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TESTS FOR RELATION TYPE - EQUIVALENCE OR TOLERANCE – BASED ON MULTIPLE BINARY COMPARISONS

by

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Abstract: The statistical procedure for determination of the relation type – equivalence or tolerance – in a finite set on the basis of multiple comparisons with random errors is presented in the paper. The procedure consists of two tests based on probabilistic inequalities; it is extension of approach presented in Klukowski 2006. The test statistics is based on a mixture of some random variables; it rests on estimated form of both relations. Two cases are examined: aggregation of multiple comparisons of each pair with the use of the average and the median. The estimates of the relations are obtained on the basis of some discrete programming tasks.

Keywords: tests for equivalence and tolerance relation, multiple pairwise comparisons

1. Introduction

The methods of estimation of the relation form - equivalence or tolerance - presented in Klukowski 1990, 2002, 2007 are based on the assumption, that the type of the relation is known. In practice it may be often not true; therefore the decision rule for determination the type of the relation (the model of data) is necessary. In the paper some statistical procedure for this purpose is proposed. It is based on two statistical tests, which rest on: estimated form of the relations and some probabilistic inequalities. The estimators of the relations exploit the idea the nearest adjoining order (see Slater P. 1961, David 1988). The test statistic is based on a mixture of some random variables; two parameters of one component of the mixture are determined: the expected value and variance evaluation. Multiple comparisons are aggregated with the use of the average or the median. Let us notice, that the tolerance relation can be regarded as a case of fuzzy relation with the membership function equal to one for some number of sets. The example of such relation is a set of patients with some disease, which can be result of many sources, e.g. sclerosis can be a result of: mode of life, nutrition, genetic structure or any conjunction of these features.

2. Basic definitions and notation

The equivalence relation (reflexive, symmetric, transitive) divides the set **X** into n_1 ($n_1 \geq 2$) subsets $\chi_r^{*(1)}$ (r=1, ..., n_1) with empty intersections, i.e.:

$$\mathbf{X} = \bigcup_{r=1}^{n_1} \chi_r^{*(1)}, \qquad \chi_r^{*(1)} \cap \chi_s^{*(1)} = \{\mathbf{0}\}, \qquad \text{for } r \neq s, \tag{1}$$

where: $\{\mathbf{0}\}$ - empty set.

The tolerance relation divides the set **X** into n_2 ($n_2 \geq 2$) subsets $\chi_r^{*(2)}$ (r=1, ..., n_2) with at least one non-empty intersection, i.e. the relation is not transitive. It satisfies the conditions: $\mathbf{X} = \bigcup_{r=1}^{n_2} \chi_r^{*(2)}$ ($n_2 \geq 2$) and there exists at least one pair of subsets $\chi_r^{*(2)}, \chi_s^{*(2)}$ ($r \neq s$) with non-empty intersection: $\chi_r^{*(2)} \cap \chi_s^{*(2)} \neq \{\mathbf{0}\}$.

The equivalence relation is characterized with the use of the function $T_1 : \mathbf{X} \times \mathbf{X} \to D$, $D = \{0, 1\}$, defined as follows:

$$T_1(x_i, x_j) = \begin{cases} 0 & \text{if there exists } \chi_q^{*(1)} \text{ satisfying the condition } (x_i, x_j) \in \chi_q^{*(1)},\ i \neq j; \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

The tolerance relation is characterized with use of the function $T_2 : \mathbf{X} \times \mathbf{X} \to D$, $D = \{0, 1\}$, defined as follows:

$$T_2(x_i, x_j) = \begin{cases} 0 & \text{if there exist } q \text{ and } s \ (q{=}s \text{ not excluded}) \text{ such, that: } (x_i, x_j) \in \chi_q^{*(2)} \cap \chi_s^{*(2)},\ i \neq j; \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

It is assumed, that the function $T_2(\cdot)$ characterizes completely the tolerance relation, i.e. there exists one-to-one relationship between the relation form and the function $T_2(\cdot)$. The requirement is satisfied e.g., when each subset $\chi_q^{*(2)}$ includes an element x_i, that is not included in any other subset $\chi_s^{*(2)}$ $(q\neq s)$ (i.e. $x_i \in \chi_q^{*(2)}$ and $x_i \notin \chi_s^{*(2)}$).

In the paper it is assumed, that the type of the relation (equivalence or tolerance) in the set $\mathbf{X}$ (i.e. the function $T_1(\cdot)$ or $T_2(\cdot)$) is not known and it has to be determined on the basis of pairwise comparisons $g_k(x_i, x_j)$ $(k=1, \ldots, N; (x_i, x_j)\in\mathbf{X}\times\mathbf{X})$ disturbed with random errors. Each comparison determines homogeneity or non-homogeneity of elements from the pair only. The homogeneity means inclusion to the same subset (also to an intersection of some subsets), non-homogeneity – inclusion into different subsets. In other words the comparisons $g_k(x_i, x_j)$ do not determine directly the type of the relation; they are only the base for inference.

The result of comparison $g_k(x_i, x_j)$ is the function:

$$g_k : \mathbf{X}\times\mathbf{X}\rightarrow D, \qquad D=\{0, 1\}, \tag{4}$$

which evaluates homogeneity of the elements x_i and x_j (inclusion to the same subset), i.e. the "true" value $T_1(x_i, x_j)$ or $T_2(x_i, x_j)$.

It is assumed, that the probability of correctness of each comparison satisfies the conditions:

$$P(g_k(x_i, x_j)=T_f(x_i, x_j)) \geq 1 - \delta, \qquad \delta\in(0, ½), \tag{5a}$$

$$P((g_k(x_i, x_j)=T_f(x_i, x_j))\cap(g_l(x_r, x_s)=T_f(x_r, x_s)))=P((g_k(x_i, x_j)=T_f(x_i, x_j))P(g_l(x_r, x_s)=T_f(x_r, x_s)), \quad (k\neq l) \tag{5b}$$

where: f - equals 1 or 2 - according to the actual relation in the set $\mathbf{X}$.

The conditions (5a), (5b) means that the probability of correct comparison is greater than incorrect one and that the random variables $g_k(x_i, x_j)$, $g_l(x_r, x_s)$ are independent for $k\neq l$.

Let us notice, that any comparison $g_k(x_i, x_j)$, which satisfy the conditions (5a, b), may be equal to $T_f(x_i, x_j)$ (f=1 or 2) or not, as a result of a random error. In particular, the comparisons obtained for the equivalence relation may be not transitive (e.g.: $g_k(x_i, x_j)=0$, $g_k(x_j, x_k)=0$ and $g_k(x_i, x_k)=1$), while comparisons for the tolerance relation may be transitive. Such comparisons may be obtained as results of statistical test, which determines the inclusion to the same subset and does "not know" the relation type.

The methods of estimation of both relations have been proposed in Klukowski (1990, 2002). In the case of multiple comparisons (N>1) the values $g_k(\cdot)$ (k=1, ..., N) are agregated with the use of the average or median. In the first case the estimated form of the equivalence relation is obtained (under the assumption, that the type of the relation is known) on the basis of the discrete optimization task:

$$\min_{\chi_1^{(1)},\dots,\chi_v^{(1)}} [\sum_{<i,j>\in I(\chi_1^{(1)},\dots,\chi_v^{(1)})} \sum_{k=1}^{N} g_k(x_i,x_j) + \sum_{<i,j>\in J(\chi_1^{(1)},\dots,\chi_v^{(1)})} \sum_{k=1}^{N} (1-g_k(x_i,x_j))], \quad (6)$$

where:

$\chi_1^{(1)},\dots,\chi_v^{(1)}$ - an element of feasible set (any form of the equivalence relation in the set **X**),

$I(\chi_1^{(1)},\dots,\chi_v^{(1)})$ - the set of all indices $<i,j>$ satisfying the conditions:

$i,j\in\{1, \dots, m\}$, $j>i$;

$<i,j>\in I(\chi_1^{(1)},\dots,\chi_v^{(1)}) \Leftrightarrow \exists q$ such, that: $(x_i, x_j)\in \chi_q^{(1)}$,

$J(\chi_1^{(1)},\dots,\chi_v^{(1)})$ - the set of all indices pairs $<i,j>$ satisfying the conditions:

$i,j\in\{1, \dots, m\}$, $j>i$;

$<i,j>\in J(\chi_1^{(1)},\dots,\chi_v^{(1)}) \Leftrightarrow$ it does not exist q such, that: $(x_i, x_j)\in \chi_q^{(1)}$.

The optimal solution of the task with the criterion function (6) (estimated form of the equivalence relation) will be denoted with the symbols $\hat{\chi}_1^{(1,av)},\dots,\hat{\chi}_{\hat{m}}^{(1,av)}$. The solution can be characterized with the function:

$$\hat{t}_1^{(av)}(x_i, x_j) = \begin{cases} 0 & \text{if there exists } q \text{ in the relation } \hat{\chi}_1^{(1,av)}, \ldots, \hat{\chi}_{\hat{m}}^{(1,av)} \text{ such, that } (x_i, x_j) \in \hat{\chi}_q^{(1,av)},\ i \neq j; \\ 1 & \text{otherwise.} \end{cases} \quad (7)$$

The minimal value of the function (6) equals zero; it is assumed in the case $g_k(x_i, x_j) = \hat{t}_1^{(av)}(x_i, x_j)$ for each $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$ and k (k=1, …, N). It should be noticed, that the estimated form of the relation may be not unique, because the number of optimal solutions of discrete problem can exceeds one; the unique estimate can be selected randomly or with the use of an additional criterion, e.g. $\min_{\chi_1^{(1)}, \ldots, \chi_v^{(1)}} [\sum_{<i,j> \in I(\chi_1^{(1)}, \ldots, \chi_v^{(1)})} \sum_{k=1}^{N} g_k(x_i, x_j)]$.

In the case of the median from comparisons $g_k(x_i, x_j)$ (k=1, …, N; N – uneven; $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$) the estimated equivalence relation is obtained (under the assumption, that the type of the relation is known) on the basis of the discrete optimization task:

$$\min_{\chi_1^{(1)}, \ldots, \chi_v^{(1)}} [\sum_{<i,j> \in I(\chi_1^{(1)}, \ldots, \chi_v^{(1)})} g^{(me,N)}(x_i, x_j) + \sum_{<i,j> \in J(\chi_1^{(1)}, \ldots, \chi_v^{(1)})} (1 - g^{(me,N)}(x_i, x_j))], \quad (8)$$

where:

$\chi_1^{(1)}, \ldots, \chi_v^{(1)}$, $I(\chi_1^{(1)}, \ldots, \chi_v^{(1)})$, $J(\chi_1^{(1)}, \ldots, \chi_v^{(1)})$ - the same as in (6);

$g^{(me,N)}(x_i, x_j)$ – the median from the comparisons $g_1(x_i, x_j)$, …, $g_N(x_i, x_j)$.

Let us notice that the median $g^{(me,N)}(x_i, x_j)$ is equal to the majority in the set $\{g_1(\cdot), \ldots, g_N(\cdot)\}$; if $\sum_{k=1}^{N} g_k(\cdot) < \frac{N}{2}$, then $g^{(me,N)}(\cdot)=0$, in the opposite case $g^{(me,N)}(\cdot)=1$.

The optimal solution of the task with the criterion function (8) will be denoted with the symbols $\hat{\chi}_1^{(1,me)}, \ldots, \hat{\chi}_{\hat{m}}^{(1,me)}$. The solution can be characterized with the function:

$$\hat{t}_1^{(me)}(x_i, x_j) = \begin{cases} 0 & \text{if there exists } q \text{ in the relation } \hat{\chi}_1^{(1,me)}, \ldots, \hat{\chi}_{\hat{m}}^{(1,me)} \text{ such, that } (x_i, x_j) \in \hat{\chi}_q^{(1,me)},\ i \neq j; \\ 1 & \text{otherwise.} \end{cases} \quad (9)$$

The minimal value of the function (8) equals zero. The estimated form of the relation $\hat{\chi}_1^{(1,me)}, \dots, \hat{\chi}_{\hat{m}}^{(1,me)}$ may be also not unique.

In case of the tolerance relation the optimization tasks assume the form - respectively:

$$\min_{\chi_1^{(2)}, \dots, \chi_v^{(2)}} \Big[\sum_{<i,j> \in I(\chi_1^{(2)}, \dots, \chi_v^{(2)})} \sum_{k=1}^{N} g_k(x_i, x_j) + \sum_{<i,j> \in J(\chi_1^{(2)}, \dots, \chi_v^{(2)})} \sum_{k=1}^{N} (1 - g_k(x_i, x_j)) \Big], \tag{10}$$

$$\min_{\chi_1^{(2)}, \dots, \chi_v^{(2)}} \Big[\sum_{<i,j> \in I(\chi_1^{(2)}, \dots, \chi_v^{(2)})} g^{(me,N)}(x_i, x_j) + \sum_{<i,j> \in J(\chi_1^{(2)}, \dots, \chi_v^{(2)})} (1 - g^{(me,N)}(x_i, x_j)) \Big], \tag{11}$$

where:

$\chi_1^{(2)}, \dots, \chi_v^{(2)}$ - an element of feasible set (any form of the tolerance relation in the set **X**),

$I(\chi_1^{(2)}, \dots, \chi_v^{(2)})$ - the set of all indices pairs $<i, j>$ satisfying the conditions:

$i, j \in \{1, \dots, m\}, \;\; j>i;$

$<i, j> \in I(\chi_1^{(2)}, \dots, \chi_v^{(2)}) \Leftrightarrow \exists q, s$ ($q=s$ not excluded) such, that: $(x_i, x_j) \in \chi_q^{(2)} \cap \chi_s^{(2)}$,

there exists at least one nonempty intersection, i.e. $\chi_q^{(2)} \cap \chi_s^{(2)}$ $(q \neq s)$;

$J(\chi_1^{(2)}, \dots, \chi_v^{(2)})$ - the set of all indices $<i, j>$ satisfying the conditions:

$i, j \in \{1, \dots, m\}, \;\; j>i;$

$<i, j> \in J(\chi_1^{(2)}, \dots, \chi_v^{(2)}) \Leftrightarrow$ there does not exist q such, that: $(x_i, x_j) \in \chi_q^{(2)}$;

$g^{(me,N)}(x_i, x_j)$ – the same as in (6).

The properties of the tasks (10), (11) are similar to the properties of the tasks (6), (8).

Optimal solution of each task corresponding to the tolerance relation will be denoted – respectively: $\hat{\chi}_1^{(2,av)}, \dots, \hat{\chi}_{\hat{m}}^{(2,av)}$ or $\hat{\chi}_1^{(2,me)}, \dots, \hat{\chi}_{\hat{m}}^{(2,me)}$. The solutions can be characterized with the use of the functions $\hat{t}_2^{(av)}(x_i, x_j)$ or $\hat{t}_2^{(me)}(x_i, x_j)$ defined as follows:

$$\hat{t}_2^{(av)}(x_i, x_j) = \begin{cases} 0, & \text{if there exist } q \text{ and } s \text{ } (q=s \text{ not excluded}) \text{ such, that: } x_i, x_j \in \hat{\chi}_q^{(2,av)} \cap \hat{\chi}_s^{(2,av)}, \; i \neq j; \\ 1 & \text{otherwise.} \end{cases} \tag{12}$$

$$\hat{t}_2^{(me)}(x_i,x_j)=\begin{cases}0, & \text{if there exist } q \text{ and } s \text{ } (q=s \text{ not excluded}) \text{ such, that: } x_i,\ x_j\in \hat{\chi}_q^{(2,me)}\cap\hat{\chi}_s^{(2,me)},\ i\neq j;\\ 1 & \text{otherwise.}\end{cases} \quad (13)$$

3. Procedure of testing of the relation type

As it was mentioned above, both types of relation can be estimated on the basis of the same pairwise comparisons $g_k(x_i, x_j)$. In the case of unknown relation type the question arises which of them is true model of data.

The procedure proposed below rests on the differences between comparisons and estimated form of equivalence and tolerance relation. The procedure consists of two statistical tests. In the case of averaged comparisons the test statistics is a function of inconsistencies between comparisons $g_k(x_i, x_j)$ (k=1, ..., N) and the functions $\hat{t}_1^{(av)}(x_i,x_j)$, $\hat{t}_2^{(av)}(x_i,x_j)$, for the pairs (x_i, x_j), which satisfy the condition $\hat{t}_1^{(av)}(x_i,x_j)\neq\hat{t}_2^{(av)}(x_i,x_j)$. In the case of medians from comparisons, the basis of the tests are inconsistencies between: $g^{(me,N)}(x_i,x_j)$ and the functions $\hat{t}_1^{(me)}(x_i,x_j)$, $\hat{t}_2^{(me)}(x_i,x_j)$, for the pairs (x_i, x_j), which satisfy the condition or $\hat{t}_1^{(me)}(x_i,x_j)\neq\hat{t}_2^{(me)}(x_i,x_j)$.

3.1. Tests for the case of averaged comparisons

The basis for the tests proposed are the random variables S_{ijk}, defined as follows:

$$S_{ijk}=\left|\hat{t}_1^{(av)}(x_i,x_j)-g_k(x_i,x_j)\right|-\left|\hat{t}_2^{(av)}(x_i,x_j)-g_k(x_i,x_j)\right| \quad (\langle i,j\rangle\in I_w^{(av)}), \quad (14)$$

where:

$I_w^{(av)}$ - the set of all pairs of indices $\langle i, j\rangle$, which satisfy the condition:

$$\hat{t}_1^{(av)}(x_i,x_j)\neq\hat{t}_2^{(av)}(x_i,x_j);$$

($\hat{t}_1^{(av)}(x_i,x_j)$ and $\hat{t}_2^{(av)}(x_i,x_j)$ defined - respectively - in (7) and (12)).

The condition, which define the set $I_w^{(av)}$, means that:

• in estimated form of the tolerance relation the elements x_i and x_j are included in an intersection of two subsets $\hat{\chi}_q^{(2,av)} \cap \hat{\chi}_s^{(2,av)}$ ($q=s$ not excluded), while in (estimated) equivalence relation they are included in different subsets

or

• in estimated form of the tolerance relation the elements x_i and x_j are not included in any intersection of subsets (also in the same subset), while in (estimated) equivalence relation they are included in the same subset.

The test statistics $S^{(av,N)}$ is the sum of random variables S_{ijk} ($<i,j> \in I_w^{(av)}$, $k=1, \ldots, N$) divided by the number of elements in the sum; it assumes the form:

$$S^{(av,N)} = \frac{1}{\#(I_w^{(av)})} \frac{1}{N} \sum_{<i,j> \in I_w^{(av)}} \sum_{k=1}^{N} S_{ijk} \ , \tag{15}$$

where: $\#(I_w^{(av)})$ – number of elements of the set $I_w^{(av)}$.

The properties of the statistics $S^{(av,N)}$ depend on actual relation type in the set **X**. Let us consider firstly the case of the tolerance relation; the expected value and the evaluation of variance of the variable $S^{(av,N)}$ are determined below.

For simplification it is assumed, that probability of an error in each comparison $g_k(x_i, x_j)$ ($k=1, \ldots, N$; $j \neq i$) is equal to δ (see (5a)). In the case, when some probabilities are lower than δ the properties of the procedure proposed (the probabilities of errors in the tests) are not worse.

In the case when tolerance relation exists in the set **X**, the estimated form of the relation is equivalent to the actual (errorless result of estimation), i.e. $\hat{\chi}_1^{(2,av)}, \ldots, \hat{\chi}_{\hat{n}_2}^{(2,av)} \equiv \chi_1^{*(2)}, \ldots, \chi_{n_2}^{*(2)}$, with the probability $P(\hat{\chi}_1^{(2,av)}, \ldots, \hat{\chi}_{\hat{n}_2}^{(2,av)} \equiv \chi_1^{*(2)}, \ldots, \chi_{n_2}^{*(2)} \mid \mathbf{R}^{(2)})$. In this case the equalities $\hat{t}_2^{(av)}(x_i, x_j) = T_2(x_i, x_j)$ ($<i,j> \in I_w^{(av)}$) hold. Moreover, each expression

$\left|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right|$ and $\left|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right|$ ($<i,j> \in I_w^{(av)}$) is zero-one random variable; its distribution can be determined on the basis of the properties of the comparison (random variable) $g_k(x_i, x_j)$. The probability function of each random variable $\left|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right|$ ($<i,j> \in I_w^{(av)}$) is determined as follows (assuming equality in (5a)):

$$P(\left|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right| = 0 \mid \hat{t}_2^{(av)}(\cdot) = T_2(\cdot)) = \\ P(g_k(x_i, x_j) = \hat{t}_2^{(av)}(x_i, x_j) \mid \hat{t}_2^{(av)}(\cdot) = T_2(\cdot)) = 1 - \delta, \tag{16a}$$

$$P(\left|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right| = 1 \mid \hat{t}_2^{(av)} = T_2(\cdot)) = P(g_k(x_i, x_j) \neq \hat{t}_2^{(av)}(x_i, x_j) \mid \hat{t}_2^{(av)} = T_2(\cdot)) = \delta, \tag{16b}$$

where: $\hat{t}_2^{(av)}(\cdot) = T_2(\cdot)$ means equality for all pairs $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$.

Under the assumption $\hat{t}_1^{(av)}(x_i, x_j) \neq \hat{t}_2^{(av)}(x_i, x_j)$, the probability function of the random variable $\left|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right|$ assumes the form:

$$P(\left|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right| = 0 \mid \hat{t}_2^{(av)}(\cdot) = T_2(\cdot)) = P(g_k(x_i, x_j) \neq \hat{t}_2^{(av)}(\cdot) \mid \hat{t}_2(\cdot) = T_2(\cdot)) = \delta, \tag{17a}$$

$$P(\left|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right| = 1 \mid \hat{t}_2^{(av)}(\cdot) = T_2(\cdot)) = \\ P(g_k(x_i, x_j) = \hat{t}_2^{(av)}(x_i, x_j) \mid \hat{t}_2^{(av)}(\cdot) = T_2(\cdot)) = 1 - \delta. \tag{17b}$$

The probabilities (16a) – (17b) result from the fact, that for $<i,j> \in I_w^{(av)}$ inequalities $\hat{t}_1^{(av)}(x_i, x_j) \neq \hat{t}_2^{(av)}(x_i, x_j)$ and implications: $g_k(x_i, x_j) = \hat{t}_1^{(av)}(x_i, x_j) \Rightarrow g_k(x_i, x_j) \neq \hat{t}_2^{(av)}(x_i, x_j)$, $g_k(x_i, x_j) \neq \hat{t}_1^{(av)}(x_i, x_j) \Rightarrow g_k(x_i, x_j) = \hat{t}_2^{(av)}(x_i, x_j)$ hold.

The equalities (16a) - (17b) indicate:

$$P(S_{ijk} = -1 \mid \hat{t}_2^{(av)}(\cdot) = T_2(\cdot)) = \\ P[(\left|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right| = 0) \cap (\left|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right| = 1) \mid \hat{t}_2^{(av)}(\cdot) = T_2(\cdot)] = \delta, \tag{18}$$

$$P(S_{ijk}=1|\ \hat{t}_2^{(av)}(\cdot)=T_2(\cdot))=$$
$$P[(|\hat{t}_1^{(av)}(x_i,x_j)-g_k(x_i,x_j)|=1)\cap(|\hat{t}_2^{(av)}(x_i,x_j)-g_k(x_i,x_j)|=0)|\ \hat{t}_2^{(av)}(\cdot)=T_2(\cdot)]=1-\delta. \quad (19)$$

It follows from (18) and (19) that in the case, when the tolerance relation exists in the set **X**, the expected value $E_2(S_{ijk})$ and variance $Var_2(S_{ijk})$ of each random variable S_{ijk}, $<i,j>\in I_w^{(av)}$, assume the form - respectively:

$$E_2(S_{ijk}) = -\delta + 1 - \delta = 1 - 2\delta, \quad (20)$$

$$Var_2(S_{ijk}) = (-1-(1-2\delta))^2\,\delta + (1-(1-2\delta))^2\,(1-\delta) = 4\delta(1-\delta). \quad (21)$$

Each random variable $S^{(av,N)}$ is the sum of the variables S_{ijk} divided by their number $\#(I_w^{(av)})N$. Therefore, the expected value of the variable $S^{(av,N)}$ equals:

$$E_2(S^{(av,N)}) = \#(I_w^{(av)})\cdot N\,(1-2\delta)\,/\,\#(I_w^{(av)})\cdot N=(1-2\delta). \quad (22)$$

The variance $Var_2(S^{(av,N)})$ of the random variable $S^{(av,N)}$ is evaluated under an assumption that any random variables S_{ijk} and S_{rsk} ($1\leq k\leq N$), which satisfy the conditions $i\neq r$, s and $j\neq r$, s, are independent (i.e. their covariance equals to zero), while remaining variables may be dependent. The number of covariances equal to zero (for individual k) is denoted $L(I_w^{(av)})$; if the assumption does not hold, then $L(I_w^{(av)})=0$. The evaluation of variance of the variable $S^{(av,N)}$ is based on the following facts: each variance of S_{ijk} is equal to $4\delta(1-\delta)$ and each non-zero covariance $C(S_{ijk}, S_{rsk})$ is not greater, than $4\delta(1-\delta)$. Moreover the number of variances $Var_2(S_{ijk})$ ($<i,j>\in I_w^{(av)}$) is equal to $\#(I_w^{(av)})$ and number of covariances (in the set $I_w^{(av)}$) is equal to $\#(I_w^{(av)})*(\ \#(I_w^{(av)})-1)/2-L(I_w^{(av)})$. As a result $Var_2(S^{(av,N)})$ satisfies the condition:

$$Var_2(S^{(av,N)}) \leq \tfrac{1}{N}\tfrac{1}{\#I(I_w^{(av)})}((\#(I_w^{(av)}))^2 - 2L(I_w^{(av)}))4\delta(1-\delta),$$

equivalent to:

$$Var_2(S^{(av,N)}) \leq \tfrac{4}{N}(\#(I_w^{(av)}) - 2L(I_w^{(av)})/\#(I_w^{(av)}))\delta(1-\delta). \quad (23)$$

The right-hand side of the inequality (23) can significantly exceed the actual variance $Var_2(S^{(av,N)})$, because covariances $C(S_{ijk}, S_{rsk})$ may be lower than $Var_2(S_{ijk})$, in particular – negative. More precise evaluation of the variance requires some additional knowledge about covariances $C(S_{ijk}, S_{rsk})$. Sometimes their values can be evaluated, e.g. when the comparisons $g_k(x_i, x_j)$ are obtained from statistical test and covariance of test statistics is known.

The properties (22) and (23) of the random variable $S^{(av,N)}$ are valid in the case of errorless estimation result of the tolerance relation (i.e. $\hat{\chi}_1^{(2,av)}, \ldots, \hat{\chi}_{\hat{m}_2}^{(2,av)} \equiv \chi_1^{*(2)}, \ldots, \chi_{n_2}^{*(2)}$). If it is not true, then the properties mentioned do not hold. The value of the variable $S^{(av,N)}$ obtained for any estimation result (errorless or not) can be treated as a realization of some mixture of the variables. However, the properties (the expected value and the evaluation of variance) of only one random variable from the mixture - corresponding to errorless estimation result - can be determined (without difficulties). Let us notice that the probability of the errorless estimation result $P(\hat{\chi}_1^{(2,av)}, \ldots, \hat{\chi}_{\hat{m}_2}^{(2,av)} \equiv \chi_1^{*(2)}, \ldots, \chi_{m_2}^{*(2)} \mid \mathbf{R}^{(2)})$ is also not easy to determine; it can be evaluated with the use of simulation approach, e.g. boot-strap techniques.

In the case, when the equivalence relation exists in the set **X** and the result of estimation is errorless the distribution of the random variable $S^{(av,N)}$ (defined in (15)) can be obtained in similar way. The distributions of the random variables $\left|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right|$ and $\left|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right|$ ($\langle i,j\rangle \in I_w^{(av)}$) are determined as follows (assuming equality in (5a)):

$$P(\left|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right| = 0 \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)) = \\ P(g_k(x_i, x_j) = \hat{t}_1^{(av)}(x_i, x_j) \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)) = 1 - \delta, \quad (24a)$$

$$P(\left|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)\right| = 1 \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)) = \\ P(g_k(x_i, x_j) \neq \hat{t}_1^{(av)}(x_i, x_j)) \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)) = \delta, \quad (24b)$$

$$P(|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)| = 0 \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)) =$$
$$P(g_k(x_i, x_j) = \hat{t}_2^{(av)}(x_i, x_j) \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)) = \delta, \tag{25a}$$

$$P(|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)| = 1 \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)) =$$
$$P(g_k(x_i, x_j) \neq \hat{t}_2^{(av)}(x_i, x_j) \mid \hat{t}_1^{(av)} = T_1(\cdot)) = 1 - \delta. \tag{25b}$$

From (24a) - (25b) it follows, that:

$$P(S_{ijk} = -1 \mid \hat{t}_1^{(av)}(x_i, x_j) = T_1(x_i, x_j)) =$$
$$P[(|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)| = 0) \cap (|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)| = 1) \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)] = 1 - \delta,$$

(26a)

$$P(S_{ijk} = 1 \mid \hat{t}_1^{(av)}(x_i, x_j) = T_1(x_i, x_j)) =$$
$$P[(|\hat{t}_1^{(av)}(x_i, x_j) - g_k(x_i, x_j)| = 1) \cap (|\hat{t}_2^{(av)}(x_i, x_j) - g_k(x_i, x_j)| = 0) \mid \hat{t}_1^{(av)}(\cdot) = T_1(\cdot)] = \delta, \tag{26b}$$

where: $\hat{t}_1^{(av)}(\cdot) = T_1(\cdot)$ means equality for all pairs $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$.

The formulas (25a) and (26b) are the basis for determination of expected value and variance of each random variable S_{ijk}:

$$E_1(S_{ijk}) = -1 + \delta + \delta = 2\delta - 1, \tag{27}$$

$$Var_1(S_{ijk}) = 4\delta(1 - \delta). \tag{28}$$

The form of above parameters allows to determine the expected value and evaluation of variance of the random variable $S^{(av,N)}$, when the equivalence relation exists in the set **X**. The expected value equals:

$$E_1(S^{(av,N)}) = 2\delta - 1, \tag{29}$$

while the variance satisfy the condition:

$$Var_1(S^{(av,N)}) \leq \tfrac{4}{N}(\#(I_w) - 2L(I_w)/\#(I_w))\delta(1-\delta), \tag{30}$$

the same as condition (23).

The properties (29) and (30) hold for the equivalence relation, when errorless estimation result occurs. However, with the probability

$1-P(\hat{\chi}_1^{(1,av)},\ldots,\hat{\chi}_{\hat{m}}^{(1,av)} \equiv \chi_1^{*(1)},\ldots,\chi_m^{*(1)} \mid \mathbf{R}^{(1)})$ the result of estimation is different than errorless one. Therefore, the value of the random variable $S^{(av,N)}$ can be treated as a realization from a mixture of distributions, with similar properties, as in the case of tolerance relation.

On the basis of expected value and evaluation of variance of the random variable $S^{(av,N)}$, for both relation types, it is possible to determine some tests for distinction them. The Tschebyshev's inequality can be used as the basis of the tests:

$$P(|X\text{-}E(X)| > \lambda\sigma) < \tfrac{1}{\lambda^2},$$

where:

X – a random variable with expected value $E(X)$ and variance σ,

λ – positive constant.

The test for verification the tolerance relation in the set **X** rests on expected value and evaluation of the variance of the random variable $S^{(av,N)}$. The null and alternative hypothesis ($H_{2,0}$ and $H_{2,1}$) of the test can be formulated in the following way:

$H_{2,0}$: $E(S^{(av,N)}) = 1 - 2\delta$,

$H_{2,1}$: $E(S^{(av,N)}) = 2\delta - 1$,

with the critical region:

$$\Lambda_2^{(av,N)} = \{ S^{(av,N)} \mid S^{(av,N)} < 1\text{-}2\delta\text{-}\lambda\sigma_S^{(av,N)} \}, \tag{31}$$

where: $\sigma_S^{(av,N)}$ – square root of the variance $Var_2(S^{(av,N)})$ evaluation, i.e.:

$$\sigma_S^{(av,N)} = (Var_2(S^{(av,N)}))^{1/2} = [\tfrac{4}{N}(\#(I_w^{(av)}) - 2L(I_w^{(av)})/\#(I_w^{(av)}))\delta(1-\delta)]^{1/2}.$$

The form of the test for the equivalence relation is "symmetric":

$H_{1,0}$: $E(S^{(av,N)}) = 2\delta - 1$,

$H_{1,1}$: $E(S^{(av,N)}) = 1 - 2\delta$,

with the critical region:

$$\Lambda_1^{(av,N)} = \{ S^{(av,N)} \mid S^{(av,N)} > 2\delta\text{-}1 + \lambda \sigma_S^{(av,N)} \} \tag{32}$$

($\sigma_S^{(av,N)}$ – the same, as in the formula (31)).

The tests may be used together or separately (one of them only). In the first case, their critical regions $\Lambda_f^{(av,N)} (f = 1, 2)$ have to be non-overlapped; the value of parameter λ may be different in each test; it leads to different probabilities of errors in the tests. In the case, when: $2\delta\text{-}1 + \lambda\sigma_S^{(av,N)} < 1\text{-}2\delta\text{-}\lambda\sigma_S^{(av,N)}$, there exists some non-decision region of the procedure. The evaluations of the probabilities of errors are determined below.

Application of one test only allows to reject hypothesis $H_{2,0}$ or $H_{1,0}$ (significance test); the alternative hypothesis can assume the form $H_{2,1}$: $E(S^{(av,N)}) < 1\text{-}2\delta$ or $H_{1,1}$: $E(S^{(av,N)}) > 2\delta\text{-}1$.

Let us notice, that if the exact values or evaluations of covariances $C(S_{ijk}, S_{rsk})$ are known, then the critical regions $\Lambda_f^{(av,N)} (f = 1, 2)$ can be determined more precisely; it improves the properties of the tests. Moreover, the critical regions of both tests are based on (two-sided) Tschebyshev inequality. Thus, the probabilities resulting from the inequality are overestimated, because the tests exploit one side of the inequality.

The properties (20), (21), (27), (28) of the statistics $S^{(av,N)}$ are valid in the case of errorless estimation result of the relation form. The probability of the errorless estimation result is lower than one; therefore, the evaluation of the first and the second type error in the tests (31) and (32) has to be corrected with the probability. Denoting the significance level of the tests with the symbol $\alpha^{(av)}$ (its value results from the term $1/\lambda^2$ in Tshebyshev's inequality) the corrected significance level can be expressed, for the equivalence relation, in the form:

$$1\text{-}(1\text{-}\alpha^{(av)}) P(\hat{\chi}_1^{(1,av)}, \ldots, \hat{\chi}_{\hat{n}}^{(1,av)} \equiv \chi_1^{*(1)}, \ldots, \chi_m^{*(1)} \mid \mathbf{R}^{(1)}). \tag{33}$$

The formula (33) results from the fact, that the probability of correct decision (the opposite event to the first type error) in the test is greater than $1\text{-}\alpha^{(av)}$, but it is valid in the case of errorless estimation result (probability of this event is equal to $P(\hat{\chi}_1^{(1,av)},...,\hat{\chi}_{\hat{m}}^{(1,av)} \equiv \chi_1^{*(1)},...,\chi_m^{*(1)} \mid \mathbf{R}^{(1)})$). Therefore, the probability 1-$\alpha^{(av)}$ is multiplied by the factor $P(\hat{\chi}_1^{(1,av)},...,\hat{\chi}_{\hat{m}}^{(1,av)} \equiv \chi_1^{*(1)},...,\chi_m^{*(1)} \mid \mathbf{R}^{(1)})$; the product expresses an evaluation of the probability of conjunction: the errorless estimate and verification result. The corrected significance level is higher than $\alpha^{(av)}$; the increase of the probability resulting from the fact, that test statistics is based on the mixture of the random variables and parameters of one random variable from the mixture are determined only. The correction of the first type error for the tolerance relation is performed in the same way.

The evaluation of the probability of the second type error is obtained under the assumption, that value of the parameter λ is the same in both tests. The probability can be evaluated for both tests in the following way.

In the case of the tolerance relation, the second type error occurs, when $H_{2,0}$ is accepted (i.e. $S^{(av,N)} \geq 1-2\delta-\lambda\sigma_S^{(av,N)}$), while the equivalence relation exists in the set **X** (i.e. $E(S^{(av,N)} \mid H_{1,0})=2\delta-1$). The probability of such event can be evaluated in the following way:

$P(S^{(av,N)} \geq 1-2\delta-\lambda\sigma_S^{(av,N)} \mid H_{1,0}) =$

$P(S^{(av,N)}-(2\delta-1) \geq 1-2\delta-(2\delta-1)-\lambda\sigma_S^{(av,N)} \mid H_{1,0}) =$

$P(S^{(av,N)}-(2\delta-1) \geq 2(1-2\delta)-\lambda\sigma_S^{(av,N)} \mid H_{1,0}) =$

$$P(S^{(av,N)}-(2\delta-1) \geq \lambda_{R0}\sigma_S^{(av,N)} \mid H_{1,0}) \leq P(\mid S^{(av,N)}-(2\delta-1) \mid \geq \mid \lambda_{1,0}^{(av)}\sigma_S^{(av,N)} \mid) \leq 1/\lambda_{1,0}^{(av)2}, \qquad (34)$$

where the value of $\lambda_{1,0}^{(av)}$ results from the relationship (the expression $\lambda_{1,0}^{(av)}\sigma_S^{(av,N)}$ is positive under assumptions made):

$$2(1-2\delta)-\lambda\sigma_S^{(av,N)} = \lambda_{1,0}^{(av)}\sigma_S^{(av,N)} \quad \Rightarrow \quad \lambda_{1,0}^{(av)} = (2(1-2\delta)-\lambda\sigma_S^{(av,N)})/\sigma_S^{(av,N)}. \qquad (35)$$

The probability of the second type error in the case of the equivalence relation (i.e. $E(S^{(av,N)} \mid H_{2,0})=1-2\delta$) is obtained in similar way:

$$P(S^{(av,N)} \leq 2\delta\text{-}1 + \lambda \sigma_S^{(av,N)} \mid \mathrm{H}_{2,0}) =$$

$$P(S^{(av,N)} - (1-2\delta) \leq 2\delta\text{-}1 - (1-2\delta) + \lambda \sigma_S^{(av,N)} \mid \mathrm{H}_{2,0}) =$$

$$P(S^{(av,N)} - (1-2\delta) \leq 2(2\delta\text{-}1) + \lambda \sigma_S^{(av,N)} \mid \mathrm{H}_{2,0}) =$$

$$P(S^{(av,N)} - (1-2\delta) \leq \lambda_{2,0}^{(av)} \sigma_S^{(av,N)} \mid H_{T0}) \leq P(\mid S^{(av,N)} - (1-2\delta) \mid \geq \mid \lambda_{2,0}^{(av)} \sigma_S^{(av,N)} \mid) \leq 1/\lambda_{2,0}^{(av)2}, \tag{36}$$

where:

$$\lambda_{2,0}^{(av)} = (2(2\delta\text{-}1) + \lambda \sigma_S^{(av,N)}) / \sigma_S^{(av,N)} = (\lambda \sigma_S^{(av,N)} - 2(1-2\delta)) / \sigma_S^{(av,N)}. \tag{37}$$

Let us notice, that the values $\lambda_{2,0}^{(av)2}$ and $\lambda_{1,0}^{(av)2}$ are equal for the same value of the parameter λ in both tests; therefore the evaluations of the second type error probabilities are also the same.

Evaluations (34), (36) correspond to the case of errorless estimation result, while the realizations of the random variable $S^{(av,N)}$ are obtained from the mixture of distributions. Therefore these evaluations have to be corrected – similarly, as in (33). Denoting the probability of the second type error, resulting from inequalities (34) and (36), with the symbol $\beta^{(av)}$, the corrected probability of this error can be expressed, for the equivalence relation, in the form:

$$1-(1-\beta^{(av)}) P(\hat{\chi}_1^{(1,av)}, \ldots, \hat{\chi}_{\hat{m}}^{(1,av)} \equiv \chi_1^{*(1)}, \ldots, \chi_m^{*(1)} \mid \mathbf{R}^{(1)}). \tag{38}$$

Let us notice, that if $\beta^{(av)}$ converges to zero and $P(\hat{\chi}_1^{(1,av)}, \ldots, \hat{\chi}_{\hat{m}}^{(1,av)} \equiv \chi_1^{*(1)}, \ldots, \chi_m^{*(1)} \mid \mathbf{R}^{(1)})$ converges to one, then the probability (38) does converge to one and the test is consistent. It seems, that the condition $N \to \infty$ guarantee consistence of the test. The correction of the error for the tolerance relation is performed in the same way.

3.2. The median approach

In the case of the median approach it is assumed that the number of comparisons is uneven N (N=2r+1; r=0, 1, ...,) and that the distribution functions of comparisons $g_k(x_i,x_j)$ (k=1, ..., N; $(x_i, x_j)\in\mathbf{X}\times\mathbf{X}$) of the pair (x_i, x_j) are the same for each k. Under these assumptions the problem is similar to the case of one comparison of each pair. Its role plays the median $g^{(me,N)}(x_i,x_j)$ from comparisons $g_1(x_i,x_j), \ldots, g_N(x_i,x_j)$ (the median is equivalent to majority in the set $\{g_1(\cdot), \ldots, g_N(\cdot)\}$ - see Klukowski 1994, point 5.2) and the probabilities $P(g^{me,N}(x_i,x_j)=T_f(x_i,x_j))$ (f equal to 1 or 2) assume the form:

$$P(g^{(me,N)}(x_i,x_j)=T_f(x_i,x_j))=\begin{cases} P(\sum_{k=1}^{N} g_k(x_i,x_j)<\frac{N}{2} \mid T_f(x_i,x_j)=0); \\ P(\sum_{k=1}^{N} g_k(x_i,x_j)>\frac{N}{2} \mid T_f(x_i,x_j)=1). \end{cases} \tag{39}$$

The value of the probability (39) may be determined on the basis of the probability function of the binomial distribution (or evaluated for large N). Let us notice that for $\delta<½$ the probability $P(g^{(me,N)}(x_i,x_j)=T_f(x_i,x_j))$ converges to one for $N\to\infty$ (because $E\left|\frac{1}{N}\sum_{k=1}^{N}(T_f(\cdot)-g_k(\cdot))\right|<½$ and $\lim_{N\to\infty} Var(\frac{1}{N}\sum_{k=1}^{N}(T_f(\cdot)-g_k(\cdot)))=0$, for $\delta<½$).

The evaluation of the probability (39) will be denoted:

$$P(g^{(me,N)}(x_i,x_j)=T_f(x_i,x_j))\geq 1-\eta_{me,N}\,. \tag{40}$$

Thus, the procedure for the median case can be constructed in the same way as in the averaging approach - using the evaluation (40) instead of (5a). The random variables and tests corresponding to the median case assume the following form:

$$S_{ij,me}=\left|\hat{t}_1^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)\right|-\left|\hat{t}_2^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)\right| \quad (<i,j>\in I_w^{(me)}), \tag{41}$$

where:

$I_w^{(me)}$ - the set of all pairs of indices $<i,j>$, which satisfy the condition:

$\hat{t}_1^{(me)}(x_i, x_j) \neq \hat{t}_2^{(me)}(x_i, x_j)$;

($\hat{t}_1^{(me)}(x_i, x_j)$ and $\hat{t}_2^{(me)}(x_i, x_j)$ defined - respectively - in (9) and (13)).

The test statistics $S^{(me,N)}$ is the sum of random variables $S_{ij,me}$ ($<i, j> \in I_w^{(me)}$) divided by the number of elements in the sum; it assumes the form:

$$S^{(me,N)} = \frac{1}{\#(I_w^{(me)})} \sum_{<i,j> \in I_w^{(me)}} S_{ij,me}\ , \tag{42}$$

where: $\#(I_w^{(me)})$ – number of elements of the set $I_w^{(me)}$.

The properties of the statistics $S^{(me,N)}$ depend on actual relation type in the set **X**. Let us consider firstly the case of the tolerance relation. The expected value and the evaluation of variance of the variable $S^{(me,N)}$ are determined below (assuming equality in (40)).

In the case when tolerance relation exists in the set **X**, the parameters of the variable $S^{(me,N)}$ are as follows (assuming equality in (40)):

$$\begin{aligned} &P(|\hat{t}_2^{(me)}(x_i, x_j) - g^{(me,N)}(x_i, x_j)| = 0 \mid \hat{t}_2^{(me)}(\cdot) = T_2(\cdot)) = \\ &P(g^{(me,N)}(x_i, x_j) = \hat{t}_2^{(me)}(x_i, x_j) \mid \hat{t}_2^{(me)}(\cdot) = T_2(\cdot)) = 1 - \eta_{me,N}, \end{aligned} \tag{43a}$$

$$\begin{aligned} &P(|\hat{t}_2^{(me)}(x_i, x_j) - g_{me}(x_i, x_j)| = 1 \mid \hat{t}_2^{(me)}(\cdot) = T_2(\cdot)) = \\ &P(g_{me}(x_i, x_j) \neq \hat{t}_2^{(me)}(x_i, x_j) \mid \hat{t}_2^{(me)}(\cdot) = T_2(\cdot)) = \eta_{me,N} \end{aligned} \tag{43b}$$

(the expression $\hat{t}_2^{(me)}(\cdot) = T_2(\cdot)$ means equality for all pairs $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$).

Under the assumption $\hat{t}_1^{(me)}(x_i, x_j) \neq \hat{t}_2^{(me)}(x_i, x_j)$, the probability function of the random variable $|\hat{t}_1^{(me)}(x_i, x_j) - g^{(me,N)}(x_i, x_j)|$ assumes the form:

$$\begin{aligned} &P(|\hat{t}_1^{(me)}(x_i, x_j) - g^{(me,N)}(x_i, x_j)| = 0 \mid \hat{t}_2^{(me)}(\cdot) = T_2(\cdot)) = \\ &P(g^{(me,N)}(x_i, x_j) \neq \hat{t}_2^{(me)}(\cdot) \mid \hat{t}_2(\cdot) = T_2(\cdot)) = \eta_{me,N}, \end{aligned} \tag{44a}$$

$$\begin{aligned} &P(|\hat{t}_1^{(me)}(x_i, x_j) - g^{(me,N)}(x_i, x_j)| = 1 \mid \hat{t}_2^{(me)}(\cdot) = T_2(\cdot)) = \\ &P(g^{(me,N)}(x_i, x_j) = \hat{t}_2^{(me)}(x_i, x_j) \mid \hat{t}_2^{(me)}(\cdot) = T_2(\cdot)) = 1 - \eta_{me,N}. \end{aligned} \tag{44b}$$

The equalities (43a) - (44b) indicate:

$$P(S_{ij,me}=-1|\ \hat{t}_2^{(me)}(\cdot)=T_2(\cdot))=$$
$$P[(|\hat{t}_1^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)|=0)\cap(|\hat{t}_2^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)|=1)|\ \hat{t}_2(\cdot)=T_2(\cdot)\,]=\eta_{me,N},\tag{45a}$$

$$P(S_{ij,me}=1|\ \hat{t}_2^{(me)}(\cdot)=T_2(\cdot))=$$
$$P[(|\hat{t}_1^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)|=1)\cap(|\hat{t}_2^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)|=0)|\ \hat{t}_2^{(me,N)}(\cdot)=T_2(\cdot)]=$$
$$1-\eta_{me,N}.\tag{45b}$$

It follows from (45a) and (45b) that in the case of the tolerance relation, the expected value $E_2(S_{ij,me})$ and variance $Var_2(S_{ij,me})$ of each random variable $S_{ij,me}$ ($<i,j>\in I_w^{(me)}$) assume the form - respectively:

$$E_2(S_{ij,me}) = -\eta_{me,N} + 1 - \eta_{me,N} = 1 - 2\eta_{me,N},\tag{46}$$

$$Var_2(S_{ij,me}) = -1-(1-2\eta_{me,N}))^2\,\delta+(1-(1-2\eta_{me,N}))^2(1-\eta_{me,N}) = 4\eta_{me,N}(1-\eta_{me,N}).\tag{47}$$

The expected value and the variance of the variable $S^{(me,N)}$ are equal:

$$E_2(S^{(me,N)}) = 1-2\eta_{me,N},\tag{48}$$

$$Var_2(S^{(me,N)})\le 4(\#(I_w^{(me)})-2L(I_w^{(me)})/\#(I_w^{(me)}))\eta_{me,N}(1-\eta_{me,N}).\tag{49}$$

The properties (48) and (49) of the random variable $S^{(me,N)}$ are similar to the case of averaging approach.

In the case, when the equivalence relation exists in the set **X** and the result of estimation is errorless the distribution of the random variable $S^{(me,N)}$ assumes the form:

$$P(|\hat{t}_1^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)|=0|\ \hat{t}_1^{(me)}(\cdot)=T_1(\cdot))=$$
$$P(g^{(me,N)}(x_i,x_j)=\hat{t}_1^{(me)}(x_i,x_j)|\ \hat{t}_1^{(me)}(\cdot)=T_1(\cdot))=1-\eta_{me,N},\tag{50a}$$

$$P(|\hat{t}_1^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)|=1|\ \hat{t}_1^{(me)}(\cdot)=T_1(\cdot))=$$
$$P(g^{(me,N)}(x_i,x_j)\ne\hat{t}_1^{(me)}(x_i,x_j)|\ \hat{t}_1^{(me)}(\cdot)=T_1(\cdot))=\eta_{me,N},\tag{50b}$$

$$P(|\hat{t}_2^{(me)}(x_i,x_j)-g^{(me,N)}(x_i,x_j)|=0|\ \hat{t}_1^{(me)}(\cdot)=T_1(\cdot))=$$
$$P(g^{(me,N)}(x_i,x_j)=\hat{t}_2^{(me)}(x_i,x_j)|\,\hat{t}_1^{(me)}(\cdot)=T_1(\cdot))=\eta_{me,N},\tag{51a}$$

$$P(|\hat{t}_2^{(me)}(x_i, x_j) - g_k(x_i, x_j)| = 1 \mid \hat{t}_1^{(me)}(\cdot) = T_1(\cdot)) =$$
$$P(g^{(me,N)}(x_i, x_j) \neq \hat{t}_2^{(me)}(x_i, x_j) \mid \hat{t}_1^{(me)}(\cdot) = T_1(\cdot)) = 1 - \eta_{me,N}, \tag{51b}$$

(the expression $\hat{t}_1^{(me)}(\cdot) = T_1(\cdot)$ means equality for each $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$).

From (50a) and (51b) it follows, that:

$$P(S_{ij,me} = -1 \mid \hat{t}_1^{(me)}(x_i, x_j) = T_1(x_i, x_j)) =$$
$$P[(|\hat{t}_1^{(me)}(x_i, x_j) - g^{(me,N)}(x_i, x_j)| = 0) \cap (|\hat{t}_2^{(me)}(x_i, x_j) - g^{(me,N)}(x_i, x_j)| = 1) \mid \hat{t}_1^{(me)}(\cdot) = T_1(\cdot)] =$$
$$1 - \eta_{me,N}, \tag{52a}$$
$$P(S_{ij,me} = 1 \mid \hat{t}_1^{(me)}(x_i, x_j) = T_1(x_i, x_j)) =$$
$$P[(|\hat{t}_1^{(me)}(x_i, x_j) - g^{(me,N)}(x_i, x_j)| = 1) \cap (|\hat{t}_2^{(me)}(x_i, x_j) - g^{(me,N)}(x_i, x_j)| = 0) \mid \hat{t}_1^{(me)}(\cdot) = T_1(\cdot)] =$$
$$\eta_{me,N}. \tag{52b}$$

The formulas (52a) and (52b) are the basis for determination of expected value and variance of each random variable $S_{ij,me}$:

$$E_1(S_{ij,me}) = -1 + \eta_{me,N} + \eta_{me,N} = 2\eta_{me,N} - 1, \tag{53}$$

$$Var_1(S_{ijk}) = 4\eta_{me,N}(1 - \eta_{me,N}). \tag{54}$$

The form of above parameters allows to determine the expected value and evaluation of variance of the random variable $S^{(me,N)}$, when the equivalence relation exists in the set **X**. The expected value assumes the form:

$$E_1(S^{(me,N)}) = 2\eta_{me,N} - 1, \tag{55}$$

while the variance satisfy the condition:

$$Var_1(S^{(me,N)}) \leq 4(\#(I_w) - 2L(I_w)/\#(I_w))\eta_{me,N}(1 - \eta_{me,N}). \tag{56}$$

The properties (55) and (56) hold for the equivalence relation, when errorless estimation result occurs. However, with the probability $1 - P(\hat{\chi}_1^{(1,me)}, \ldots, \hat{\chi}_1^{(1,me)} \equiv \chi_1^{*(1)}, \ldots, \chi_m^{*(1)} \mid \mathbf{R}^{(1)})$ the result of estimation is different than errorless one. Therefore, the value of the random variable $S^{(me,N)}$ is a realization of a mixture of random variables, with similar properties as in the case of the tolerance relation.

The test for verification the tolerance relation in the set **X** rests on expected value (equal to 1-2$\eta_{me,N}$) and evaluation (56) of variance of the random variable $S^{(me,N)}$. The null and alternative hypothesis ($H_{2,0}$ and $H_{2,1}$) of the test can be formulated in the following way:

$H_{2,0}$: $E(S^{(me,N)}) = 1 - 2\eta_{me,N}$,

$H_{2,1}$: $E(S^{(me,N)}) = 2\eta_{me,N} - 1$,

with the critical region:

$$\Lambda_2^{(me,N)} = \{ S^{(me,N)} \mid S^{(me,N)} < 1-2\eta_{me,N} - \lambda\sigma_S^{(me,N)}), \quad (57)$$

where: $\sigma_S^{(me,N)}$ – square root of the variance $Var_2(S^{(me,N)})$ evaluation, i.e.:

$$\sigma_S^{(me,N)} = (Var_2(S^{(me,N)}))^{1/2} = [4(\#(I_w^{(me)}) - 2L(I_w^{(me)})/\#(I_w^{(me)}))\eta_{me,N}(1-\eta_{me,N})]^{1/2}.$$

The form of the test for the equivalence relation is "symmetric":

$H_{1,0}$: $E(S^{(me,N)}) = 2\eta_{me,N} - 1$,

$H_{1,1}$: $E(S^{(me,N)}) = 1 - 2\eta_{me,N}$,

with the critical region:

$$\Lambda_1^{(me,N)} = \{ S^{(me,N)} \mid S^{(me,N)} > 2\eta_{me,N} - 1 + \lambda\sigma_S^{(me,N)}) \quad (58)$$

($\sigma_S^{(me,N)}$ – the same, as in the formula (57)).

In the case of the tolerance relation, the second type error occurs, when $H_{2,0}$ is accepted (i.e. $S^{(me,N)} \geq 1-2\eta_{me,N} - \lambda\sigma_S^{(me,N)}$), while the equivalence relation is true (i.e. $E(S^{(me,N)} \mid H_{1,0}) = 2\eta_{me,N} - 1$). The probability of the event can be evaluated in the following way:

$P(S^{(me,N)} \geq 1-2\eta_{me,N} - \lambda\sigma_S^{(me,N)} \mid H_{1,0}) =$

$P(S^{(me,N)} - (2\eta_{me,N} - 1) \geq 1-2\eta_{me,N} - (2\eta_{me,N} - 1) - \lambda\sigma_S^{(me,N)} \mid H_{1,0}) =$

$P(S^{(me,N)} - (2\eta_{me,N} - 1) \geq 2(1-2\eta_{me,N}) - \lambda\sigma_S^{(me,N)} \mid H_{1,0}) =$

$$P(S^{(me,N)}-(2\eta_{me,N}-1)\geq\lambda_{1,0}^{(me)}\sigma_S^{(me,N)}\mid H_{1,0})\leq P(\mid S^{(me,N)}-(2\eta_{me,N}-1)\mid\geq\mid\lambda_{1,0}^{(me)}\sigma_S^{(me,N)}\mid)\leq$$

$$\leq 1/\lambda_{1,0}^{(me)2}, \tag{59}$$

where the value of $\lambda_{1,0}^{(me)}$ is determined in the following way (the expression $\lambda_{1,0}^{(me)}\sigma_S^{(me,N)}$ is positive under the assumptions made):

$$2(1-2\eta_{me,N})-\lambda\sigma_S^{(me,N)}=\lambda_{1,0}^{(me)}\sigma_S^{(av,N)} \Rightarrow \lambda_{1,0}^{(me)}=(2(1-2\eta_{me,N})-\lambda\sigma_S^{(me,N)})/\sigma_S^{(me,N)}. \tag{60}$$

The probability of the second type error in the case the equivalence relation ($E(S^{(me,N)}\mid H_{2,0})=1-2\eta_{me,N}$) is obtained in similar way:

$$P(S^{(me,N)}\leq 2\eta_{me,N}-1+\lambda\sigma_S^{(me,N)}\mid H_{2,0})=$$

$$P(S^{(me,N)}-(1-2\eta_{me,N})\leq 2\eta_{me,N}-1-(1-2\eta_{me,N})+\lambda\sigma_S^{(me,N)}\mid H_{2,0})=$$

$$P(S^{(me,N)}-(1-2\eta_{me,N})\leq 2(2\eta_{me,N}-1)+\lambda\sigma_S^{(me,N)}\mid H_{2,0})=$$

$$P(S^{(me,N)}-(1-2\eta_{me,N})\leq\lambda_{2,0}^{(me)}\sigma_S^{(me,N)}\mid H_{2,0})\leq P(\mid S^{(me,N)}-(1-2\eta_{me,N})\mid\geq\mid\lambda_{2,0}^{(me)}\sigma_S^{(me,N)}\mid)\leq$$

$$\leq 1/\lambda_{2,0}^{(me)2}, \tag{61}$$

where:

$$\lambda_{2,0}^{(me)}=(2(2\eta_{me,N}-1)+\lambda\sigma_S^{(me,N)})/\sigma_S^{(me,N)}=(\lambda\sigma_S^{(me,N)}-2(1-2\eta_{me,N}))/\sigma_S^{(me,N)}. \tag{62}$$

The probabilities of errors in the tests have to be corrected with the factor $P(\hat{\chi}_1^{(1,me)},\ldots,\hat{\chi}_{\hat{m}}^{(1,me)}\equiv\chi_1^{*(1)},\ldots,\chi_m^{*(1)}\mid \mathbf{R}^{(1)})$:

- the corrected significance level for the equivalence relation assumes the form:

$$1-(1-\alpha^{(me)})P(\hat{\chi}_1^{(1,me)},\ldots,\hat{\chi}_{\hat{m}}^{(1,me)}\equiv\chi_1^{*(1)},\ldots,\chi_m^{*(1)}\mid \mathbf{R}^{(1)}), \tag{63}$$

- the corrected probability of the second type error for the equivalence relation assumes the form:

$$1-(1-\beta^{(me)})P(\hat{\chi}_1^{(1,me)},\ldots,\hat{\chi}_{\hat{m}}^{(1,me)}\equiv\chi_1^{*(1)},\ldots,\chi_m^{*(1)}\mid \mathbf{R}^{(1)}). \tag{64}$$

The corrected probabilities of errors for the tolerance relation assume the same form as in the case of the equivalence relation.

4. Summary

The tests for distinction of the equivalence and tolerance relation in the case of multiple comparisons are based on weak assumptions and simple probabilistic inequalities and evaluations. Therefore the results of its application can be also of rough type; it is a "cost" of non-restricted assumptions about comparisons. However, the tests provide some progress in comparison to arbitrary decision. The properties of the tests require further investigation, especially with the use of simulation approach.

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