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Reliability analysis of a flow network with inflow and outflow points and series-parallel-reducible structure

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Abstract

In this work we consider a flow network with directed links and three types of nodes: inflow points, transit-only nodes, and outflow points. Its structure can be reduced to a single component by means of the so-called series-parallel aggregation. The network's components are binary (i.e. they can be either operable or failed) and repairable, have constant failure and repair rates, and their states are mutually independent. Each operable component has a non-zero integer throughput, a failed component has zero throughput.

It is natural to measure the network's performance by the total demand satisfied (TDS) at all outflow points vs. the total demand required (TDR) at these points. Clearly, as each component undergoes failures and repairs, TDS can change over the [0, TDR] interval. Thus, the probability that TDS is no smaller than a given value d, where d≤TDR, referred to as d-availability can be regarded the basic reliability parameter of the considered network. An algorithm developed by the author, computing this parameter, is presented. This algorithm is a recursive procedure executed in the course of the series–parallel aggregation, operates on integer numbers, and has relatively low numerical complexity. The author has also defined a number of other reliability parameters characterizing the network's dynamically changing ability to satisfy the demands at the outflow points. These parameters are calculated using the network's d-availability and the d-importances (a multi-state generalization of the Birnbaum importance) of individual components. The presented results can be applied e.g. in the reliability analysis of water supply networks, gas pipeline systems, power transmission and/or distribution networks, etc.

1. Introduction

The current paper is a continuation of Malinowski (2013), where the capacity of a series-parallel-reducible system, defined as a function of its components' capacities, was considered. The subject matter considered is a flow network which delivers a commodity (water, oil, gas, electric power, etc.) from inflow to outflow points. The paper can be classified as a contribution to the field of reliability of flow networks regarded as multi-state repairable systems. Multi-state systems with binary or multi-state components have been studied with increasing interest during the recent decades. A broad overview of multi-state reliability models and methods used for their analysis, along with the comprehensive survey of relevant literature can be found in Lisnianski et al. (2010), and Lisnianski & Levitin (2003). Methods of computing failure/repair frequencies, intensities, and availabilities of multi-state systems are investigated in Chang et al. (2004), Druault-Vicard & Tanguy (2006), and Korczak (2007). As far as series-parallel flow networks are concerned, the literature on this subject is rather scarce, however some interesting results, related to the topic of this paper, can be found in Klinz & Woeginger (2004) and Krumke & Zeck (2013).

The considered network is assumed to have the following features:

1) It is composed of two-state independent components: inflow points, transit-only nodes, outflow points, and directed links. Clearly, links are transit-only components.

2) It has a series-parallel-reducible structure that can be reduced to a single component by means of stepwise series-parallel aggregation.

3) To the inflow points only outbound links are connected, to the transit nodes – both inbound and outbound links, a node to which only inbound links are connected is called terminal. An outflow point can be either a transit or a terminal one.

4) Each operable component has integer-valued throughput. The amount of flow fed into a component cannot exceed its throughput, and is divided into the demand satisfied at the component, and its feeding capacity (both integer-valued), as shown in Fig.1. Obviously, demand can be satisfied at outflow points alone, and terminal outflow points have no feeding capacity. A failed component's throughput is equal to zero.

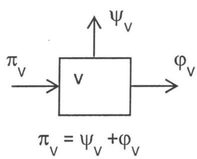

Fig. 1. The division of flow fed into a component

Let v be a component, and let c_v, π_v, d_v, ψ_v, and φ_v respectively denote the throughput of v, the amount of flow fed into v ($\pi_v \leq c_v$), the demand required at v, the demand satisfied at v, and the feeding capacity of v. Clearly, $d_v = 0$ for inflow points and transit-only components. The following equalities, based on the flow conservation law, are assumed to hold:

$\varphi_v = \max(\pi_v - d_v, 0)$: for inflow points and transit components (1)

$\varphi_v = 0$: for terminal outflow nodes (2)

$\psi_v = \min(d_v, \pi_v)$: for all components (3)

5) When a component fails its repair is started immediately. The time-to-failure and time-to-repair of each component are independent, exponentially distributed random variables.

6) The basic reliability measure of the considered network, referred to as d-availability, is the probability that the total demand satisfied (TDS) at all outflow points is greater or equal to an integer value d, where $d \in [0, D]$, D being the total demand.

In Fig. 2 the reliability block diagram (RBD) of a small exemplary network is presented; e_1 and e_6 are inflow points, e_2, e_3, e_4, e_7 – transit-outflow points, e_5, e_8 – transit-only nodes, and e_9 is a terminal outflow node. This network could be a fragment of a bigger one, if e_9 were a transit-outflow rather than a terminal outflow node. For simplicity, pipes are not included in the RBD, otherwise, additional boxes should be inserted between the existing ones.

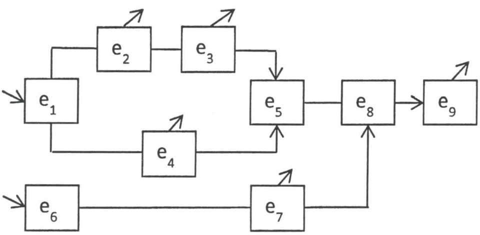

Figure 2. The RBD of a small exemplary network

The above described systems can be used as reliability models of various types of flow networks with series-parallel-reducible structures. Systems of this kind are usually assigned multiple degrees of functionality related to the percentage of TDS at the customer supply points. Their reliability analysis is thus based on parameters related to the time-points in which specific values of TDS are reached from above or below during the system operation. We will consider three such parameters, namely the already mentioned d-availability and the two d-intensities. They correspond to analogous parameters widely used in the reliability analysis of two-state systems. The first parameter, denoted A(d,t), is the probability that at time t the TDS is greater or equal to d. The remaining two, denoted Λ^+(d,t) and Λ^-(d,t), are the intensities with which the TDS crosses level d from below or above respectively. They will be referred to as transition intensities, where a transition takes place between the states "≥d" and "<d". It will be shown how to compute asymptotic (t→∞) values of these parameters, and how other essential reliability parameters of repairable flow networks can be derived from the asymptotic intensities. The presented method is illustrated by the calculation of A(d,t) for an exemplary network.

2. Detailed assumptions and notation

Let the adopted assumptions be divided into the three following groups:

<u>Regarding the network components</u>

e_i – the i-th component; i∈S, where S is the set of indices of the network's components.

c_i – the throughput of the operable e_i ; c_i is a fixed integer number

d_i – the demand required at e_i ; d_i is a fixed integer number

x_i : the state of e_i; $x_i = c_i$ or $x_i = 0$ depending whether e_i is operable or failed.

$\underline{x}$ – the vector of x_i, i∈S, i.e. $\underline{x} = [x_i, i\in S]$.

$X_i(t)$ – the (random) state of e_i at time t.

$\underline{X}(t)$ – the (random) vector of the components' states at time t, i.e. $\underline{X}(t) = [X_i(t), i\in S]$.

λ_i, μ_i – the failure or repair intensity of e_i , i.e.

$$\lambda_i = \lim_{u\to 0} \Pr[X_i(t+u) = 0 | X_i(t) = c_i] / u ,$$

$$\mu_i = \lim_{u\to 0} \Pr[X_i(t+u) = c_i | X_i(t) = 0] / u \qquad (4)$$

It is assumed that both intensities are constant, i.e. they do not depend on t. Clearly, this occurs if the time-to-failure and time-to-repair of e_i are exponentially distributed.

$p_i(t)$, $q_i(t)$ – the probability of e_i being operable or failed at time t, i.e.

$$p_i(t) = \Pr[X_i(t) = c_i]\,, \quad q_i(t) = \Pr[X_i(t) = 0] = 1 - p_i(t) \tag{5}$$

As shown in Barlow & Proschan (1975), the above probabilities are related to the failure/repair intensities of e_i in the following way:

$$p_i(t) = \frac{\mu_i}{\lambda_i+\mu_i} + \frac{\lambda_i}{\lambda_i+\mu_i}\exp[-(\lambda_i + \mu_i)t]\,, \quad q_i(t) = \frac{\lambda_i}{\lambda_i+\mu_i} - \frac{\lambda_i}{\lambda_i+\mu_i}\exp[-(\lambda_i + \mu_i)t] \tag{6}$$

<u>Regarding a two-state general system</u>

$\Phi(\underline{x})$ – a binary function expressing the system state in relation to the states of its components; $\Phi(\underline{x}) = 1$ or $\Phi(\underline{x}) = 0$ depending whether the system is operable or failed.

A system is called coherent if Φ is non-decreasing, and each component is relevant. See Barlow & Proschan (1975) for details. For a coherent system we can define a number of parameters given below.

Z(t) – the system state at time t, i.e. $Z(t) = \Phi(\underline{X}(t))$.

$I_i^B(t)$ – the Birnbaum importance of e_i at time t, defined as follows:

$$I_i^B(t) = \frac{\partial Pr[Z(t)=1]}{\partial p_i(t)} = \Pr[Z(t) = 1 | X_i(t) = c_i] - Pr[Z(t) = 1 | X_i(t) = 0] \tag{7}$$

$I_i^B(t)$ can be interpreted as the probability that the failure of e_i results in the failure of the system.

A(t), U(t) – the system availability or unavailability at time t, i.e.

$$A(t) = E[\mathrm{Z(t)}] = \Pr[\mathrm{Z(t)} = 1], \quad U(t) = 1 - E[Z(\mathrm{t})] = \Pr[\mathrm{Z(t)} = 0] \tag{8}$$

$N_f(s,t)$, $N_r(s,t)$ – the (random) number of times the system respectively fails or is repaired in the interval (s,t].

V(t), W(t) – the instantaneous system failure or repair frequency at time t, i.e.

$$V(t) = \lim_{\mathrm{u}\to 0} \mathrm{E}\left[N_f(\mathrm{t},\mathrm{t}+\mathrm{u})\right]/u,$$

$$W(t) = \lim_{\mathrm{u}\to 0} \mathrm{E}[N_r(\mathrm{t},\mathrm{t}+\mathrm{u})]/u \tag{9}$$

$\Lambda_f(t)$, $\Lambda_r(t)$ – the system failure or repair intensity at time t, i.e.

$$\Lambda_f(t) = \lim_{\mathrm{u}\to 0} \Pr[\mathrm{Z(t+u)} = 0 \mid \mathrm{Z(t)} = 1]/\mathrm{u},$$

$$\Lambda_r(t) = \lim_{\mathrm{u}\to 0} \Pr[\mathrm{Z(t+u)} = 1 \mid \mathrm{Z(t)} = 0]/\mathrm{u} \tag{10}$$

Clearly, (9) and (10) hold if the respective limits exist.

<u>Regarding a multi-state flow network</u>

C(t) – the network's throughput (i.e. the maximal flow that can run through it) at time t

$\Psi(t)$ – TDS at all outflow points when flow of magnitude C(t) runs through the network

A(d,t) – the network d-availability at time t, defined as follows:

$$A(d,t) = \Pr[\Psi(t) \geq d] \tag{11}$$

$I_i^B(d,t)$ – the Birnbaum d-importance of e_i at time t, defined as follows:

$$I_i^B(d,t) = \frac{\partial Pr[\Psi(t) \geq d]}{\partial p_i(t)} = Pr[\Psi(t) \geq d | X_i(t) = c_i] - Pr[\Psi(t) \geq d | X_i(t) = 0] \tag{12}$$

$I_i^B(d,t)$ can be interpreted as the probability of the event {a failure of e_i causes the TDS to fall from a level ≥d to a level <d}.

E_d^+ – the event {TDS rises from a level <d to a level ≥d}.

E_d^- – the event {TDS falls from a level ≥d to a level <d}.

$N_d^+(s,t)$, $N_d^-(s,t)$ – the (random) number of times the event E_d^+ or E_d^- occurs in the interval (s,t].

V(d,t), W(d,t) – the instantaneous frequency with which the event E_d^+ or E_d^- respectively occurs at time t, i.e.

$$V(d,t) = \lim_{u \to 0} \frac{1}{u} E[N_d^-(t, t+u)]$$

$$W(d,t) = \lim_{u \to 0} \frac{1}{u} E[N_d^+(t, t+u)] \tag{13}$$

$\Lambda^-(d,t)$, $\Lambda^+(d,t)$ – the intensity with which the event E_d^- or E_d^+ occurs at time t, i.e.

$$\Lambda^-(d,t) = \lim_{u \to 0} \Pr[\Psi(t+u) < d \mid \Psi(t) \geq d]/u$$

$$\Lambda^+(d,t) = \lim_{u \to 0} \Pr[\Psi(t+u) \geq d \mid \Psi(t) < d]/u \tag{14}$$

Clearly, (13) and (14) hold if the respective limits exist. It should also be noted that the above defined frequencies and intensities are functions of t, unlike the failure/repair intensities of single components.

3. Formulas for the transition frequencies and intensities

Let us first consider a two-state repairable system with independent two-state components. The instantaneous failure and repair frequencies of such a system are given by the following formulas:

$$V(t) = \sum_{i\in S} \lambda_i p_i(t) I_i^B(t)\,, \quad W(t) = \sum_{i\in S} \mu_i q_i(t) I_i^B(t) \tag{15}$$

The failure and repair intensities are related to the repair and failure frequencies in the following way:

$$\Lambda_f(t) = V(t)/A(t)\,, \quad \Lambda_r(t) = W(t)/U(t) \tag{16}$$

The proofs of (15) and (16) can be found in Schneeweiss (1981).

The quantities defined in Section 2 for multi-state flow networks can be expressed in terms of binary systems. For this purpose it is sufficient to assume that the system fails or its repair is completed when the event E_d^- or E_d^+ occurs, and use Ψ instead of Z in (7), (8), and (10). In consequence, (15) and (16) are transformed to the following formulas:

$$V(d,t) = \sum_{i\in S} \lambda_i p_i(t) I_i^B(d,t)\,, \quad W(d,t) = \sum_{i\in S} \mu_i q_i(t) I_i^B(d,t) \tag{17}$$

$$\Lambda^-(d,t) = V(d,t)/A(d,t)\,, \quad \Lambda^+(d,t) = W(\mathrm{d},t)/U(d,t) \tag{18}$$

To apply (17) and (18) for the purpose of computing $\Lambda^{-}(d,t)$ and $\Lambda^{+}(d,t)$, it is first necessary to find A(d,t), and $I_i^B(d,t)$: $i \in S$ (we assume that λ_i, μ_i, and, in view of (6), $p_i(t)$ for $i \in S$ are known). The knowledge of $\Lambda^{-}(d,t)$ and $\Lambda^{+}(d,t)$ has a key significance, because, as will be shown in Section 6, other essential reliability parameters of flow networks can be calculated based on $\Lambda^{-}(d,t)$ and $\Lambda^{+}(d,t)$. Let us also note that A(d,t) and $I_i^B(c,t)$: $i \in S$, are functions of $p_i(t)$: $i \in S$, thus their asymptotic values for $t \to \infty$, denoted by A(d) and $I_i^B(d)$: $i \in S$, are functions of $p_i = \lim_{t \to \infty} p_i(t)$: $i \in S$. As follows from (6), $p_i(t)$ quickly converges to p_i, hence these asymptotic values characterize, with good accuracy, the network's behavior in the long time horizon. In the next section it will be shown how to compute A(d) and $I_i^B(d)$ for series-parallel-reducible structures.

4. A method to compute A(d)

Let a module be a group of components or sub-modules connected in series or in parallel. We will denote a module by M_X, where X is the set of indices of the module's components. Let us adopt the following notation:

C_X – throughput of M_X, i.e. the maximal flow that can run through M_X

Ψ_X – total demand satisfied (TDS) at the outflow nodes of M_X when flow of magnitude C_X runs through M_X

Φ_X – feeding capacity of M_X when flow of magnitude C_X runs through M_X

C_X^{max}, Ψ_X^{max}, Φ_X^{max} – maximal values of C_X, Ψ_X, and Φ_X respectively

C_X, Ψ_X, and Φ_X are random variables whose values depend on the states of the components of M_X, while C_X^{max}, Ψ_X^{max} and Φ_X^{max} are constants. Obviously, the following equality holds:

$\Psi_X + \Phi_X = C_X$

In order to compute the d-availability for a network with series-parallel-reducible structure it will be necessary to find the probabilities $\Pr(\Phi_S = c, \Psi_S = d)$, $c = 0,\ldots,\Phi_S^{max}$, $d = 0,\ldots,\Psi_S^{max}$, where S denotes the set of indices of the network's components. In other words, our aim is to find all elements of the following matrix:

$P_S^{=} = [\ \Pr(\Phi_S = c, \Psi_S = d)\ ]$, $0 \le c \le \Phi_S^{max}$, $0 \le d \le \Psi_S^{max}$

Once $P_S^{=}$ is found, A(d) will be computed from the following equality:

$$A(d) = \Pr(\Psi_S \ge d) = \Pr(\Phi_S \ge 0, \Psi_S \ge d) =$$

$$= \sum_{b=d}^{\Psi_S^{max}} \sum_{a=0}^{\Phi_S^{max}} P_S^{=}(a,b)\,,\ \ d = 1, \ldots, \Psi_S^{max} \qquad (19)$$

i.e. all elements in the columns from d to Ψ_S^{max} are added.

In order to obtain $P_S^{=}$ we will use a technique called series-parallel aggregation. This is a stepwise process, in each step of which a group of components and/or sub-modules arranged in series or in parallel is aggregated into a single module. For example, the network in Fig. 2 is aggregated in the following steps:

Step 1: $e_2, e_3 \rightarrow M_{\{2,3\}} = ser(e_2, e_3)$

Step 2: $M_{\{2,3\}}, e_4 \rightarrow M_{\{2,3,4\}} = par(M_{\{2,3\}}, e_4)$

Step 3: $e_1, M_{\{2,3,4\}}, e_5 \rightarrow M_{\{1,\ldots,5\}} = ser(e_1, M_{\{2,3,4\}}, e_5)$

Step 4: $e_6, e_7 \rightarrow M_{\{6,7\}} = ser(e_6, e_7)$

Step 5: $M_{\{1,\ldots,5\}}, M_{\{6,7\}} \rightarrow M_{\{1,\ldots,7\}} = par(M_{\{1,\ldots,5\}}, M_{\{6,7\}})$

Step 6: $M_{\{1,\ldots,7\}}, e_8, e_9 \rightarrow M_{\{1,\ldots,9\}} = ser(M_{\{1,\ldots,7\}}, e_8, e_9)$

In order that $P_S^{=}$ can be found, the matrix

$P_Z^{=} = [\ Pr(\Phi_Z = c,\ \Psi_Z = d)\ ]\ ,\ 0 \leq c \leq \Phi_Z^{max}\ ,\ 0 \leq d \leq \Psi_Z^{max}$

is found in each step of the aggregation process, where M_Z denotes the module obtained in one such step. We will also need the matrix $P_Z^{\geq}$ defined as follows:

$P_Z^{=} = [\ Pr(\Phi_Z \geq c,\ \Psi_Z = d)\ ]\ ,\ 0 \leq c \leq \Phi_Z^{max}\ ,\ 0 \leq d \leq \Psi_Z^{max}$

$P_Z^{\geq}$ can be obtained from $P_Z^{=}$, and vice versa, by row-wise summation or subtraction. The elements of $P_Z^{=}$ or $P_Z^{\geq}$ are calculated with the use of the following theorems:

Theorem 1 (the matrix $P_{\{i\}}^{=}$ for the component e_i)

If e_i is a non-terminal component, then $P_{\{i\}}^{=}$ is calculated as follows:

For $c_i > d_i$:

$$\Pr(\varphi_i = a,\ \psi_i = b) = \begin{cases} p_i, & a = c_i - d_i \ \wedge\ b = d_i \\ q_i, & a = 0 \ \wedge\ b = 0 \\ 0, & otherwise \end{cases} \tag{20}$$

For $c_i \leq d_i$:

$$\Pr(\varphi_i = a,\ \psi_i = b) = \begin{cases} p_i, & a = 0 \ \wedge\ b = c_i \\ q_i, & a = 0 \ \wedge\ b = 0 \\ 0, & otherwise \end{cases} \tag{21}$$

If e_i is a terminal outflow node, then $P_{\{i\}}^{=}$ is a one-row matrix computed as follows:

For $c_i > d_i$:

$$\Pr(\varphi_i = 0,\ \psi_i = b) = \begin{cases} p_i, & b = d_i \\ q_i, & b = 0 \\ 0, & otherwise \end{cases} \tag{22}$$

For $c_i \le d_i$:

$$\Pr(\varphi_i = a,\ \psi_i = b) = \begin{cases} p_i, & b = c_i \\ q_i, & b = 0 \\ 0, & otherwise \end{cases} \qquad (23)$$

Proof:

Let us assume that e_i is a non-terminal component. The feeding capacity of operable e_i is equal to $\max(c_i - d_i, 0)$, and is equal to zero for failed e_i. The demand satisfied by operable e_i is equal to $\min(c_i, d_i)$, and is equal to 0 for failed e_i. As a consequence, the first two equalities in both (20) and (21) are obtained. The last equalities in (20) and (21) are obvious. (22) and (23) are proved analogously.

Theorem 2 (The matrix $P_{X\cup Y}^{=}$ or $P_{X\cup Y}^{\geq}$ for the module aggregated from M_X and M_Y)

If the modules M_X and M_Y are connected in parallel, then we have:

$$\Pr(\Phi_{X\cup Y} = c,\ \Psi_{X\cup Y} = d) =$$

$$= \sum_{\substack{a=0,\dots,c \\ b=0,\dots,d}} \Pr(\Phi_X = c - a,\ \Psi_X = d - b) \Pr(\Phi_Y = a,\ \Psi_Y = b) \qquad (24)$$

If the modules M_X and M_Y are connected in series, then it holds that

$$\Pr(\Phi_{X\cup Y} \geq c,\ \Psi_{X\cup Y} = d) =$$

$$= \sum_{b=0}^{d} \Pr(\Phi_X \geq c + b,\ \Psi_X = d - b) \Pr(\Phi_Y \geq c,\ \Psi_Y = b),\ c \geq 1,\ d \geq 0 \qquad (25)$$

and

$$\Pr(\Phi_{X\cup Y} \geq 0,\ \Psi_{X\cup Y} = d) =$$

$$= \Pr(\Phi_{X\cup Y} = 0,\ \Psi_{X\cup Y} = d) + \Pr(\Phi_{X\cup Y} \geq 1,\ \Psi_{X\cup Y} = d),\ d \geq 0 \tag{26}$$

where the second component on the right-hand side of (26) is found from (25), and the first one – from the following equality:

$$\Pr(\Phi_{X\cup Y} = 0,\ \Psi_{X\cup Y} = d) =$$

$$= \Pr(\Phi_X = 0,\ \Psi_X = d) + \Pr(\Phi_X \geq 1,\ \Psi_X = d)\Pr(\Phi_Y = 0,\ \Psi_Y = 0) +$$

$$+ \textstyle\sum_{b=1}^{d} \Pr(\Phi_X = b,\ \Psi_X = d - b)\Pr(\Psi_Y \geq b) +$$

$$+ \textstyle\sum_{b=1}^{d} \Pr(\Phi_X > b,\ \Psi_X = d - b)\Pr(\Phi_Y = 0,\ \Psi_Y = b),\ d \geq 0 \tag{27}$$

Note that the sum from b=1 to d is equal to zero for d=0.

Proof:

For the modules M_X and M_Y connected in parallel the following equalities hold:

$$\Phi_{X\cup Y} = \Phi_X + \Phi_Y,\ \Psi_{X\cup Y} = \Psi_X + \Psi_Y \tag{28}$$

If, in turn, M_X and M_Y are connected in series, then we have

$$\Phi_{X\cup Y} = \min[\max(\Phi_X - \Psi_Y, 0), \Phi_Y] \tag{29}$$

$$\Psi_{X\cup Y} = \Psi_X + \min(\Phi_X, \Psi_Y) \tag{30}$$

The equalities (28) – (30) are simple consequences of the flow conservation law. Clearly, (24) follows directly from (28) and the independence of M_X and M_Y. For the proof of (25) let us note that for c≥1 we have:

$$\Pr(\Phi_{X\cup Y} \geq c,\ \Psi_{X\cup Y} = d) =$$

$$= \sum_{b=0}^{d} \Pr(\Phi_{X\cup Y} \geq c,\ \Psi_X = d - b,\ \Psi_Y = b),\ c \geq 1,\ d \geq 0 \tag{31}$$

From (29) and (30) it follows that if $\Psi_X = d - b$ and $\Psi_Y = b$, then $\Phi_X \geq c + b$ and $\Phi_Y \geq c$ must hold so that $\Phi_{X\cup Y} \geq c$. Otherwise (i.e. $\Phi_X < c + b$ or $\Phi_Y < c$) $\Phi_{X\cup Y} < c$ according to (29). We thus have:

$$\sum_{b=0}^{d} \Pr(\Phi_{X\cup Y} \geq c,\ \Psi_X = d - b,\ \Psi_Y = b) =$$

$$= \sum_{b=0}^{d} \Pr(\Phi_X \geq c + b,\ \Phi_Y \geq c,\ \Psi_X = d - b,\ \Psi_Y = b) \tag{32}$$

In view of the independence of M_X and M_Y, (32) is equivalent to (25).

As (26) is obvious, it remains to prove (27). We have:

$$\Pr(\Phi_{X\cup Y} = 0,\ \Psi_{X\cup Y} = d) =$$

$$= \Pr(\Phi_{X\cup Y} = 0,\ \Psi_{X\cup Y} = d,\ \Phi_X = 0,\ \Psi_X = d) +$$

$$+ \Pr(\Phi_{X\cup Y} = 0,\ \Psi_{X\cup Y} = d,\ \Phi_X \geq 1,\ \Psi_X = d) +$$

$$+ \sum_{b=1}^{d} \Pr(\Phi_{X\cup Y} = 0,\ \Psi_{X\cup Y} = d,\ \Psi_X = d - b),\ d \geq 0 \tag{33}$$

Based on (29) and (30) the following argument is carried out: If $\Phi_X = 0$ and $\Psi_X = d$, then $\Phi_{X\cup Y} = 0$ and $\Psi_{X\cup Y} = d$ regardless of the values of Φ_Y and Ψ_Y. If $\Phi_X \geq 1$ and $\Psi_X = d$, then $\Phi_Y = 0$ and $\Psi_Y = 0$ must hold so that $\Phi_{X\cup Y} = 0$ and $\Psi_{X\cup Y} = d$ (see the figure below).

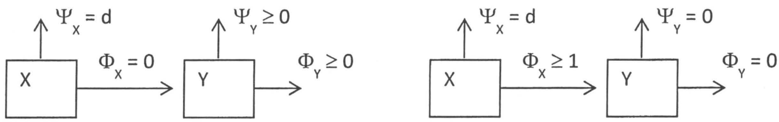

Fig. 3. Illustration of the case "$\Psi_X = d$"

If $\Psi_X = d - b$, $b\in\{1,\ldots,d\}$, then $\Phi_X \geq b$ must hold so that $\Phi_{X\cup Y} = 0$ and $\Psi_{X\cup Y} = d$. Otherwise (i.e. $\Phi_X < b$) $\Psi_{X\cup Y} < d$ according to (30). If $\Phi_X = b$, then $\Psi_Y \geq b$ (Φ_Y can have arbitrary value) must hold, and if $\Phi_X > b$, then $\Phi_Y = 0$ and $\Psi_Y = b$ must hold so that $\Phi_{X\cup Y} = 0$ and $\Psi_{X\cup Y} = d$ (see the figure below).

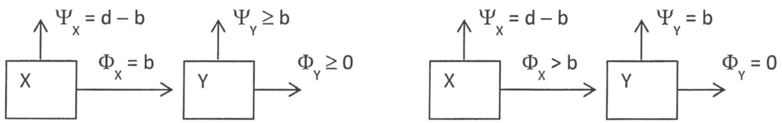

Fig. 4. Illustration of the case "$\Psi_X = d - b$, $b\in\{1,\ldots,d\}$"

Finally, we obtain:

$$\Pr(\Phi_{X\cup Y}=0,\ \Psi_{X\cup Y}=d)=$$

$$=\Pr(\Phi_X=0,\ \Psi_X=d)+\Pr(\Phi_X\geq 1,\ \Psi_X=d,\ \Phi_Y=0,\ \Psi_Y=d,\ )+$$

$$+\textstyle\sum_{b=1}^{d}\Pr(\Phi_X=b,\ \Psi_X=d-b,\ \Psi_Y\geq b)+$$

$$+\textstyle\sum_{b=1}^{d}\Pr(\Phi_X>b,\ \Psi_X=d-b,\ \Phi_Y=0,\ \Psi_Y=b),\ d\geq 0 \qquad (34)$$

In view of the independence of M_X and M_Y, (34) is equivalent to (27). The whole proof is thus completed.

Remark: let us note that for serially connected M_X and M_Y, the cases "c≥1" (formula 25) and "c=0" (formula 26) have to be considered separately, as the second case involves more underlying events expressed according to a different pattern.

Clearly, before computing the elements of the matrix $P_{X\cup Y}^{=}$ or $P_{X\cup Y}^{\geq}$, we need to specify its dimensions expressed by $\Phi_{X\cup Y}^{max}$ and $\Psi_{X\cup Y}^{max}$. The bounds on these dimensions are given in the following theorem:

Theorem 3

If M_X and M_Y are connected in series, then

$$\Phi_{X\cup Y}^{max}\leq\min(\Phi_X^{max},\Phi_Y^{max}) \qquad (35)$$

If, in turn, they are connected in parallel, then

$$\Phi_{X\cup Y}^{max}\leq\Phi_X^{max}+\Phi_Y^{max} \qquad (36)$$

Regardless of how M_X and M_Y are connected it holds that

$$\Psi_{X\cup Y}^{max}\leq\Psi_X^{max}+\Psi_Y^{max} \qquad (37)$$

Proof:

(35) – (37) follow directly from (28) – (30).

5. An illustrative example

To illustrate the method presented in Section 4 let us consider a small five-component network presented in Fig. 5. We assume that $c_1 = 5$, $c_2 = 4$, $c_3 = 4$, $c_4 = 5$, $c_5 = 3$, and $d_1 = 2$, $d_2 = 1$, $d_3 = 2$, $d_4 = 2$, $d_5 = 1$. Our aim is to compute A(d) – the probability that TDS at e_i: $i \in \{1,\dots,5\}$ is greater than d, provided that the network accommodates the maximum possible flow.

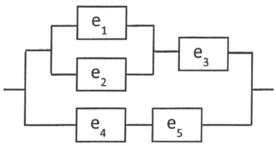

Fig. 5. A small exemplary network

Remark: the network has no inflow points or terminal outflow nodes, so it can be regarded as a module of a bigger network.

Using the formulas (24) – (27) we will compute the elements of $P_S^=$ in the following steps:

Step 1:

For parallel aggregation of e_1 and e_2 we use:

$$P_{\{1\}}^{=} = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_1 \end{bmatrix}, \quad P_{\{2\}}^{=} = \begin{bmatrix} q_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & p_2 \end{bmatrix}$$

obtained from (20), and the accordingly modified formula (24):

$$\Pr\left(\Phi_{\{1,2\}}=c,\ \Psi_{\{1,2\}}=d\right)=\sum_{\substack{a=0,\dots,c\\ b=0,\dots,d}}\Pr(\varphi_1=c-a,\ \psi_1=d-b)\Pr(\varphi_2=a,\ \psi_2=b)$$

which yield:

$$P_{\{1,2\}}^{=}=\begin{bmatrix} q_1q_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & q_1p_2 & p_1q_2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & p_1p_2 \end{bmatrix},\quad P_{\{1,2\}}^{\geq}=\begin{bmatrix} q_1q_2 & q_1p_2 & p_1q_2 & p_1p_2\\ 0 & q_1p_2 & p_1q_2 & p_1p_2\\ 0 & q_1p_2 & p_1q_2 & p_1p_2\\ 0 & q_1p_2 & p_1q_2 & p_1p_2\\ 0 & 0 & 0 & p_1p_2\\ 0 & 0 & 0 & p_1p_2\\ 0 & 0 & 0 & p_1p_2 \end{bmatrix}$$

Clearly,

$$P_{\{1,2\}}^{=}[c,d]=P_{\{1,2\}}^{\geq}[c,d]=0,\ c>\Phi_{\{1,2\}}^{max}=6,\ d>\Psi_{\{1,2\}}^{max}=3$$

Step 2:

For serial aggregation of $M_{\{1,2\}}$ and e_3 we use:

$$P_{\{3\}}^{=}=\begin{bmatrix} q_3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & p_3 \end{bmatrix},\ P_{\{3\}}^{\geq}=\begin{bmatrix} q_3 & 0 & p_3\\ 0 & 0 & p_3\\ 0 & 0 & p_3 \end{bmatrix}$$

obtained from (20), and the accordingly modified formulas (25) and (27):

$$\Pr\left(\Phi_{\{1,2,3\}}\geq c,\ \Psi_{\{1,2,3\}}=d\right)=$$

$$=\sum_{b=0}^{d}\Pr\left(\Phi_{\{1,2\}}\geq c+b,\ \Psi_{\{1,2\}}=d-b\right)\Pr(\varphi_3\geq c,\ \psi_3=b),\ c\geq 1,\ d\geq 0$$

$$\Pr\left(\Phi_{\{1,2,3\}}=0,\ \Psi_{\{1,2,3\}}=d\right)=$$

$$=\Pr\left(\Phi_{\{1,2\}}=0,\ \Psi_{\{1,2\}}=d\right)+\Pr\left(\Phi_{\{1,2\}}\geq 1,\ \Psi_{\{1,2\}}=d\right)\Pr(\varphi_3=0,\ \psi_3=0)+$$

$$=\sum_{b=1}^{d}\Pr\left(\Phi_{\{1,2\}}\geq b,\ \Psi_{\{1,2\}}=d-b\right)\Pr(\varphi_3=0,\ \psi_3=b),\ d\geq 0$$

which yield:

$$P_{\{1,2,3\}}^{\geq}=\begin{bmatrix} q_1q_2 & q_1p_2q_3 & p_1q_2q_3 & p_1p_2q_3+q_1p_2p_3 & p_1q_2p_3 & p_1p_2p_3\\ 0 & 0 & 0 & q_1p_2p_3 & p_1q_2p_3 & p_1p_2p_3\\ 0 & 0 & 0 & 0 & 0 & p_1p_2p_3 \end{bmatrix},$$

and

$$P^{=}_{\{1,2,3\}} = \begin{bmatrix} q_1q_2 & q_1p_2q_3 & p_1q_2q_3 & p_1p_2q_3 & 0 & 0 \\ 0 & 0 & 0 & q_1p_2p_3 & p_1q_2p_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_1p_2p_3 \end{bmatrix}$$

Step 3:

For serial aggregation of e_4 and e_5 we use:

$$P^{\geq}_{\{4\}} = \begin{bmatrix} q_4 & 0 & p_4 \\ 0 & 0 & p_4 \\ 0 & 0 & p_4 \\ 0 & 0 & p_4 \end{bmatrix}, \quad P^{\geq}_{\{5\}} = \begin{bmatrix} q_5 & p_5 \\ 0 & p_5 \\ 0 & p_5 \end{bmatrix}$$

obtained from (20), and the accordingly modified formulas (25) and (27):

$$\Pr\left(\Phi_{\{4,5\}} \geq c,\ \Psi_{\{4,5\}} = d\right) =$$

$$= \sum_{b=0}^{d} \Pr(\varphi_4 \geq c+b,\ \psi_4 = d-b) \Pr(\varphi_5 \geq c,\ \psi_5 = b),\ c \geq 1,\ d \geq 0$$

$$\Pr\left(\Phi_{\{4,5\}} = 0,\ \Psi_{\{4,5\}} = d\right) =$$

$$= \Pr(\varphi_4 = 0,\ \psi_4 = d) + \Pr(\varphi_4 \geq 1,\ \psi_4 = d) \Pr(\varphi_5 = 0,\ \psi_5 = 0) +$$

$$= \sum_{b=1}^{d} \Pr(\varphi_4 \geq b,\ \psi_4 = d-b) \Pr(\varphi_5 = 0,\ \psi_5 = b),\ d \geq 0$$

As a result we have:

$$P^{\geq}_{\{4,5\}} = \begin{bmatrix} q_4 & 0 & p_4q_5 & p_4p_5 \\ 0 & 0 & 0 & p_4p_5 \\ 0 & 0 & 0 & p_4p_5 \end{bmatrix}, \text{ and } P^{=}_{\{4,5\}} = \begin{bmatrix} q_4 & 0 & p_4q_5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_4p_5 \end{bmatrix}$$

Step 4:

For parallel aggregation of $M_{\{1,2,3\}}$ and $M_{\{4,5\}}$ we use the accordingly modified formula (24):

$$\Pr\left(\Phi_{\{1,2,3,4,5\}} = c,\ \Psi_{\{1,2,3,4,5\}} = d\right) =$$

$$= \sum_{\substack{a=0,\dots,c \\ b=0,\dots,d}} \Pr\left(\Phi_{\{1,2,3\}} = a,\ \Psi_{\{1,2,3\}} = b\right) \Pr\left(\Phi_{\{4,5\}} = c-a,\ \Psi_{\{4,5\}} = d-b\right)$$

It thus holds that

$$P^{=}_{\{1,\dots,5\}} = \begin{bmatrix} q_1q_2q_4 & q_1p_2q_3q_4 & (q_1p_4q_5 + p_1q_3q_4)q_2 & (q_1p_4q_5 + p_1q_4)p_2q_3 \\ 0 & 0 & 0 & q_1p_2p_3q_4 \\ 0 & 0 & 0 & q_1q_2p_4p_5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} p_1q_2q_3p_4q_5 & p_1p_2q_3p_4q_5 & 0 & 0 & 0 \\ p_1q_2p_3q_4 & q_1p_2p_3p_4q_5 & p_1q_2p_3p_4q_5 & 0 & 0 \\ q_1p_2q_3p_4p_5 & p_1(q_2q_3p_4p_5 + p_2p_3q_4) & p_1p_2q_3p_4p_5 & p_1p_2p_3p_4q_5 & 0 \\ 0 & 0 & q_1p_2p_3p_4p_5 & p_1q_2p_3p_4p_5 & 0 \\ 0 & 0 & 0 & 0 & p_1p_2p_3p_4p_5 \end{bmatrix}$$

Due to lack of space the first four columns of $P^{=}_{\{1,\dots,5\}}$ are placed above the remaining five.

Now A(d) can be computed from (19).

6. Computing d-intensities and other reliability parameters

From Section 4 we recall that once all the elements of $P^{=}_{S}$ are computed, A(d) is found from (19). Subsequently, $I_i^B(d)$ can be calculated from the following equality:

$$I_i^B(d) = A(d)|_{p_i=1} - A(d)|_{p_i=0} \tag{38}$$

which is a direct consequence of (12). Thus, to obtain the d-importance of e_i, one has to find A(d) for $p_i=1$ and $p_i=0$ (p_j being fixed for $j \neq i$), and apply (38). When A(d) and $I_i^B(d)$: $i \in S$ are known, $\Lambda^+(d)$, and $\Lambda^-(d)$ can be calculated from (17) and (18).

It has to be emphasized that in case of a larger network it is possible to compute only the numerical values of the elements of $P^{=}_{S}$ wherefrom the numerical values of A(d) and $I_i^B(d)$: $i \in S$ can be obtained. Clearly, deriving a closed formula as in the example given in Section 5, would be too complex a task in such a case.

It turns out that, using the intensities $\Lambda^+(d)$ and $\Lambda^-(d)$, it is possible to compute other important reliability parameters characterizing the fluctuations of TDS over time. Recalling that $\Psi(t)$ is the TDS at time t, and E_d^- and E_d^+ are events related to crossing level d by the TDS, let us adopt the following definitions:

$T_{k,d}$ – length of the k-th period during which $\Psi(t)\geq d$,

$R_{k,d}$ – length of the k-th period in which $\Psi(t)<d$,

$\tau_{k,d}$, $\rho_{k,d}$ – expected lengths of $T_{k,d}$ or $R_{k,d}$ respectively,

$n_d^-(t)$, $n_d^+(t)$ – the expected number of occurrences of the event E_d^- or E_d^+ respectively in a time interval of length t (e.g. in a month), provided that the system has long been in operation at the beginning of this interval.

Following the argument from Malinowski (2013) it can be easily proved that $T_{k,d}$ and $R_{k,d}$ have asymptotically ($k\to\infty$) exponential distributions with the parameters $\Lambda^-(d)$ and $\Lambda^+(d)$ respectively, hence it holds that

$$\tau_d = \lim_{k\to\infty} \tau_k = 1/\Lambda^-(c)\,,$$
$$\rho_d = \lim_{k\to\infty} \rho_k = 1/\Lambda^+(c) \qquad (39)$$

In the same way as in the above cited reference, it can be shown that

$$n_d^-(t) = n_d^+(t) = \frac{t}{\frac{1}{\Lambda^+(d)} + \frac{1}{\Lambda^-(d)}} = \frac{t}{\rho_d + \tau_d} \qquad (40)$$

Remark: It is important to consistently use the same units of time for the parameters in (39) and (40). For example, if $\Lambda^-(d)$ and $\Lambda^+(d)$ are given in h^{-1} (hour to the power –1), then $n_d^-(t)$

is the expected number of occurrences of E_d^- in t hours. In such a case the expected number of occurrences of E_d^- in a year is equal to $n_d^-(8760)$.

Conclusion

The paper deals with the reliability analysis of a flow network with directed links, multiple inflow and outflow points, and a structure that can be reduced to a single component by series-parallel aggregation. For such a network three basic reliability parameters were defined: d-availability A(d) – the probability with which the total demand satisfied at all outflow points (TDS) is greater or equal to d, and two d-intensities $\Lambda^+(d)$ and $\Lambda^-(d)$ with which TDS is respectively reached from below or above in the course of the failure-repair process. Based on the assumption that the individual components' throughputs are integer valued, a method of computing these parameters was developed. This method has rather low numerical complexity (polynomial with degree 3) compared to most known methods for network reliability calculation. It was also demonstrated how other parameters, frequently used in the reliability analysis of flow networks, can be found using the three basic parameters – A(d), $\Lambda^+(d)$ and $\Lambda^-(d)$. The obtained results can be applied to water supply networks, hydraulic systems, oil or gas pipeline systems, electric power networks, etc.

It should be noted that the matrix $P_Z^=$, where MZ is a network module obtained by series-parallel aggregation, can be used to calculate the probabilities $\Pr(\Phi_Z = c)$, $c=1,\ldots,\Phi_Z^{max}$, i.e.

$$\Pr(\Phi_Z = c) = \sum_{d=0}^{\Psi_Z^{max}} P_S^=(c,d), \quad c = 1, \ldots, \Phi_Z^{max} \qquad (41)$$

These probabilities can also be useful in the reliability analysis of the considered networks, as it may be essential to know the probabilities of the feeding capacities of modules which provide input at the specific network junctions. E.g., in the network depicted in Fig. 2, the modules $M_{\{1,\ldots,5\}}$ and $M_{\{6,7\}}$ provide input to the module $M_{\{8,9\}}$. Clearly, the demand satisfied at e_9 depends on the total feeding capacities of $M_{\{1,\ldots,5\}}$ and $M_{\{6,7\}}$. This example shows that the probabilities $\Pr(\Phi_Z = c)$ are essential for the analysis of individual outflow points.

The considered network model can be extended by adding one more type of node, i.e. transit-inflow node presented in Fig. 6. Let b_v be the capability of the source connected to v, and ψ_v – the amount of flow fed into v from this source. The remaining parameters are defined in Section 1. We have:

$$\psi_v = \min(b_v, c_v - \pi_v) \tag{42}$$

$$\varphi_v = \pi_v + \psi_v \tag{43}$$

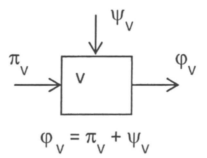

Fig. 6. A diagram of a transit-inflow node

By adopting this extension one is not restricted to networks where inflows can only take place at the initial nodes (i.e. nodes to which only outbound links are connected – see Assumption 4 in Section 1).

The presented method can also be applied to networks with arbitrary structures (modeled by directed graphs), but it has to be combined with the factorization method transforming a graph into a number of series-parallel derivative graphs. Combining these two methods along with taking transit-inflow nodes into consideration will be a topic of further research.

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