

41/2011

Raport Badawczy
Research Report

RB/35/2011

**Probabilistic analysis
of the Two-Constraint Binary
Knapsack Problem;
cases of small and large constraints
right-hand-sides values**

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Warszawa 2011

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December 5, 2011

Abstract

The paper deals with the Two-Constraint Binary Knapsack Problem, which is special case of Multi-Constraint Knapsack Problem, with 2 constraints only. It is assumed that some of the problem coefficients are realizations of mutually independent random variables. Asymptotical probabilistic properties of selected problem characteristics are investigated for the 2 specific cases corresponding to large and small values of the Lagrange multipliers.

1 Introduction

Let us consider a Two-Constraint Binary Knapsack Problem in the following formulation:

$$\begin{aligned} z_{OPT}(n) &= \max \sum_{i=1}^n c_i \cdot x_i \\ \text{subject to} \quad & \sum_{j=1}^2 a_{ji} \cdot x_i \leq b_j(n) \\ \text{where } j &= 1, 2, \quad x_i = 0 \text{ or } 1 \end{aligned} \tag{1}$$

Without restricting the generality of considerations it may be also assumed that:

$$b_1(n) \leq b_2(n).$$

It is assumed that:

$$c_i > 0, a_{ji} > 0, 0 < b_j(n) \leq \sum_{i=1}^n a_{ji}, \quad i = 1, \dots, n, \quad j = 1, 2.$$

The assumptions that $c_i, a_{ji} > 0, 0 < b_j(n) \leq \sum_{i=1}^n a_{ji}, i = 1, \dots, n, j = 1, 2$, are supposed to avoid the trivial and degenerated problems. More precisely interpretation of the $a_{ji} = 0$ or $c_i = 0$ is far not obvious. When $b_j(n) > \sum_{i=1}^n a_{ji}$ then the corresponding constraint is always fulfilled and therefore it may be removed from the problem formulation, otherwise if $b_j(n) = 0$ then (1) has only the trivial solution i.e. $z_{OPT}(n) = 0$.

Two-Constraint Binary Knapsack Problem is special case of the binary multiconstraint knapsack problem, also known as m -constraint knapsack problem, see Nemhauser and Wolsey [10] and Martello and Toth [7], where in general case there is arbitrary number m of constraints, i.e. $j = 1, \dots, m$. Another important special case of the multiconstraint knapsack problem is classical (single constraint) or, in other words, Binary Knapsack Problem, which have only one constraint, i.e. $j = 1$ (see Martello and Toth [7]). In the Szkatula's papers see [13] and [14] probabilistic analysis results of the different cases of the binary multiconstraint knapsack problem were presented. Moreover full case of the classical (single constraint) Binary Knapsack Problem was considered in the paper [14].

The Multi-Constraint Knapsack Problem is well known to be \mathcal{NP} hard, moreover, when $m \geq 2$, it is \mathcal{NP} hard in the strong sense (see Garey and Johnson [3]). It does mean that Two-Constraint Binary Knapsack Problem (1) is also \mathcal{NP} hard in the strong sense. Classical (one-constraint) Binary Knapsack Problem is \mathcal{NP} hard combinatorial optimization problem, however not in the strong sense.

The papers by Frieze and Clarke [2], Mamer and Schilling [6], Schilling [11] and [12] investigate the asymptotic value of $z_{OPT}(n)$ for the random model of Multi-Constraint Knapsack Problem, where $b_j(n) = 1, j = 1, \dots, m$. Papers by Szkatula [13] and [14] were dealing with the random model of the Multi-Constraint Knapsack Problem, where $b_j(n)$ are not restricted to be equal to 1. Papers by Meanti, Rinnooy Kan, Stougie and Vercellis [9], Lee and Oh [4] consider more general random models of Multi-Constraint Knapsack Problem but only for $j = 1, 2$ some analytical results describing the growth of $z_{OPT}(n)$ were obtained.

The aim of the present paper is to analyze the growth of the asymptotic value of $z_{OPT}(n)$ for the class of random Two-Constraint Binary Knapsack Problems (1) with possibly full spectrum of the constraints right-hand-sides values. Two-Constraint Binary Knapsack Problem is important special case of the general Multi-Constraint Knapsack Problem, see Martello and Toth [8]. It is \mathcal{NP} hard in the strong sense combinatorial optimization problem. Results of the probabilistic analysis of this problem may allow to describe asymptotic behavior of the $z_{OPT}(n)$ for practically all combinations of values of $b_1(n)$ and $b_2(n)$ as well as other problem coefficients (considered as realizations of the random variables). Those results may help to better understand number of the theoretical issues related to Two-Constraint Binary Knapsack Problems as well as enable construction of more efficient algorithms for solving the practical instances of the (1).

2 Definitions

The following definitions are necessary for the further presentation:

Definition 1 We denote $V_n \approx Y_n$, where $n \rightarrow \infty$, if

$$Y_n \cdot (1 - o(1)) \leq V_n \leq Y_n \cdot (1 + o(1))$$

when V_n, Y_n are sequences of numbers, or

$$\lim_{n \rightarrow \infty} P\{Y_n \cdot (1 - o(1)) \leq V_n \leq Y_n \cdot (1 + o(1))\} = 1$$

when V_n is a sequence of random variables and Y_n is a sequence of numbers or random variables, where $\lim_{n \rightarrow \infty} o(1) = 0$ as it is usually presumed.

Definition 2 We denote $V_n \preceq Y_n (V_n \succeq W_n)$ if

$$V_n \leq (1 + o(1)) \cdot Y_n \quad (V_n \geq (1 - o(1)) \cdot W_n)$$

when $V_n, Y_n (W_n)$ are sequences of numbers, or

$$\lim_{n \rightarrow \infty} P\{V_n \leq (1 + o(1)) \cdot Y_n\} = 1 \quad (\lim_{n \rightarrow \infty} P\{V_n \geq (1 - o(1)) \cdot W_n\} = 1)$$

when V_n is a sequence of random variables and $Y_n (W_n)$ is a sequence of numbers or random variables, where $\lim_{n \rightarrow \infty} o(1) = 0$.

Definition 3 We denote $V_n \cong Y_n$ if there exist constants $c'' \geq c' > 0$ such that

$$c' \cdot Y_n \preceq V_n \preceq c'' \cdot Y_n$$

where Y_n, V_n are sequences of numbers or random variables.

The following random model of (1) will be considered in the paper:

- $n \rightarrow \infty, i = 1, \dots, n, j = 1, 2$.
- c_i, a_{ji} are realizations of mutually independent random variables and moreover c_i, a_{ji} are uniformly distributed over $(0, 1]$.
- $0 < \delta \leq b_1(n) \leq b_2(n) \leq n/2, b_j(n) \leq b_j(n+1)$, for every $n \geq 1$ and all $b_j(n), j = 1, 2$, are deterministic, where δ is a constant.

Under the assumptions made about c_i, a_{ji} and $b_j(n)$ the following always hold

$$0 \leq z_{OPT}(n) \leq \sum_{i=1}^n c_i \leq n, \quad \delta \leq b_j(n) \leq \sum_{i=1}^n a_{ji} \leq n, \quad j = 1, 2. \quad (2)$$

Moreover, from the strong law of large numbers it follows that

$$\sum_{i=1}^n c_i \approx E(c_1) \cdot n = n/2, \quad \sum_{i=1}^n a_{ji} \approx E(a_{11}) \cdot n = n/2.$$

Therefore, it is justified to enhance formula (2) in the following way:

$$0 \leq z_{OPT}(n) \preceq n/2, \quad 0 < \delta \leq b_1(n) \leq b_2(n) \preceq n/2 \quad (3)$$

Formula (3) shows that random model of the Two-Constraint Binary Knapsack Problem (1) is complete in the sense that nearly all possible instances of

the problem are considered. In this respect the model where $b_1(n) = b_2(n) = 1$ is just a very special case. Taking into account that $\sum_{i=1}^n a_{ji} \approx n/2$ assumption that $b_j(n) \leq b_j(n+1)$, for all $n \geq 1$, is quite logical.

The growth of $z_{OPT}(n)$ - value of the optimal solution of the problem (1) may be influenced by the problem coefficients, namely:

$$n, c_i, a_{ji}, b_1(n), b_2(n), \text{ where } i = 1, \dots, n.$$

It is assumed that c_i, a_{ji} are realizations of the random variables and therefore their impact on the $z_{OPT}(n)$ growth is in this case indirect. Moreover, we have assumed that $n \rightarrow \infty$. The aim of the probabilistic analysis is to investigate asymptotic behavior of $z_{OPT}(n)$ when $n \rightarrow \infty$. The impact of the right-hand-side values - $b_1(n), b_2(n)$ - is well illustrated by the Lagrange function and the problem dual to (1), see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [9], Szkatula [13] and [14]. Due to the very complicated formulas, impossible to handle in the general case, the papers by Szkatula [13] and [14] investigate only two important special cases of values of constraints right hand sides in the case of Multi-Constraint Knapsack Problem.

3 Lagrange and dual estimations

When the general knapsack type problem, with one or many constraints, is considered then Lagrange function and the corresponding dual problems, see Averbakh [1], Meanti, Rinnooy Kan, Stougie and Vercellis [9], Szkatula [13] and [14] are very useful tools to perform various kind of analyses of the original problem. In the specific case of the Two-Constraint Binary Knapsack Problem Lagrange function of the problem (1) may be formulated as follows:

$$\begin{aligned} L_n(x) &= \sum_{i=1}^n c_i \cdot x_i + \sum_{j=1}^2 \lambda_j \cdot \left(b_j(n) - \sum_{i=1}^n a_{ji} \cdot x_i \right) = \\ &= \sum_{j=1}^2 \lambda_j + \sum_{i=1}^n \left(c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} \right) \cdot x_i \end{aligned}$$

where $x = [x_1, \dots, x_n]$ and $\Lambda = [\lambda_1, \lambda_2]$ - vector of Lagrange multipliers. Moreover, let for every $\Lambda, \lambda_j \geq 0, j = 1, 2$:

$$\phi_n(\Lambda) = \max_{x \in \{0,1\}^n} L_n(x, \Lambda) = \max_{x \in \{0,1\}^n} \left\{ \sum_{j=1}^2 \lambda_j \cdot b_j(n) + \sum_{i=1}^n \left(c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} \right) x_i \right\}.$$

Using the following notation:

$$\begin{aligned}
 x_i(\Lambda) &= \begin{cases} 1 & \text{if } c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \\
 c_i(\Lambda) &= \begin{cases} c_i & \text{if } c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \\
 a_{ji}(\Lambda) &= \begin{cases} a_{ji} & \text{if } c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned} \tag{4}$$

we have for every Λ , $\lambda_j \geq 0$, $j = 1, 2$:

$$\begin{aligned}
 \phi_n(\Lambda) &= \sum_{j=1}^2 \lambda_j \cdot b_j(n) + \sum_{i=1}^n \left(c_i - \sum_{j=1}^2 \lambda_j \cdot a_{ji} \right) \cdot x_i(\Lambda) = \\
 &= \sum_{j=1}^2 \lambda_j \cdot b_j(n) + \sum_{i=1}^n \left(c_i(\Lambda) - \sum_{j=1}^2 \lambda_j \cdot a_{ji}(\Lambda) \right)
 \end{aligned}$$

Obviously for $i = 1, \dots, n$, $j = 1, 2$,

$$c_i(\Lambda) = c_i \cdot x_i(\Lambda), \quad a_{ji}(\Lambda) = a_{ji} \cdot x_i(\Lambda).$$

Dual problem to Two-Constraint Binary Knapsack Problem (1) maybe formulated as follows:

$$\Phi_n^* = \min_{\Lambda \geq 0} \phi_n(\Lambda). \tag{5}$$

For every $\Lambda \geq 0$ the following holds:

$$z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \sum_{j=1}^2 \lambda_j (b_j(n) - s_j(\Lambda)). \tag{6}$$

Let us denote:

$$\begin{aligned}
 z_n(\Lambda) &= \sum_{i=1}^n c_i \cdot x_i(\Lambda) = \sum_{i=1}^n c_i(\Lambda), \quad s_j(\Lambda) = \sum_{i=1}^n a_{ji} \cdot x_i(\Lambda) = \sum_{i=1}^n a_{ji}(\Lambda), \\
 S_n(\Lambda) &= \sum_{j=1}^2 \lambda_j \cdot s_j(\Lambda), \quad B(\Lambda) = \sum_{j=1}^2 \lambda_j \cdot b_j(n).
 \end{aligned}$$

By definition of $c_i(\Lambda)$ and $a_{ji}(\Lambda)$, see (4), we have:

$$c_i(\Lambda) \geq \sum_{j=1}^2 \lambda_j \cdot a_{ji}(\Lambda), \quad i = 1, \dots, n,$$

and therefore

$$z_n(\Lambda) \geq S_n(\Lambda). \quad (7)$$

For certain Λ , $x_i(\Lambda)$ given by (4) may provide feasible solution of (1), i.e.:

$$s_j(\Lambda) \leq b_j(n) \quad \text{for every } j = 1, 2. \quad (8)$$

Then:

$$z_n(\Lambda) \leq z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + B(\Lambda) - S_n(\Lambda). \quad (9)$$

If (8) holds, then the below inequality also holds:

$$B(\Lambda) - S_n(\Lambda) \geq 0.$$

From (7) we get:

$$\frac{\phi_n(\Lambda)}{z_n(\Lambda)} = \frac{z_n(\Lambda)}{z_n(\Lambda)} + \frac{B(\Lambda) - S_n(\Lambda)}{z_n(\Lambda)} \leq 1 + \frac{B(\Lambda) - S_n(\Lambda)}{S_n(\Lambda)}.$$

Therefore if (8) holds, then the following inequality also holds:

$$1 \leq \frac{z_{OPT}(n)}{z_n(\Lambda)} \leq \frac{\Phi_n^*}{z_n(\Lambda)} \leq \frac{\phi_n(\Lambda)}{z_n(\Lambda)} \leq \frac{B(\Lambda)}{S_n(\Lambda)}. \quad (10)$$

Formula (10) shows, that if there exists such a set of Lagrange multipliers $\Lambda(n)$ which is fulfilling the formula (8) and if the formula below holds:

$$\lim_{n \rightarrow \infty} \frac{B(\Lambda(n))}{S_n(\Lambda(n))} = 1 \quad (11)$$

then, due to (10), $\lim_{n \rightarrow \infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1$ and therefore $x_i(\Lambda(n))$, $i = 1, \dots, n$, given by (4), is the asymptotically sub-optimal solution of the Two-Constraint Binary Knapsack Problem (1). Moreover the value of $z_n(\Lambda(n))$ is an asymptotical approximation of the optimal solution value of the Two-Constraint Binary Knapsack Problem i.e. $z_{OPT}(n)$.

4 Probabilistic analysis

In the present section of the paper some probabilistic properties of the Two-Constraint Binary Knapsack Problem (1) will be investigated. We have assumed that c_i , a_{ji} , $i = 1, \dots, n$, $j = 1, 2$ are realizations of mutually independent random variables and moreover c_i , a_{ji} are uniformly distributed over $(0, 1]$. Moreover we have assumed that $0 < \delta \leq b_1(n) \leq b_2(n) \leq n/2$, $b_j(n) \leq b_j(n+1)$. In addition we will assume that Lagrange multipliers λ_1 and λ_2 , $\lambda_2 \leq \lambda_1$, $\Lambda = (\lambda_1, \lambda_2)$ are also deterministic. Monotonicity of constraints right hand sides, $b_1(n) \leq b_2(n)$, is in this case determining monotonicity of the Lagrange multipliers, $\lambda_2 \leq \lambda_1$. This is often used in the literature probabilistic model of the general knapsack problems and it suits very well also to Two-Constraint Binary Knapsack Problem (1).

Let us first observe that due to the assumptions made the following holds, for $i = 1, \dots, n, j = 1, 2$:

$$P(a_{ji} < x) = \begin{cases} 0 & \text{when } x \leq 0 \\ x & \text{when } 0 < x \leq 1 \\ 1 & \text{when } x \geq 1 \end{cases}, P(c_i < x) = \begin{cases} 0 & \text{when } x \leq 0 \\ x & \text{when } 0 < x \leq 1 \\ 1 & \text{when } x \geq 1 \end{cases}. \quad (12)$$

In order to proceed with probabilistic analysis of the Two-Constraint Binary Knapsack Problem (1) it is necessary to consider probabilistic distribution of the following random variables

$$\sum_{j=1}^k \lambda_j \cdot a_{ji}, \quad k = 1 \text{ or } 2$$

Let $(x)_+ = \frac{|x| + x}{2} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$, $j^* = \begin{cases} 1 & \text{if } j = 2 \\ 2 & \text{if } j = 1 \end{cases}$, Then for or $i = 1, \dots, n, j = 1, 2$, the following holds:

$$\begin{aligned} F_1(x, \lambda_j) &= P\{\lambda_j \cdot a_{ji} < x\} = \frac{1}{\lambda_j} ((x)_+ - (x - \lambda_j)_+), \\ F_2(x, \Lambda) &= P\{\lambda_1 \cdot a_{1i} + \lambda_2 \cdot a_{2i} < x\} = \frac{1}{\lambda_j} \int_0^1 F_1(x - \lambda_j \cdot t, \Lambda \setminus \lambda_j) dt = (13) \\ &= \frac{1}{\lambda_1 \cdot \lambda_2} ((x)_+^2 - (x - \lambda_1)_+^2 - (x - \lambda_2)_+^2 + (x - \lambda_1 - \lambda_2)_+^2) \end{aligned}$$

The distribution functions of the random variables $a_{ji}(\Lambda)$, $c_i(\Lambda)$, $i = 1, \dots, n$, $j = 1, 2$ are:

$$\begin{aligned} G_{ji}(x, \Lambda) &= P\{a_{ji}(\Lambda) < x\} = \\ &= P\left\{a_{ji} < x \cup a_{ji} \geq x \cap \sum_{k=1}^2 \lambda_k \cdot a_{ik} \geq c_i\right\} = \quad (14) \\ &= 1 - \int_x^1 \int_0^1 F_1(r - \lambda_j \cdot t, \Lambda \setminus \lambda_j) dr dt \end{aligned}$$

$$\begin{aligned} H_i(x, \Lambda) &= P\{c_i(\Lambda) < x\} = \\ &= P\left\{c_i < x \cup c_i \geq x \cap \sum_{k=1}^2 \lambda_k \cdot a_{ik} \geq c_i\right\} = \quad (15) \\ &= 1 - \int_x^1 F_2(t, \Lambda) dt, \end{aligned}$$

Using above formulas (14) and (15) expectations of the $a_{ji}(\Lambda)$, $c_i(\Lambda)$ could be expressed as follows:

$$\begin{aligned}
E(a_{ji}(\Lambda)) &= \int_0^1 x dG_{ji}(x, \Lambda) = \int_0^1 x \int_0^1 F_1(r - \lambda_j \cdot x, \Lambda \setminus \lambda_j) dr dx = \quad (16) \\
&= \frac{1}{\lambda_j} \left(\int_0^1 x \int_0^1 ((r - x \cdot \lambda_j)_+ - (r - x \cdot \lambda_j - \lambda_j)_+) dr dx \right)
\end{aligned}$$

$$\begin{aligned}
E(c_i(\Lambda)) &= \int_0^1 x dH_i(x, \Lambda) = \int_0^1 x \cdot F_2(x, \Lambda) dx = \quad (17) \\
&= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \int_0^1 x \cdot ((x)_+ - (x - \lambda_1)_+^2 - (x - \lambda_2)_+^2 + (x - \lambda_1 - \lambda_2)_+^2) dx = \\
&= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \left(\frac{1}{4} - \int_0^1 x \cdot ((x - \lambda_1)_+^2 + (x - \lambda_2)_+^2 - (x - \lambda_1 - \lambda_2)_+^2) dx \right).
\end{aligned}$$

It is easy to observe that above formulas (16) and (17) may take different values, depending on the mutual relations between λ_1, λ_2 and x, r since several items of the formulas above may become 0 or be strongly positive. 4 specific cases could be distinguished for $i = 1, \dots, n, j = 1, 2$:

1. Case of "large" values of the Lagrange multipliers $1 \leq \lambda_2 \leq \lambda_1$. In this case:

$$\begin{aligned}
E(a_{ji}(\Lambda)) &= \frac{1}{\lambda_j} \int_0^{1/\lambda_j} x \int_{x \cdot \lambda_j}^1 (r - x \cdot \lambda_j) dr dx = \frac{1}{24 \cdot \lambda_j^2 \cdot \lambda_j} \quad (18) \\
E(c_i(\Lambda)) &= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \int_0^1 x^3 dx = \frac{1}{8 \cdot \lambda_1 \cdot \lambda_2}.
\end{aligned}$$

2. Case of "mixed" values of the Lagrange multipliers $\lambda_2 \leq 1 \leq \lambda_1$. In this

case:

$$\begin{aligned}
E(a_{1i}(\Lambda)) &= \frac{1}{\lambda_2} \left(\int_0^{1/\lambda_1} x \int_{x \cdot \lambda_1}^1 (r - x \cdot \lambda_1) dr dx - \right. \\
&\quad \left. - \int_0^{\frac{1-\lambda_2}{\lambda_1}} x \int_{(x \cdot \lambda_1 + \lambda_2)}^1 (r - x \cdot \lambda_1 - \lambda_2) dr dx \right) = \\
&= \frac{1}{24} \frac{4 - 6\lambda_2 - \lambda_2^3 + 4\lambda_2^2}{\lambda_1^2}, \\
E(a_{2i}(\Lambda)) &= \frac{1}{\lambda_1} \left(\int_0^1 x \int_{x \cdot \lambda_2}^1 (r - x \cdot \lambda_2) dr dx \right) = \frac{1}{24} \frac{3\lambda_2^3 - 8\lambda_2 + 6}{\lambda_1}, \\
E(c_i(\Lambda)) &= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \left(\frac{1}{4} - \int_{\lambda_2}^1 x \cdot (x - \lambda_2)^2 dx \right) = \\
&= \frac{1}{24\lambda_1} (\lambda_2^3 - 6\lambda_2 + 8).
\end{aligned} \tag{19}$$

3. Case of "moderate" values of the Lagrange multipliers $\lambda_2 \leq \lambda_1 \leq 1$, $\lambda_2 + \lambda_1 \geq 1$. In this case:

$$\begin{aligned}
E(a_{ji}(\Lambda)) &= \frac{1}{\lambda_{j^*}} \left(\int_0^1 x \int_{x \cdot \lambda_j}^1 (r - x \cdot \lambda_j) dr dx - \right. \\
&\quad \left. - \int_0^{(1-\lambda_{j^*})/\lambda_j} x \int_{(x \cdot \lambda_j + \lambda_{j^*})}^1 (r - x \cdot \lambda_j - \lambda_{j^*}) dr dx \right) = \\
&= \frac{1}{24} \frac{3\lambda_j^4 - 8\lambda_j^3 + 6\lambda_j^2 - 6\lambda_j^2 \cdot \lambda_{j^*} + 4\lambda_j \cdot \lambda_{j^*}^2 + 4\lambda_{j^*}^3 - 1}{\lambda_j^2 \lambda_{j^*}}, \\
E(c_i(\Lambda)) &= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \left(\frac{1}{4} - \int_{\lambda_1}^1 x \cdot (x - \lambda_1)^2 dx - \int_{\lambda_2}^1 x \cdot (x - \lambda_2)^2 dx \right) = \\
&= \frac{1}{24\lambda_1 \lambda_2} (\lambda_1^4 - 6\lambda_1^2 + 8\lambda_1 + \lambda_2^4 - 6\lambda_2^2 + 8\lambda_2 - 3).
\end{aligned} \tag{20}$$

4. Case of "small" values of the Lagrange multipliers $\lambda_2 \leq \lambda_1 \leq 1$,

$\lambda_2 + \lambda_1 \leq 1$. In this case:

$$\begin{aligned}
E(a_{j_i}(\Lambda)) &= \frac{1}{\lambda_j} \left(\int_0^1 x \int_{x-\lambda_j}^1 (r-x \cdot \lambda_j) dr dx - \right. \\
&\quad \left. - \int_0^1 x \int_{(x-\lambda_j+\lambda_{j'})}^1 (r-x \cdot \lambda_j - \lambda_{j'}) dr dx \right) = \\
&= \frac{1}{2} - \frac{1}{3} \lambda_j - \frac{1}{4} \lambda_{j'}, \\
E(c_i(\Lambda)) &= \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \left(\frac{1}{4} - \int_{\lambda_1}^1 x \cdot (x - \lambda_1)^2 dx - \int_{\lambda_2}^1 x \cdot (x - \lambda_2)^2 dx + \right. \\
&\quad \left. + \int_{\lambda_1+\lambda_2}^1 x \cdot (x - \lambda_1 - \lambda_2)^2 dx \right) = \\
&= \frac{1}{2} - \frac{1}{6} \lambda_1^2 - \frac{1}{4} \lambda_1 \lambda_2 - \frac{1}{6} \lambda_2^2.
\end{aligned} \tag{21}$$

Probabilistic, or in other words average case, analysis consists in determining such Lagrange multipliers $\lambda_1(n)$, $\lambda_2(n)$ that when $n \rightarrow \infty$, $x_i(\Lambda(n))$, $i = 1, \dots, n$, defined by (4) will provide solutions of the Two-Constraint Binary Knapsack Problem (1) which are, in the sense of convergence in probability, see Loeve [5], providing solutions which are asymptotically feasible, i.e. $s_j(\Lambda(n))$ is satisfying (8) and moreover if $S_n(\Lambda(n))$ is fulfilling (11) then, due to (10), $\lim_{n \rightarrow \infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1$ and $z_n(\Lambda(n))$ is suboptimal solution of the (1) and moreover

$$z_{OPT}(n) \approx z_n(\Lambda(n)) \approx E(z_n(\Lambda(n))).$$

The above goal may be achieved by determining $\Lambda(n)$ as the solution of the following system of equations:

$$E(s_1(\Lambda(n))) = b'_1(n), \quad E(s_2(\Lambda(n))) = b'_2(n), \tag{22}$$

where $b'_1(n) < b_1(n)$ and $b'_2(n) < b_2(n)$ and $\Lambda(n)$ is fulfilling both (8) and (11). Each of the 4 cases mentioned above should be considered independently. Let us observe that $E(s_j(\Lambda(n))) = n \cdot E(a_{j_1}(\Lambda(n)))$, $E(z_n(\Lambda(n))) = n \cdot E(c_1(\Lambda(n)))$.

Lemma 1 *If c_i, a_{j_i} $i = 1, \dots, n$, $j = 1, 2$, are realizations of mutually independent random variables uniformly distributed over $(0, 1)$, and if $b'_1(n) \leq b'_2(n)$ and $1 \leq \lambda_2(n) \leq \lambda_1(n)$ then*

$$\lambda_1(n) = \frac{1}{b'_1(n)} \sqrt{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}}, \quad \lambda_2(n) = \frac{1}{b'_2(n)} \sqrt{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}} \tag{23}$$

is the solution of (22) and

$$E(z_n(\Lambda(n))) = 3 \cdot \sqrt{\frac{n \cdot b'_1(n) \cdot b'_2(n)}{24}}. \tag{24}$$

Proof. Above formulas follow immediately from the (18) and (22). ■
From the Lemma 1 assumptions and from (23) it follows that when:

$$\delta \leq b'_1(n) \leq b'_2(n) \leq \frac{n}{24}, \text{ where } \delta \text{ is a constant}$$

then condition $1 \leq \lambda_2(n) \leq \lambda_1(n)$ holds.

Lemma 2 *If c_i, a_{ji} $i = 1, \dots, n, j = 1, 2$, are realizations of mutually independent random variables uniformly distributed over $(0, 1)$, and if $b'_1(n) \leq b'_2(n)$ and $\lambda_2(n) + \lambda_1(n) \leq 1$ then*

$$\begin{aligned} \lambda_1(n) &= \frac{1}{7} \cdot \left(36 \cdot \frac{b'_2(n)}{n} - 48 \cdot \frac{b'_1(n)}{n} + 6 \right), \\ \lambda_2(n) &= \frac{1}{7} \cdot \left(36 \cdot \frac{b'_1(n)}{n} - 48 \cdot \frac{b'_2(n)}{n} + 6 \right) \end{aligned} \quad (25)$$

is the solution of (22) and

$$\begin{aligned} E(z_n(\Lambda(n))) &= \frac{1}{7} \cdot \left(\frac{n}{2} + 6 \cdot b'_1(n) + 6 \cdot b'_2(n) + \right. \\ &\quad \left. + 36 \cdot \frac{b'_1(n) \cdot b'_2(n)}{n} - 24 \cdot \frac{(b'_1(n))^2}{n} - 24 \cdot \frac{(b'_2(n))^2}{n} \right) \end{aligned} \quad (26)$$

Proof. From the (21) it could be obtained that $\lambda_1(n)$ and $\lambda_2(n)$ given by formula (25) are solving the equation (22). ■

In this case the condition $\lambda_2(n) + \lambda_1(n) \leq 1$ and formula (3) are providing that following right-hand-sides of the constraints:

$$\frac{5 \cdot n}{12} \leq b'_1(n) + b'_2(n) \leq n,$$

are fulfilling assumptions of the Lemma 2.

5 Concluding remarks

In the present paper results describing probabilistic properties of the Two-Constraint Binary Knapsack Problem (1) in the case of smaller values of constraints right-hand-sides (when $1 \leq \lambda_2(n) \leq \lambda_1(n)$, i.e. large) as well as large values of constraints right-hand-sides (when $\lambda_2(n) + \lambda_1(n) \leq 1$, corresponding to small values of $\lambda_1(n)$ and $\lambda_2(n)$) are considered.

In the paper distribution functions of the various random variables representing important problems characteristics are presented.

The future research will be aimed at investigation of 2 remaining cases (mixed and moderate values) of the mutual relations between $\lambda_1(n)$ and $\lambda_2(n)$, feasibility of the received solutions and estimations of the Two-Constraint Binary Knapsack Problem (1) optimal solution values $z_{OPT}(n)$ growth, when $n \rightarrow \infty$.

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the 1990s, the number of people in the world who are illiterate has increased from 400 million to 500 million. The number of illiterate people in the world is expected to reach 600 million by the year 2015 (UNESCO 2003).

Illiteracy is a global problem. It is a major barrier to economic and social development. It is a major cause of poverty and social exclusion. It is a major cause of ill health and poor living conditions. It is a major cause of unemployment and underemployment. It is a major cause of social inequality and discrimination. It is a major cause of social instability and conflict.

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the 1990s, the number of people in the UK who are aged 65 and over has increased from 10.5 million to 13.5 million (15.5% of the population).

There is a growing awareness of the need to address the needs of older people, and the Government has set out a strategy for the 21st century in the White Paper on *Ageing Better: A Strategy for the 21st Century* (Department of Health 1999). This strategy is based on the following principles:

- Older people should be able to live independently and actively in their own homes.
- Older people should be able to live in their own communities.
- Older people should be able to live in their own homes and communities for as long as possible.

The White Paper also sets out a number of key objectives for the 21st century, including:

- To ensure that older people are able to live independently and actively in their own homes.
- To ensure that older people are able to live in their own communities.
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