\frac{\beta}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \hat{\mu}_n + z_{1-\frac{\beta}{2}} \frac{\sigma}{\sqrt{n}}\right) \geq 1 - \beta \tag{7}$$

Here, $\hat{\mu}_n$ is the sample mean – a commonly used estimator of μ, i.e.

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{8}$$

and $z_{1-\beta/2}$ is the 1 – β/2 quantile of the standardized normal distribution, i.e.

$$\Pr\left(Z \leq z_{1-\frac{\beta}{2}}\right) = 1 - \frac{\beta}{2} \tag{9}$$

where Z is normally distributed with mean 0 and variance 1. The interval defined by (7) is called a confidence interval; it contains μ with probability $1-\beta$ called the confidence level. Hence, if ε is an arbitrarily chosen small number, then the minimum sample size for which the interval $(\hat{\mu}_n - \varepsilon,\ \hat{\mu}_n + \varepsilon)$ contains μ with probability greater or equal to $1-\beta$ is given by the following formula:

$$n = \left\lfloor \left(\frac{\sigma}{\varepsilon} z_{1-\frac{\beta}{2}}\right)^2 \right\rfloor + 1 \tag{10}$$

where $\lfloor x \rfloor$ denotes the integer part of x. Thus, n is the smallest sample size for which the desired estimation accuracy, expressed by ε and β, is attained. As σ is usually unknown, for practical purposes it can be replaced in (10) by the (unbiased) sample variance of X, i.e.

$$\hat{\sigma}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}_n)^2 \tag{11}$$

In our case $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ are the sample means which estimate the expected values $E[\ln(\hat{\lambda})]$ and $E(1/\hat{\alpha})$. Therefore, the confidence intervals for $\ln(\lambda)$ and $1/\alpha$ are defined using $Var[\ln(\hat{\lambda})]$ and $Var(1/\hat{\alpha})$ respectively. In consequence, the minimum sample sizes, guaranteeing that $\ln(\lambda)$ and $1/\alpha$ are estimated with the desired accuracy specified by ε (half-width of the confidence interval) and 1−β (the confidence level) are respectively given as follows:

$$n_\lambda = \left\lfloor Var[\ln(\hat{\lambda})] \left(z_{1-\frac{\beta}{2}}/\varepsilon\right)^2 \right\rfloor + 1\;,\; n_\alpha = \left\lfloor Var(1/\hat{\alpha}) \left(z_{1-\frac{\beta}{2}}/\varepsilon\right)^2 \right\rfloor + 1 \tag{12}$$

For computational purposes $Var[\ln(\hat{\lambda})]$ and $Var(1/\hat{\alpha})$ can be replaced in (12) with the respective sample variances.

Assuming that (11) is a good approximation of σ^2, we may conclude that the ability to find an analytical expression for σ^2 has a purely theoretical significance. However, there are two reasons why this is not the case. Firstly, having an analytical expression for at least an upper bound of σ^2 allows to assess the numerical complexity of the estimation problem. Secondly, it can be checked whether $\sigma^2 < \infty$, i.e. whether the estimation is numerically tractable.

In view of the above considerations, we need to find $Var\left[\ln(\hat{\lambda})\right]$ and $Var(1/\hat{\alpha})$, or at least their upper bounds. Before that, upper bounds for $E\left[\left(\ln(\hat{\lambda})\right)^2\right]$ and $E[(1/\hat{\alpha})^2]$ will be found in the next section.

2. Upper bounds for $E\left[\left(\ln(\hat{\lambda})\right)^2\right]$ and $E[(1/\hat{\alpha})^2]$

First, the formulas for the moment generating functions of $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$ will be derived. We have

$$G_{1/\hat{\alpha}}(u) = E\left(e^{u\cdot\left[\ln(t_1+\cdots+t_m)-\frac{1}{m}\sum_{i=1}^{m}\ln(t_1+\cdots+t_i)\right]}\right) =$$

$$= E\left[\left(e^{\sum_{i=1}^{m}[\ln(t_1+\cdots+t_m)-\ln(t_1+\cdots+t_i)]}\right)^{u/m}\right] =$$

$$= E\left[\left(\prod_{i=1}^{m} e^{\ln[(t_1+\cdots+t_m)/(t_1+\cdots+t_i)]}\right)^{u/m}\right] =$$

$$= E\left[\left(\prod_{i=1}^{m}\frac{t_1+\cdots+t_m}{t_1+\cdots+t_i}\right)^{u/m}\right] = E[X^u] \qquad (13)$$

where

$$X = \left(\prod_{i=1}^{m}\frac{t_1+\cdots+t_m}{t_1+\cdots+t_i}\right)^{1/m} \qquad (14)$$

As $Var[\ln(\hat{\lambda})] = Var[-\ln(\hat{\lambda})]$, and it will be more convenient to operate on $-\ln(\hat{\lambda})$ rather than $\ln(\hat{\lambda})$, the formula for the MGF of the former will now be derived.

$$G_{-\ln(\hat{\lambda})}(u) = E\left(e^{u\cdot[-\ln(\hat{\lambda})]}\right) =$$

$$= E\left[\left(e^{\ln(t_1+\cdots+t_m)-\frac{1}{\hat{\alpha}}\ln(m)}\right)^{u}\right] =$$

$$= E\left[(t_1+\cdots+t_m)^{u} / \left(\prod_{i=1}^{m-1} \frac{t_1+\cdots+t_m}{t_1+\cdots+t_i}\right)^{\frac{u\ln(m)}{m}}\right] =$$

$$= E\left[\left((t_1+\cdots+t_m) / \prod_{i=1}^{m-1} \left(\frac{t_1+\cdots+t_m}{t_1+\cdots+t_i}\right)^{\frac{\ln(m)}{m}}\right)^{u}\right] = E[Y^u] \tag{15}$$

where

$$Y = (t_1+\cdots+t_m) / \prod_{i=1}^{m-1} \left(\frac{t_1+\cdots+t_m}{t_1+\cdots+t_i}\right)^{\frac{\ln(m)}{m}} \tag{16}$$

The MGF of an arbitrary random variable V has the following properties:

$$E(V) = \left.\frac{dG_V(u)}{du}\right|_{u=0}, \quad E(V^2) = \left.\frac{d^2G_V(u)}{du^2}\right|_{u=0} \tag{17}$$

We thus have:

$$E[(1/\hat{\alpha})^2] = \left.\frac{d^2G_{1/\hat{\alpha}}(u)}{du^2}\right|_{u=0} = \left.\frac{d^2E(X^u)}{du^2}\right|_{u=0} \tag{18}$$

$$E\left[\left(-\ln(\hat{\lambda})\right)^2\right] = \left.\frac{d^2G_{-\ln(\hat{\lambda})}(u)}{du^2}\right|_{u=0} = \left.\frac{d^2E(Y^u)}{du^2}\right|_{u=0} \tag{19}$$

Moreover, as proved in [2], for every non-negative random variable V such that $V \geq a \geq 0$ and $E(V^u)$ exists, it holds that

$$E(V^u) = a^u + u\int_a^\infty x^{u-1}\,\Pr(V > x)\,dx \tag{20}$$

Using the above equalities, second derivatives of $E(X^u)$ and $E(Y^u)$ will now be computed. As it holds that

$$\frac{da^u}{du} = \ln(a)\,a^u\,,\ \frac{dx^{u-1}}{du} = \ln(x)\,x^{u-1}\,, \tag{21}$$

the product rule yields:

$$\frac{dE(V^u)}{du} = \ln(a)\,a^u + \int_a^\infty x^{u-1}\,\Pr(V > x)\,dx + u\int_a^\infty \ln(x)\,x^{u-1}\,\Pr(V > x)\,dx \tag{22}$$

$$\frac{d^2E(V^u)}{du^2} = [\ln(a)]^2 a^u +$$

$$+2\int_a^\infty \ln(x)\,x^{u-1}\,\Pr(V > x)\,dx + u\int_a^\infty ln^2(x)\,x^{u-1}\,\Pr(V > x)\,dx \tag{23}$$

Theorem 1

$$E\left[\left(\ln(\hat{\lambda})\right)^2\right] \leq E[(\ln(S_m))^2] \tag{24}$$

Proof: As Y≥0, applying (23) to Y yields:

$$\left.\frac{d^2E(Y^u)}{du^2}\right|_{u=0} = 2\int_0^{\infty} \ln(x)\, x^{-1} \Pr(Y > x)\, dx \leq$$

$$\leq 2\int_1^{\infty} \ln(x)\, x^{-1} \Pr(Y > x)\, dx \tag{25}$$

As $S_m > S_i$ for i=1,...,m – 1, it follows from (16) that $Y < S_m$ and

$$\Pr[Y > x] \leq \Pr[s_m > x] = \Pr[\ln(s_m) > \ln(x)] \tag{26}$$

for x≥0. It thus holds that

$$\int_1^{\infty} \ln(x)\, x^{-1} \Pr(Y > x)\, dx \leq \int_1^{\infty} \ln(x)\, x^{-1} \Pr[\ln(s_m) > \ln(x)]\, dx =$$

$$= \int_1^{\infty} \ln(x) \frac{d\ln(x)}{dx} \Pr[\ln(s_m) > \ln(x)]\, dx \tag{27}$$

Substituting ln(x) by the variable s we obtain

$$\int_1^{\infty} \ln(x) \frac{d\ln(x)}{dx} \Pr[\ln(s_m) > \ln(x)]\, dx = \int_0^{\infty} s \Pr[\ln(s_m) > s]\, ds \leq$$

$$\leq \int_0^{\infty} s \Pr[|\ln(s_m)| > s]\, ds = \frac{1}{2} E[(\ln(s_m))^2] \tag{28}$$

where the last equality is a consequence of the fact, proved in [2], that for any random variable V such that $E(|V|^u) < \infty$, where u>0, it holds that

$$E(|V|^u) = u \int_a^{\infty} x^{u-1} \Pr(|V| > x)\, dx \tag{29}$$

Finally, from (19) and (25)-(28), we conclude that

$$E\left[\left(\ln(\hat{\lambda})\right)^2\right] = E\left[\left(-\ln(\hat{\lambda})\right)^2\right] \le E[(\ln(S_m))^2] \tag{30}$$

which completes the proof.

Theorem 2

$$E[(1/\hat{\alpha})^2] \le \left[1 + \sqrt{E[(\ln(S_m))^2]} + \sqrt{E[(\ln(T_1))^2]}\right]^2 \tag{31}$$

Proof: As X≥1, using (23) and integration by substitution we obtain

$$\left.\frac{d^2 E(X^u)}{du^2}\right|_{u=0} = 2\int_1^{\infty} \ln(x)\, x^{-1} \Pr(X > x)\, dx =$$

$$= 2\int_1^{\infty} \ln(x) \Pr(\ln(X) > \ln(x)) \frac{d \ln(x)}{dx} dx = 2\int_0^{\infty} s \Pr(\ln(X) > s)\, ds \tag{32}$$

Thus, in view of (18), we have the following formula:

$$E[(1/\hat{\alpha})^2] = 2\int_0^{\infty} s \Pr(\ln(X) > s)\, ds \tag{33}$$

Let us note that

$$X = \left(\prod_{i=1}^{m} \frac{S_m}{S_i}\right)^{1/m} < \left(\prod_{i=1}^{m} \frac{S_m}{T_1}\right)^{1/m} = \frac{S_m}{T_1} \tag{34}$$

hence

$$\Pr[\ln(X) > s] \leq \Pr\left[\ln\left(\frac{S_m}{T_1}\right) > s\right],\ s \geq 0 \tag{35}$$

It thus holds that

$$\int_0^\infty s \Pr[\ln(X) > s]\, ds \leq \int_0^\infty s \Pr\left[\ln\left(\frac{S_m}{T_1}\right) > s\right] ds \tag{36}$$

Substituting the variable s by ln(x) we obtain

$$\int_0^\infty s \Pr\left[\ln\left(\frac{S_m}{T_1}\right) > s\right] ds = \int_1^\infty \ln(x) \Pr\left[\ln\left(\frac{S_m}{T_1}\right) > \ln(x)\right] \frac{d\ln(x)}{dx} dx =$$

$$= \int_1^\infty \ln(x)\, x^{-1} \Pr\left(\frac{S_m}{T_1} > x\right) dx \tag{37}$$

Clearly, T_1 has Weibull distribution with density function $w_{\alpha,\lambda}$. Conditioning on T_1 yields:

$$\Pr\left(\frac{S_m}{T_1} > x\right) = \int_0^\infty \Pr\left(\frac{S_m}{T_1} > x \middle| T_1 = t\right) w_{\alpha,\lambda}(t) dt = \int_0^\infty \Pr\left(1 + \frac{S_{2,\dots,m}}{t} > x\right) w_{\alpha,\lambda}(t) dt \tag{38}$$

where $S_{2,\dots,m} = T_2 + \cdots + T_m$. Hence, by changing the order of integration, the last expression in (37) is transformed as follows:

$$\int_1^\infty \ln(x)\, x^{-1} \Pr\left(\frac{S_m}{T_1} > x\right) dx = \int_0^\infty \int_1^\infty \ln(x)\, x^{-1} \Pr\left(1 + \frac{S_{2,\dots,m}}{t} > x\right) dx\, w_{\alpha,\lambda}(t) dt \qquad (39)$$

We have:

$$\int_1^\infty \ln(x)\, x^{-1} \Pr\left(1 + \frac{S_{2,\dots,m}}{t} > x\right) dx =$$

$$= \int_1^\infty \ln(x)\, x^{-1} \Pr\left[\ln\left(1 + \frac{S_{2,\dots,m}}{t}\right) > \ln(x)\right] dx =$$

$$= \int_0^\infty s \Pr\left[\ln\left(1 + \frac{S_{2,\dots,m}}{t}\right) > s\right] ds =$$

$$= \int_0^1 s \Pr\left[\ln\left(1 + \frac{S_{2,\dots,m}}{t}\right) > s\right] ds + \int_1^\infty s \Pr\left[\ln\left(1 + \frac{S_{2,\dots,m}}{t}\right) > s\right] ds \qquad (40)$$

It is easy to check that the following implications hold for any positive random variable V:

$$\ln(1 + V) > s \geq 1 \;\Rightarrow\; V > 1 \;\Rightarrow\; \ln(1 + V) < \ln(V) + 1 \;\Rightarrow$$

$$\Rightarrow\; \Pr[\ln(1 + V) > s] \leq \Pr[\ln(V) + 1 > s] \qquad (41)$$

From (40) and (41) it follows that

$$\int_1^\infty \ln(x)\, x^{-1} \Pr\left(1 + \frac{S_{2,\dots,m}}{t} > x\right) dx \le \int_0^1 s ds + \int_1^\infty s \Pr\left[\ln\left(\frac{S_{2,\dots,m}}{t}\right) > s - 1\right] ds \qquad (42)$$

Substituting s–1 by r, using the fact that $S_m > S_{2,\dots,m}$, and applying (29) we obtain:

$$\int_0^1 s ds + \int_1^\infty s \Pr\left[\ln\left(\frac{S_{2,\dots,m}}{t}\right) > s - 1\right] ds \le \frac{1}{2} + \int_0^\infty (r+1) \Pr\left[\ln\left(\frac{S_m}{t}\right) > r\right] dr \le$$

$$\le \frac{1}{2} + \int_0^\infty (r+1) \Pr\left[\left|\ln\left(\frac{S_m}{t}\right)\right| > r\right] dr = \frac{1}{2} + \frac{1}{2} E\left[\left(\ln\left(\frac{S_m}{t}\right)\right)^2\right] + E\left[\left|\ln\left(\frac{S_m}{t}\right)\right|\right] =$$

$$= \frac{1}{2} + \frac{1}{2} E[(\ln(S_m) - \ln(t))^2] + E(|\ln(S_m) - \ln(t)|) \le$$

$$\le \frac{1}{2} + \frac{1}{2} E[(\ln(S_m))^2] + |\ln(t)| E(|\ln(S_m)|) + \frac{1}{2}[\ln(t)]^2 +$$

$$+ E(|\ln(S_m)|) + |\ln(t)| \qquad (43)$$

As T_1 has Weibull distribution, from (36)-(43) we obtain

$$\int_0^\infty s \Pr[\ln(X) > s]\, ds \le$$

$$\le \int_0^\infty \left[\frac{1}{2} + \frac{1}{2} E[(\ln(S_m))^2] + E(|\ln(S_m)|)\right] w_{\alpha,\lambda}(t) dt +$$

$$+ \int_0^\infty \left[|\ln(t)| E(|\ln(S_m)|) + \frac{1}{2}[\ln(t)]^2 + |\ln(t)|\right] w_{\alpha,\lambda}(t) dt =$$

$$= \frac{1}{2} + \frac{1}{2} E[(\ln(S_m))^2] + E(|\ln(S_m)|) +$$

$$+E(|\ln(S_m)|)E(|\ln(T_1)|) + \frac{1}{2} E[(\ln(T_1))^2] + E(|\ln(T_1)|) \tag{44}$$

It can be easily shown that

$$E(|V|) \leq \sqrt{E(V^2)} \tag{45}$$

for every random variable V such that $E(V^2) < \infty$. Now (31) holds in view of (33), (44), and (45), thus the proof is completed.

To compute the bounds defined by (24) and (31), we need a formula for $E[(\ln(S_m))^2]$, that will be derived in the next section.

3. A formula for $E[(\ln(S_m))^2]$

It was proved in [1] that

$$G_{\ln(S_m)}(u) = \frac{1}{\lambda^u} \Gamma\left(\frac{u}{\alpha} + 1\right) + \frac{u}{\alpha \lambda^u} \sum_{k=1}^{m-1} \frac{\Gamma\left(\frac{u}{\alpha} + k\right)}{k!} \tag{46}$$

Differentiation of the first component yields:

$$d \frac{1}{\lambda^u} \Gamma\left(\frac{u}{\alpha} + 1\right) / du = -\ln(\lambda)\, \lambda^{-u} \Gamma\left(\frac{u}{\alpha} + 1\right) + \lambda^{-u} \frac{1}{\alpha} \Gamma'\left(\frac{u}{\alpha} + 1\right) =$$

$$= \left[\frac{1}{\alpha}\Gamma'\left(\frac{u}{\alpha}+1\right) - \ln(\lambda)\,\Gamma\left(\frac{u}{\alpha}+1\right)\right]\lambda^{-u} \tag{47}$$

hence

$$d^2\frac{1}{\lambda^u}\Gamma\left(\frac{u}{\alpha}+1\right)/du^2 = \left[\frac{1}{\alpha^2}\Gamma''\left(\frac{u}{\alpha}+1\right) - \ln(\lambda)\frac{1}{\alpha}\Gamma'\left(\frac{u}{\alpha}+1\right)\right]\lambda^{-u} +$$

$$-\left[\frac{1}{\alpha}\Gamma'\left(\frac{u}{\alpha}+1\right) - \ln(\lambda)\,\Gamma\left(\frac{u}{\alpha}+1\right)\right]\ln(\lambda)\,\lambda^{-u} =$$

$$= \left[\frac{1}{\alpha^2}\Gamma''\left(\frac{u}{\alpha}+1\right) - 2\ln(\lambda)\frac{1}{\alpha}\Gamma'\left(\frac{u}{\alpha}+1\right) + [\ln(\lambda)]^2\Gamma\left(\frac{u}{\alpha}+1\right)\right]\lambda^{-u} \tag{48}$$

It thus holds that

$$d^2\frac{1}{\lambda^u}\Gamma\left(\frac{u}{\alpha}+1\right)/du^2\Big|_{u=0} = \frac{1}{\alpha^2}\Gamma''(1) - 2\ln(\lambda)\frac{1}{\alpha}\Gamma'(1) + [\ln(\lambda)]^2\Gamma(1) \tag{49}$$

Differentiation of the second component yields:

$$d\,\frac{u}{\alpha\lambda^u}\sum_{k=1}^{m-1}\frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!}/du = d\,\frac{u}{\alpha\lambda^u}/du\sum_{k=1}^{m-1}\frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} + \frac{u}{\alpha\lambda^u}\sum_{k=1}^{m-1}\frac{1}{\alpha}\frac{\Gamma'\left(\frac{u}{\alpha}+k\right)}{k!} \tag{50}$$

hence

$$d^2\frac{u}{\alpha\lambda^u}\sum_{k=1}^{m-1}\frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!}/du^2 = d^2\frac{u}{\alpha\lambda^u}/du^2\sum_{k=1}^{m-1}\frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} + d\,\frac{u}{\alpha\lambda^u}/du\sum_{k=1}^{m-1}\frac{1}{\alpha}\frac{\Gamma'\left(\frac{u}{\alpha}+k\right)}{k!} +$$

$$+ d\frac{u}{\alpha\lambda^u}/du \sum_{k=1}^{m-1}\frac{1}{\alpha}\frac{\Gamma'\left(\frac{u}{\alpha}+k\right)}{k!} + \frac{u}{\alpha\lambda^u}\sum_{k=1}^{m-1}\frac{1}{\alpha^2}\frac{\Gamma''\left(\frac{u}{\alpha}+k\right)}{k!} \tag{51}$$

We have:

$$d\left(\frac{u}{\alpha\lambda^u}\right)/du = \frac{\alpha\lambda^u - u\alpha\lambda^u \ln(\lambda)}{\alpha^2\lambda^{2u}} = \frac{1-u\ln(\lambda)}{\alpha\lambda^u} \tag{52}$$

$$d^2\left(\frac{u}{\alpha\lambda^u}\right)/du^2 = \frac{-\ln(\lambda)\alpha\lambda^u - [1-u\ln(\lambda)]\alpha\lambda^u\ln(\lambda)}{\alpha^2\lambda^{2u}} = \frac{-\ln(\lambda) - [1-u\ln(\lambda)]\ln(\lambda)}{\alpha\lambda^u} \tag{53}$$

hence

$$d\left(\frac{u}{\alpha\lambda^u}\right)/du\bigg|_{u=0} = \frac{1}{\alpha}, \quad d^2\left(\frac{u}{\alpha\lambda^u}\right)/du^2\bigg|_{u=0} = \frac{-2\ln(\lambda)}{\alpha} \tag{54}$$

It thus holds that

$$d^2\frac{u}{\alpha\lambda^u}\sum_{k=1}^{m-1}\frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!}/du^2\Bigg|_{u=0} = -\frac{2\ln(\lambda)}{\alpha}\sum_{k=1}^{m-1}\frac{\Gamma(k)}{k!} + \frac{2}{\alpha^2}\sum_{k=1}^{m-1}\frac{\Gamma'(k)}{k!} \tag{55}$$

Finally, from (49) and (55) we obtain

$$E[(\ln(S_m))^2] = d^2 G_{\ln(S_m)}(u)/du^2 = \frac{1}{\alpha^2}\Gamma''(1) + \frac{2\ln(\lambda)}{\alpha}\left(\gamma - \sum_{k=1}^{m-1}\frac{1}{k}\right) + [\ln(\lambda)]^2 +$$

$$+\frac{2}{\alpha^2}\sum_{k=1}^{m-1}\frac{\Gamma'(k)}{k!} \tag{56}$$

where the sums over k=1,...,m-1 are assumed to be equal to 0 for m=1. Thus, the above formula also holds for $S_1=T_1$.

4. Upper bounds for $Var[\ln(\hat{\lambda})]$ and $Var(1/\hat{\alpha})$

Combining the results of the two previous sections we obtain:

$$Var[\ln(\hat{\lambda})] = E\left[\left(\ln(\hat{\lambda})\right)^2\right] - \left[E\left(\ln(\hat{\lambda})\right)\right]^2 \leq$$

$$\leq E[(\ln(S_m))^2] - \left[\ln(\lambda) - \frac{1}{\alpha}\left(\frac{\ln(m)}{m} - \frac{1}{m} + \Gamma'(1) - \ln(m) + \sum_{j=1}^{m}\frac{1}{j}\right)\right]^2 \tag{57}$$

$$Var(1/\hat{\alpha}) = E[(1/\hat{\alpha})^2] - [E(1/\hat{\alpha})]^2 \leq$$

$$\leq \left[1 + \sqrt{E[(\ln(S_m))^2]} + \sqrt{E[(\ln(T_1))^2]}\right]^2 - \left(\frac{m-1}{m\alpha}\right)^2 \tag{58}$$

where $E\left(\ln(\hat{\lambda})\right)$ and $E(1/\hat{\alpha})$ have been found from (4) and (5), while $E[(\ln(S_m))^2]$ and $E[(\ln(T_1))^2]$ are given by (56).

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