Raport Badawczy

Research Report

RB/17/2016

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Warszawa 2016

# Monitoring series of dependent observations using the sXWAM control chart for residuals

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**Abstract** Control charts for monitoring residuals are the main tools for statistical process control of autocorrelated streams of data. X chart for residuals calculated from a series of individual observations is probably the most popular, but its statistical characteristics are not satisfactory, especially for charts designed using limited amount of data. In order to improve these characteristics Hryniewicz and Kaczmarek proposed a new chart for residuals, XWAM chart, using the concept of weighted model averaging. Unfortunately, the design of the XWAM chart is rather complicated, and requires significant computational effort. In this paper we propose its simplification, named sXWAM chart, which is simpler to design, and in some practically important cases has similar statistical properties.

## 1 Introduction

Control charts are nowadays the most frequently used tool of Statistical Quality Control (SQC). They were introduced in the 1920th by Shewhart who at that time worked for an American company Western Electric. Since that time many statistical procedures which have their origins in Shewhart's works have been developed, and their usage in practice is known under a common name of Statistical Process Control (SPC). In the majority of applications control charts are used for monitoring production processes when long series of quality-related measurements are observed. In such a case a well grounded mathematical theory have been established. Later on, control charts have been also applied in cases of short production runs. In

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this case the mathematical theory of control charts is still under development, and analyses based on computer simulations are frequently used by researchers. Control charts may be used for monitoring of all kind of processes, both univariate and multivariate. However, the most popular of them are designed under the assumption of independence of consecutive observations. Unfortunately, serious problems arise when consecutive observations are statistically dependent. Such situations take place quite frequently when we use control charts for monitoring continuous process (e.g., chemical) data or are used to monitor data related to human health. Pioneering works in the area of process control in presence of dependent (autocorrelated) data were published in the 1970th (e.g., [7]). Since that time many papers devoted to this problem have been published, and they can be, in general, divided into two groups. Authors of the first group of papers, such as, e.g., Vasilopoulos and Stamboulis [25], Montgomery and Mastrangelo [22], Maragah and Woodall [21], Yashchin [28], Schmid [23] or Zhang [30], propose to adjust design parameters of classical control charts (Shewhart, CUSUM, EWMA) in order to take into account the impact of autocorrelation in data on chart's statistical properties. The second group of papers is originated by the paper by Alwan and Roberts [3] who proposed a control chart for residuals. In order to develop a control chart for residuals we have to build a mathematical model of the observed process using the methodology developed for the analysis of time series. The deterministic part of this model is used for the computation of predicted values of observations, and differences between predicted and observed values of the process, named *residuals*, are used as observations plotted on a control chart. Properties of different control charts for residuals have been investigated by many authors, such as, e.g., Wardell et al. [27], Zhang [29], Kramer and Schmid [19]. Both approaches have been compared in many papers, such as, e.g., Lu and Reynolds [20]. It has to be noted, however, that the applicability of the charts for residuals in SPC was a matter of discussion (see, e.g., the paper by Runger [24]), but now this approach seems to be prevailing.

The concept of residuals is a quite general one. In theory, any mathematical model used for prediction purposes can be used for the calculation of residuals. However, in practice only relatively simple autoregression models are used. For such models there exist analytical formulae that allow to implement a control chart for residuals relatively simply. However recently, more sophisticated control charts based on the idea of residuals have been proposed. For example, the ARMA chart, based on the autoregressive moving average model, proposed by Jiang et al. [17], the chart proposed by Chin and Apley [11] based on second-order linear filters, the chart proposed by Apley and Chin [4] based on general linear filters or the PCA-based procedure for the monitoring multidimensional processes proposed by De Ketelaere et al. [12]. Unfortunately, these charts can used in practice only with the help of professional software dedicated to the analysis of time series or specially dedicated programs (e.g., written by the community which develops the openly accessed programming language R).

A proper design of a control chart for autocorrelated data requires the knowledge of the mathematical model of the monitored process. When we have enough data for precise identification of the process model few solutions have already been proposed for the calculation of such characteristics of a control chart as the Average Run Length (ARL) in the in-control state, when the monitored process remains in stable conditions. However, serious problems arise when we want to calculate control chart's characteristics when the monitored process goes out of control. The situation is even worse when the amount of available data is not sufficient for the identification of the underlying model of dependence. In such a case only few analytical results exist (see, e.g., the paper by Kramer and Schmid [19] or the paper by Apley and Lee [6]). The main reason of these difficulties is the fact that for imprecisely estimated model of dependence not only observations, but residuals as well, are autocorrelated. Unfortunately, this happens in practice when we have to design a control chart after the observation of limited amount of available process data, such as, e.g., taken from the process being in its prototype phase or when we monitor patients in a health-care system.

The performance of the control charts for residuals strongly depends upon proper identification of the dependence model. In other words, if we want to have an efficient control chart for residuals we have to use good predictor for future observations of the monitored process. When we have enough data for building a good model, i.e., when the available time series is not too short, one can use methods described in textbooks on time series. However, when available data is not sufficient for building such good predictors we have to use more sophisticated methods, such as those developed by econometricians for prediction purposes in short economic time series. In such cases efficient Bayesian methods combined with the Markov Chain Monte Carlo (MCMC) simulation methodology have been developed. For very good description of this approach, see the book by Geweke [13]. Many Bayesian models used for the prediction in short time series are based on the concept of model averaging. This concept consists in taking into account not only prior knowledge about model parameters, but also prior knowledge about several possible models that can be used for prediction purposes. In practice, non-informative priors are used, and MCMC simulations are used for the evaluation of predictive posterior distributions. Better results could be obtained if take more informative probabilities (weights) of the considered models. However, finding such weights remains a serious practical problem. To overcome this, Hryniewicz and Kaczmarek [14] proposed to use some computational intelligence methods for the construction of the prior distribution on the pre-chosen set of models. Their algorithm appears to be highly competitive when compared to the best available algorithms used for the prediction in short time series. In their recent papers [15] and [16] they have adopt a similar approach for the construction of a new Shewhart-type control chart for residuals, named the XWAM chart.

This paper presents further development of the ideas described in [15] and [16]. In particular, a new and much simpler method for the construction of the XWAM chart (with only one model that is alternative to the estimated one) has been proposed. The properties of this new version of the XWAM chart, named the sXWAM (simplified XWAM) chart, have been investigated using extensive computer simulations.

The paper is organized as follows. In the next section we describe the assumed mathematical model of the monitored process. In the third section we present the algorithm for the construction of the XWAM chart, and its simplified sXWAM version. The results of computer simulations which have been performed with the aim to evaluate statistical characteristics of the sXWAM control chart are presented in Section 4. The paper is concluded in the last section, where we also outline possible areas of future investigations.

## 2 Shewhart X chart for residuals

Consider random observations described by a series of possibly dependent random variables $X_1, X_2, \ldots$. In statistical quality control these random variables may describe individual observations or observed values of sample statistics, such as, e.g., averages plotted on a popular Shewhart $\bar{X}$-chart. The most frequently used control charts are designed under the assumption of mutual independence of the observations (measurements) of monitored processes. In many practical cases, however, the assumption of independence does not hold and the full mathematical description of such a series of observations can be done using a multivariate (possibly infinitely-dimensional) probability distribution. Unfortunately, in practice this usually cannot be done. Therefore, statisticians introduced simpler and easier tractable mathematical models which are well described in textbooks devoted to the analysis of time series. In the most popular model of this kind the random variable representing the current observation is given as the sum of a deterministic part depending on the observed values of previous observations, and a random part whose probability distribution does not depend upon the previously observed values, i.e.,

$$X_i = f(x_1, \ldots x_{i-1}) + \varepsilon_i, i = 1, \ldots. \tag{1}$$

In the simplest version of (1) we usually assume that random variables $\varepsilon_i, i = 1, \ldots$ (called sometimes innovations) are mutually independent and identically distributed with the expected value equal to zero. On the other hand, we often assume that the deterministic part $f(x_1, \ldots, x_{i-1})$ has a form that assures stationarity of the time series $X_1, X_2, \ldots$ (for the definition of stationarity, see any textbook on time series, e.g., [9] ). In this paper we make even stronger assumption that

$$X_i = a_1 x_{i-1} + \ldots + a_p x_{i-p} + \varepsilon_i, \tag{2}$$

where $\varepsilon_i, i = 1, \ldots$ are normally distributed independent random variables with the expected value equal to zero, and the same finite standard deviation. Thus, our assumed model describes a classical autoregressive stochastic process of the $p$th order $AR(p)$. The comprehensive description of the $AR(p)$ process can be found in every textbook devoted to the analysis of time series, e.g., in the seminal book by Box and Jenkins [8] or a popular textbook by Brockwell and Davis [9]. It is worth noting that more complicated models have been recently proposed for monitoring processes with dependent data. These general models, such as, e.g., the $ARMA(p,q)$ which are also special cases of (1) and are widely used in the statistical analysis of time series, are also described in the aforementioned books.

Estimation of the model $AR(p)$, given by (2), is relatively simple when we know the order of the model $p$. In order to do this we have to calculate first $p$ sample autocorrelations $r_1, r_2, \ldots, r_p$, defined as

$$r_i = \frac{n\sum_{t=1}^{n-i}(x_t-\hat{\mu})(x_{t+i}-\hat{\mu})}{(n-i)\sum_{t=1}^{n}(x_t-\hat{\mu})^2}, i=1,\ldots,p, \tag{3}$$

where $n$ is the number of observations (usually, it is assumed that $n \geq 4p$) in the sample, and $\hat{\mu}$ is the sample average. Then, the parameters $a_1,\ldots,a_p$ of the $AR(p)$ model are calculated by solving the Yule-Walker equations (see, [9])

$$\begin{aligned} r_1 &= a_1 + a_2 r_1 + \ldots + a_p r_{p-1} \\ r_2 &= a_1 r_1 + a_2 + \ldots + a_p r_{p-2} \\ &\ldots \\ r_p &= a_1 r_{p-1} + a_2 r_{p-2} + \ldots + a_p \end{aligned} \tag{4}$$

In practice, however, we do not know the order of the autoregression process, so we need to estimate $p$ from data. In order to do this let us first define a random variable, called the *residual*.

$$Z_i = X_i - (a_1 x_{i-1} + \ldots + a_p x_{i-p}), i = p+1,\ldots,N. \tag{5}$$

The probability distribution of residuals is the same as the distribution of random variables $\varepsilon_i, i = 1,\ldots$ in (2), and its variance can be used as a measure of the accuracy of predictions of future values of the process. For given sample data of size $n$ the variance of residuals is decreasing with the increasing values of $p$. However, the estimates of $p$ models parameters $a_1,\ldots,a_p$ become less precise, and thus the overall precision of prediction with future data deteriorates. As the remedy to this effect several optimization criteria with a penalty factor which discourages the fitting of models with too many parameters have been proposed. In this research we use the $BIC$ criterion proposed by Akaike [1], and defined as

$$BIC = (n-p)\ln[n\hat{\sigma}^2/(n-p)] + n(1+\ln\sqrt{2\pi}) + p\ln[(\sum_{t=1}^{n} x_t^2 - n\hat{\sigma}^2)/p], \tag{6}$$

where $x_t$ are process observations transformed in such a way that their expected values are equal to zero, and $\hat{\sigma}^2$ is the observed variance of residuals. The fitted model, i.e., the estimated order $p$ and parameters of the model $\hat{a}_1,\ldots,\hat{a}_p$ minimizes the value of $BIC$ calculated according to (6).

SPC for processes with autocorrelated data using a control chart for residuals was firstly proposed by Alwan and Roberts [3]. Their methodology is applicable for any class of processes, so it is also applicable for the $AR(p)$ process considered in this paper. According to the methodology proposed by Alwan and Roberts [3] the deterministic part of (1) is estimated from sample data, and then used for the calculation of residuals. This methodology is also known under the name "filtering". The residuals are used for the design of a control chart for residuals, and are used as transformed process observations that are plotted on the control chart. If we have enough observations, and the model of our process is estimated sufficiently precisely, the calculated residuals are approximately independent, and we can use standard control charts (Shewhart, CUSUM, EWMA, etc.) for process control.

It is a well known fact that the accuracy of prediction in time series strongly depends upon the number of available observations. Moreover, problems arise with the identification of the probability distribution of residuals, as in the case of the inaccurately estimated prediction model the residuals become dependent. The effect of this dependency was investigated theoretically by Kramer and Schmid [19]. Therefore, the usage of classical control charts (with assumed independent observations) for monitoring residuals becomes questionable. In the context of SPC we may face this problem when we have to design a control chart for a short production run. In such a case the accuracy of the estimated model of a monitored process may be completely insufficient if we follow recommendations applicable in the case of a control chart for independent observations.

The problem of insufficient information used for the design of a control chart is typical not only for the control charts for residuals. In the case of Shewhart control charts for original, but independent, observations this problem was considered, e.g., in the paper by Albers and Kallenberg [2], who proposed some corrections to the control limits of a chart, or in the recent paper by Chakraborti [10], who proposed a method for exact calculation of the characteristics of a Shewhart control chart with estimated control limits.

An interesting comparison of the behavior of the classical Shewhart X chart for individual observations and the Shewhart X chart for residuals has been presented in the recent paper of the authors of this paper [16]. They have performed an extensive simulation experiment in which $N = 50000$ (200000 in the case of independent observations) charts were designed, and for each of them $N_R = 5000$ process runs of maximum $M_R = 500000$ observations (curtailment value) were simulated. The charts of both types (i.e., for original observations and residuals) have been designed using the information coming from the simulated samples of $n$ items. For each of the considered charts they have calculated the average run length (ARL), and the median run length (MRL). Below, we present the results of that comparison only for the case of the average run length (average time to the alarm signal) ARL. In Table 1 taken from [16] we present the comparison of the following characteristics of the probability distribution of ARL: average of the distribution of ARL's (AvgARL), standard deviation of the distribution of ARL's (StdARL), median the distribution of of ARL's (MedARL), skewness of the distribution of ARL's (SkewARL). All the presented results have been obtained for independent observations.

The results of simulations presented in Table 1 confirm many of well known facts. For example, in the case of the X chart for individual and independent observations (columns 2–5) the distribution of ARL's (over a set of possible control charts) for small samples is extremely positively skewed. Averaging of ARL's yields for small samples strongly positively biased estimators of the theoretical value of the ARL

**Table 1** Characteristics of the ARL distributions for X charts, and X charts for residuals - independent observations

| | X-chart | | | | X-chart (residuals) | | | |
|---|---|---|---|---|---|---|---|---|
| n | AvgARL | StdARL | MedARL | SkewARL | AvgARL | StdARL | MedARL | SkewARL |
| 20 | 1554.7 | 9228.1 | 256.9 | 23.0 | 485.6 | 5106.4 | 66.5 | 47.5 |
| 30 | 863.7 | 3326.9 | 287.6 | 37.2 | 369.9 | 1924.4 | 114.4 | 66.2 |
| 40 | 674.0 | 1730.8 | 306.3 | 30.5 | 342.9 | 801.6 | 150.2 | 14.4 |
| 50 | 578.1 | 988.5 | 315.4 | 10.9 | 341.2 | 637.1 | 180.0 | 13.8 |
| 100 | 455.9 | 401.2 | 342.3 | 4.1 | 345.8 | 315.1 | 258.1 | 5.7 |
| 200 | 408.6 | 225.0 | 355.6 | 2.1 | 356.2 | 194.6 | 309.5 | 2.0 |
| 500 | 385.0 | 125.3 | 364.9 | 1.2 | 365.9 | 118.8 | 346.4 | 1.2 |
| 1000 | 377.2 | 85.1 | 367.0 | 0.8 | 369.4 | 83.1 | 359.2 | 0.8 |
| 2000 | 374.2 | 59.1 | 369.2 | 0.5 | 371.4 | 58.3 | 366.6 | 0.5 |

equal to 370.4. This means that the rate of false alarms is lower than expected (a positive effect), but on the other hand, the ability of a chart to detect process deterioration becomes significantly lower than needed (a strongly negative effect). When we consider the X chart for residuals (columns 6–9) the situation is somewhat different. In this case the uncertainty related to imprecisely calculated control limits (positive bias) is combined with the uncertainty related to the computation of residuals (negative, as it was proved in [19]). The total bias of the estimators of ARL, based on averaging, is not a monotonic function of the sample size $n$, and attains its minimum at $n$ approximately equal to 40.

In Tables 2–3, taken from [16], we present the results of similar simulation experiments for autocorrelated data when the autocorrelation is described by the autoregression model of the first order – AR(1) model. Four cases of the strength of dependence are considered, described by the autocorrelation coefficients equal to -0.9 , -0.5 , 0.5 , and 0.9, respectively. In columns 3 and 7 values of another characteristic, median of the Median Run Length (MedMRL), are also presented.

**Table 2** Properties of the X chart for residuals with dependent observations - negative autocorrelation

| | $\rho = -0.9$ | | | | $\rho = -0.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| n | AvgARL | MedARL | MedMRL | SkewARL | AvgARL | MedARL | MedMRL | SkewARL |
| 20 | 2212.6 | 165.7 | 115.0 | 19.3 | 763.5 | 84.4 | 59.5 | 34.4 |
| 30 | 928.1 | 203.0 | 141.0 | 42.7 | 503.5 | 132.3 | 92.0 | 57.1 |
| 40 | 589.6 | 225.3 | 157.0 | 91.6 | 419.8 | 171.7 | 120.0 | 30.3 |
| 50 | 524.4 | 244.4 | 204.0 | 97.5 | 388.5 | 199.7 | 139.0 | 18.5 |
| 100 | 395.1 | 293.8 | 204.0 | 3.9 | 368.7 | 273.8 | 190.0 | 3.9 |
| 200 | 377.5 | 327.3 | 227.0 | 2.3 | 369.6 | 320.7 | 223.0 | 2.1 |
| 500 | 371.4 | 351.0 | 244.0 | 1.1 | 370.8 | 351.2 | 244.0 | 1.2 |
| 1000 | 370.7 | 360.6 | 250.0 | 0.8 | 370.9 | 360.8 | 251.0 | 0.8 |
| 2000 | 371.0 | 366.7 | 254.0 | 0.5 | 371.1 | 366.2 | 254.0 | 0.5 |

The interpretation of the results presented in Tables 2–3 is similar to that in the case of independent data. We can see that the expected values of ARL are more

**Table 3** Properties of the X chart for residuals with dependent observations - positive autocorrelation

| | $\rho = 0.9$ | | | | $\rho = 0.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| n | AvgARL | MedARL | MedMRL | SkewARL | AvgARL | MedARL | MedMRL | SkewARL |
| 20 | 1304.8 | 70.5 | 49.0 | 24.7 | 629.3 | 67.7 | 47.3 | 40.5 |
| 30 | 654.3 | 116.0 | 81.0 | 48.3 | 457.3 | 119.1 | 83.0 | 70.5 |
| 40 | 461.7 | 155.3 | 108.0 | 83.8 | 398.2 | 160.4 | 112.0 | 31.8 |
| 50 | 390.9 | 184.4 | 128.0 | 183.4 | 374.8 | 191.2 | 133.0 | 19.3 |
| 100 | 361.4 | 266.3 | 185.0 | 4.3 | 364.9 | 270.5 | 188.0 | 3.9 |
| 200 | 365.4 | 318.2 | 221.0 | 2.2 | 368.4 | 319.6 | 222.0 | 2.1 |
| 500 | 369.2 | 349.7 | 243.0 | 1.2 | 370.6 | 350.5 | 243.5 | 1.2 |
| 1000 | 370.5 | 360.0 | 250.0 | 0.8 | 370.9 | 360.7 | 250.5 | 0.8 |
| 2000 | 371.3 | 366.3 | 254.0 | 0.5 | 371.1 | 366.2 | 254.0 | 0.5 |

sensitive to the strength of dependence in the case of negative dependence. However, in case of the positive dependence the observed bias practically does not depend on the strength of dependence (except for very small sample sizes). From Tables 1–3 we can also notice extremely high values of skewness of the distribution of ARL's when sample sizes are small. In practice it means that from time to time (but rather seldom) we can face the case of a control chart with too wide control limits. Such charts are unable to detect changes of the process level. Therefore, when our chart is designed using a small sample of observations, we can never be sure that the lack of an alarm is due to good behavior of a monitored process, and not the consequence of too wide control limits. To alleviate this problem Hryniewicz and Kaczmarek [16] propose to curtail the length of a process run. When the number of consecutive observations (residuals) reaches the curtailment limit, say $M_R$, without observing an alarm, we should stop charting, and should recalculate control limits. Extreme skewness of the distributions of ARL's has a very negative impact on the investigations based on computer simulations. If we use averages (over a set of simulated control charts) for the estimation purposes even in the case of thousands of simulated charts few outlying cases, that make the value of skewness so high, may dramatically change the results of estimation.

As it could be expected, introduction of curtailment for a monitored process may significantly change operational characteristics of control charts. This influence of curtailment on X control charts for residuals was investigated in [16]. An example of such analysis is presented in Table 4, taken from [16].

**Table 4** Properties of the X chart for residuals with dependent observations - positive autocorrelation, runs curtailed at 1000

| | $\rho = 0.9$ | | | | $\rho = 0.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| n | AvgARL | MedARL | MedMRL | SkewARL | AvgARL | MedARL | MedMRL | SkewARL |
| 20 | 190.8 | 70.0 | 49.0 | 1.7 | 172.5 | 67.8 | 48.0 | 1.9 |
| 30 | 220.0 | 117.4 | 82.0 | 1.4 | 214.1 | 119.2 | 83.0 | 1.5 |
| 40 | 241.0 | 156.8 | 109.0 | 1.3 | 249.1 | 169.3 | 113.0 | 1.3 |
| 50 | 255.0 | 187.4 | 131.0 | 1.1 | 259.7 | 193.7 | 135.0 | 1.1 |
| 100 | 296.8 | 262.8 | 187.0 | 0.82 | 299.5 | 265.8 | 189.0 | 0.83 |

The results of simulation experiments presented in [16] show that the average values of ARL's are, especially for small sample sizes, significantly smaller than those observed for processes without curtailment imposed on the run length. It is worth noting than in the case of a classical Shewhart control charts with known process parameters the curtailment to 1000 observations decreases the ARL from 370 to 345. From Table 4 we can see that in the case of small samples the impact of curtailment may be even more significant. Therefore, when we design a X control chart for residuals using information coming from small samples, we should not use 345 as the design target value for the ARL.

## 3 XWAM control chart for residuals and its sXWAM modification

The results published by many authors, including the results of computer simulations presented in [15] and [16], and recalled in the previous Section, show undoubtedly that properties of the X chart for residuals designed using small amount of data are unsatisfactory from a practical point of view. Large values of the average ARL presented in Tables 1 - 3 for non-curtailed run lengths signal the possibility of non-detecting process disorders. On the other hand, when we impose curtailment we observe a contrary effect: the average ARL is too small, and thus, the rate of false alarms is too high. The possible reason of these effects is related to imprecise estimation of the process model, necessary for calculation of residuals.

The problem of small sample sizes in the design of control charts is not new. However, autocorrelated data have been usually considered for production processes with large number of available observations. Possible application of control charts in areas where data are correlated and scarce, such as data encountered in health care systems, changes this situation. Thus, we have to look for better solutions in the areas where short time series are common, namely in the analysis of economic data. The problem of prediction in short economic time series is of great importance for econometricians. They have introduced many effective methods, especially those based on the Bayesian paradigm and model averaging. A good description of the Bayesian approach in the analysis of time series can be found in the monograph by Geweke [13]. The concept of model averaging, promoted in [13] and other papers, inspired Hryniewicz and Kaczmarek [14] who proposed a prediction algorithm in which model weights are computed using some methods of intelligent computing and data mining. The proposed prediction algorithm has appeared to be very competitive in comparison to many others considered as good ones (see, [14]). A similar approach was used by the same authors when they introduced in [15] a new control chart, named XWAM (X Weighted Average Model chart) for residuals.

Let $M_0$ be the process model estimated from a sample, and its parameters estimated from this sample denote by a vector $(a_{1,0},\ldots.a_{p_0,0})$. We assign to this estimated model a certain weight $w_0 \in [0,1]$. We also consider $k$ alternative models $M_j, j = 1,\ldots,k$, each described by a vector of parameters $(a^0_{1,j},\ldots,a^0_{p_j,j})$. In general, any model with known parameters can be used as an alternative one, but in this paper we restrict ourselves to autoregression models of maximum fourth order. Let $w'_1,\ldots,w'_k$ denote the weights assigned to models $M_1,\ldots,M_k$. Then, in the construction of the XWAM chart we assign to each alternative models a weight $w_j = (1-w_0)w'_j, j = 1,\ldots,k$.

When we model our process using $k+1$ models (estimated, and $k$ alternative) each process observation generates $k+1$ residuals. When all considered models belong to the class of autoregression processes $AR(p)$ residuals are calculated using the following formula

$$z_{i,j} = x_i - (a_{1,j}x_{i-1} + \ldots + a_{p_j,j}x_{i-p_j}), j = 0,\ldots,k; i = p_j + 1,\ldots. \quad (7)$$

Let $i_{min} = \max(p_0,\ldots,p_k)+1$. For the calculation of the parameters of the XWAM control chart we use $n - i_{min} + 1$ weighted residuals calculated from the formula

$$z_i^\star - \sum_{j=0}^{k} w_j z_{i,j}, i = i_{min},\ldots,n. \quad (8)$$

The central line of the chart is calculated as the mean of $z_i^\star$, and the control limits are equal to to the mean plus/minus three standard deviations of $z_i^\star$, respectively.

The weights $w'_1,\ldots,w'_k$ were calculated in [15] and [16] using a methodology known from data mining of time series. For each of the considered alternative models $M_j, j = 1,\ldots,k$ a long template time series was generated. Then, the observed sample was compared to each template series using the Dynamic Time Warping (DTW) method introduced by Berndt and Clifford [5]. The result of this comparison is expressed as a certain distance $dist_i, i = 1,\ldots,J$. Having evaluated distances between the sample and each template series generated from the respective alternative models $\{M_1,\ldots,M_J\}$, we select $k$ models characterized by the smallest distance. Then, the weights $\{w'_1,\ldots,w'_k\}$ are calculated from the formula

$$w'_i = \frac{dist_i}{\sum_{h=1}^{k} dist_h}. \quad (9)$$

It is worth noting that any method of comparison can be used for this purposes. For the description of other methods, we refer the reader to a recent survey and experimental comparison of representation methods and distance measures for time series data provided by Wang et al. [26].

The properties of the XWAM chart for residuals have been extensively investigated in [16]. In Table 5, taken from [16], we present the average ARL's for processes with independent observations ($\rho = 0$). We consider here different weights $w$ of the estimated model, and different shifts of the monitored process level expressed as multiples of the process standard deviation (real, not estimated). The simulations were performed for a very small sample size, $n = 20$.

Table 5 shows a very interesting feature of the XWAM chart for residuals. When we decrease the weight $w$ assigned to the estimated model the respective values of the average ARL's are increasing when shifts of the process mean are either not

**Table 5** Average ARL for different weights assigned to the estimated model and different shifts of the process level, $\rho = 0$, n=20

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 4.6 | 4.9 | 4.8 | 4.8 | 4.8 | 4.8 | 4.8 | 4.9 | 4.9 | 5.0 | 5.2 |
| -2 | 9.8 | 9.9 | 9.0 | 10.0 | 10.1 | 10.3 | 10.6 | 11.0 | 11.5 | 12.3 | 13.4 |
| -1 | 46.9 | 49.5 | 52.4 | 55.7 | 60.0 | 64.0 | 69.2 | 75.2 | 82.0 | 90.0 | 99.2 |
| 0 | 177.4 | 190.4 | 204.6 | 220.2 | 237.3 | 255.9 | 276.1 | 297.3 | 318.6 | 339.2 | 358.0 |
| 1 | 50.0 | 52.5 | 55.3 | 58.4 | 62.0 | 66.1 | 70.8 | 76.3 | 82.6 | 90.1 | 98.9 |
| 2 | 10.4 | 10.5 | 10.5 | 10.6 | 10.7 | 10.9 | 11.1 | 11.5 | 12.0 | 12.6 | 13.6 |
| 3 | 5.0 | 4.9 | 4.9 | 4.9 | 4.8 | 4.8 | 4.9 | 4.9 | 5.0 | 5.1 | 5.2 |

present (the in-control state) or are small (e.g., the shift of one standard deviation). For larger shifts these values for the XWAM chart are similar or may be even smaller (!) than in the case of the classical X chart for residuals ($w = 1$). It means that for the XWAM chart for residuals the rate of false alarms is smaller than in the case of the X chart for residuals, and similar or even smaller for large shifts of the process level. Thus, the proposed XWAM chart seems to be more effective than the Shewhart X chart for residuals.

Interesting behavior has been observed in [16] for autocorrelated data. For example, in Table 6, taken from [16], we show the average values of ARL when the process observations are positively, but not very strongly, correlated ($\rho = 0.5$). For non-shifted processes and processes with small shifts of the expected value the decreased weight of the estimated model results with larger values of the ARL0 (a positive effect) but also with larger values of the ARL for shifted processes (a negative effect). However, for larger shifts of the process level the situation becomes different; the value of ARL becomes smaller (a very positive effect) with decreased values of $w$. Therefore, one can think about an "optimal" value of $w$ for which we have a low rate of false alarms (large ARL0), and a high rate of alarms (low ARL) for significantly shifted process levels.

**Table 6** Average ARL for different weights assigned to the estimated model and different shifts of the process level, $\rho = 0.5$, n=20

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 16.7 | 16.3 | 15.7 | 15.0 | 14.2 | 13.5 | 12.8 | 12.2 | 11.8 | 11.4 | 11.2 |
| -2 | 43.0 | 47.0 | 45.8 | 46.4 | 46.6 | 46.4 | 45.9 | 45.1 | 44.1 | 43.1 | 42.2 |
| -1 | 107.2 | 117.2 | 126.4 | 135.0 | 142.7 | 149.2 | 154.3 | 158.0 | 160.3 | 161.4 | 161.5 |
| 0 | 165.8 | 185.0 | 204.2 | 222.7 | 239.7 | 254.5 | 267.1 | 275.0 | 284.8 | 289.5 | 291.4 |
| 1 | 105.2 | 115.3 | 124.6 | 132.7 | 139.5 | 144.6 | 148.1 | 150.2 | 151.3 | 151.5 | 151.0 |
| 2 | 42.0 | 44.0 | 45.3 | 46.0 | 46.3 | 46.1 | 45.7 | 44.9 | 43.9 | 42.6 | 41.4 |
| 3 | 16.2 | 15.9 | 15.4 | 14.7 | 14.0 | 13.3 | 12,5 | 11.9 | 11.4 | 11.1 | 10.8 |

From Tables 5 - 6, and others presented in [16], we can see that the XWAM chart has better discriminative power than the classical X chart for residuals. Taking into account that too large values of the ARL for shifts of small and medium sizes may be not acceptable, we can set parameter $w$ to such value that the average time to a false alarm is not smaller than a given value (e.g., equal to 250), or to find its "optimal" value, defined, e.g., as the value of $w$ for which we have the highest rate of alarms (i.e., the lowest value of the ARL) for large shifts of the process level. In the cases considered in [16], such an "optimal" value of $w$ seems to be close to 0.5.

The method for the construction of the XWAM chart was firstly proposed by Hryniewicz and Kaczmarek in [15] taking in mind the possibility to use several sets of real data series as templates. However, when we consider a large number of alternative models (for example, all stationary $AR(p), p = 1,..,4$ autoregression models) the proposed method for finding alternative models and their weights is time consuming. Therefore, there is a practical need to simplify it, and thus make it easier for implementation. In this paper we propose a modification of the XWAM chart, coined as the sXWAM (simplified XWAM). In this modification we use only one alternative model. To find this model we do not compare directly the observed sample and the stored template time series, but their summarizations provided in terms of the autocorrelation functions of the $p$th order. Let $r_1, r_2, \ldots, r_p$ be the consecutive $p$ values of the sample autocorrelation function calculated using (3). Similarly, let $r_{1,i}, r_(2,i), \ldots, r_{p,i}, i = 1, \ldots, J$ be the consecutive $p$ values of the autocorrelation function of the alternative model. For given parameters of the alternative autoregression process $a_{1,i}, \ldots, a_{p,i}, i = 1, \ldots, J$ the values of $r_{1,i}, r_(2,i), \ldots, r_{p,i}, i = 1, \ldots, J$ can be found by solving the Yule - Walker equations (4). In general, the consecutive values of $r_p$ can be computed using the following recursion equation

$$r_p = a_1 * r_{p-1} + a_2 * r_{p-2} + \cdots + a_p \tag{10}$$

In this paper we consider only processes of the maximum fourth order. In such a case, by doing some simple but tedious algebra, we can obtain the following explicit formulae for the first three autoregression coefficients:

$$r_1 = A_1, \tag{11}$$

$$r_2 = a_1 A_1 + a_2, \tag{12}$$

$$r_3 = \frac{a_1 B_1 + a_3 + (a_2 + a_4)(A_1 + A_2 B_1)}{1 - a_1 B_2 - (a_2 + a_4)(A_2 B_2 + A_3)}, \tag{13}$$

where

$$A_1 = \frac{a_1}{1 - a_2},$$

$$A_2 = \frac{a_3}{1 - a_2},$$

$$A_3 = \frac{a_4}{1 - a_2},$$

$$B_1 = \frac{A_1(a_1 + a_3) + a_2}{1 - (a_1 + a_3)A_2 - a_4},$$

$$B_2 = \frac{A_3(a_1 + a_3)}{1 - (a_1 + a_3)A_2 - a_4}.$$

Hence, the consecutive values of $r_4, r_5, \ldots$ can be directly computed from (10).

As the measure of distance between the estimated autocorrelations $r_1, r_2, \ldots, r_p$ and the correlations calculated for the $i$th alternative model $r_{1,i}, r_{2,i}, \ldots, r_{p,i}, i = 1, \ldots, J$ we use a simple sum of absolute differences (called the Manhattan distance in the community of data mining)

$$dist_{i,MH} = \sum_{k=1}^{p} |r_k - r_{k,i}|, i = 1, \ldots, J. \quad (14)$$

The autoregression model with the lowest value of $dist_{i,MH}$ is chosen as the alternative model with the weight equal to $1 - w$.

The design of the sXWAM chart for residuals is thus much simpler. The values of the autoregression functions for different alternative models can be computed in advance, and stored in an external file. This file can be red by a computer program, and used for choosing the model that fits to the observed sample (and its estimated autoregression function). The properties of the sXWAM chart have been investigated in extensive computer simulations. The results of these investigations are presented in the next Section.

# 4 Properties of the sXWAM chart - numerical experiments

The properties of the XWAM chart, extensively investigated in [15] and [16] and recalled in Section 3, show that statistical properties of this new control chart are undoubtedly better than the properties of the classical X control chart for residuals. The price for this improvement is a significant complication of the design algorithm. The sXWAM chart, introduced in this paper, does not require such computation efforts, and from a practical point of view is simpler for implementation. However, this simplification may result in worsening of its statistical properties. In this Section we investigate this problem using Monte Carlo simulations. All results presented in this Section have been obtained from simulations of 10000 control charts for residuals. Each simulated control chart was designed using a simulated sample of either $n = 20$ or $n = 100$ observations. Samples were generated using a chosen autoregression process with innovations having the standard normal distribution. In the design process for calculation of residuals we used the estimated (from the sample) autoregression model. Then, for each chart we simulated 5000 runs, each curtailed after 1000 observations. In this experiment many different characteristics were evaluated. However, in this paper we restrict ourselves to average (averaged over the set of all simulated charts) values of the ARL (Average Run Length).

The most important characteristic of a control chart is its ARL for a process being in an in-control state, usually denoted by ARL0. For the XWAM chart this value, as it is demonstrated in Section 3, is growing with increased values of the weight assigned to alternative models. For the sXWAM chart, as it has been shown in Tables 7 - 8, this looks somewhat different.

**Table 7** Average in-control ARL, ARL0, for different weights assigned to the estimated model, n=20

| Model/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| AR(-0.9) | 258.3 | 270.2 | 277.6 | 280.8 | 280.6 | 278.0 | 274.0 | 269.5 | 264.8 | 260.4 | 256.4 |
| AR(-0.5) | 201.0 | 202.5 | 203.9 | 205.3 | 206.5 | 207.7 | 208.7 | 209.7 | 210.4 | 211.1 | 211.7 |
| AR(0) | 169.5 | 169.8 | 170.0 | 170.1 | 170.0 | 169.8 | 169.6 | 169.2 | 168.8 | 168.4 | 167.9 |
| AR(0.5) | 175.8 | 176.5 | 177.1 | 177.7 | 178.2 | 178.6 | 178.9 | 179.2 | 179.4 | 179.5 | 179.5 |
| AR(0.9) | 146.5 | 149.3 | 151.0 | 151.8 | 151.8 | 151.4 | 150.9 | 150.2 | 149.6 | 149.0 | 148.4 |
| AR(0.7,-0.9) | 205.6 | 207.2 | 205.7 | 201.6 | 195.6 | 188.7 | 186.7 | 174.1 | 167.1 | 160.5 | 154.2 |
| AR(0.7,-0.9,0.1) | 196.3 | 200.3 | 202.8 | 203.9 | 203.4 | 201.6 | 198.8 | 195.2 | 191.0 | 186.4 | 181.6 |
| AR(0.7,-0.9,0.1,-0.2) | 209.5 | 214.2 | 217.4 | 219.0 | 219.0 | 217.5 | 214.8 | 211.1 | 206.7 | 201.9 | 196.8 |

**Table 8** Average in-control ARL, ARL0, for different weights assigned to the estimated model, n=100

| Model/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| AR(-0.9) | 323.6 | 341.5 | 351.0 | 352.7 | 349.0 | 342.3 | 335.2 | 329.0 | 324.3 | 321.2 | 319.4 |
| AR(-0.5) | 310.1 | 311.0 | 311.7 | 312.2 | 312.5 | 312.7 | 312.6 | 312.4 | 312.0 | 311.4 | 310.7 |
| AR(0) | 288.1 | 288.5 | 288.7 | 288.6 | 288.2 | 287.7 | 286.9 | 285.2 | 284.6 | 283.2 | 281.5 |
| AR(0.5) | 307.3 | 308.0 | 308.7 | 309.1 | 309.4 | 309.5 | 309.5 | 309.3 | 309.0 | 308.5 | 307.9 |
| AR(0.9) | 302.6 | 314.2 | 320.3 | 322.0 | 320.6 | 317.7 | 314.8 | 312.6 | 311.2 | 310.8 | 311.1 |
| AR(0.7,-0.9) | 302.4 | 304.6 | 297.3 | 282.5 | 263.2 | 242.3 | 221,9 | 203.2 | 186.5 | 171.8 | 158.8 |
| AR(0.7,-0.9,0.1) | 296.6 | 302.9 | 305.6 | 304.8 | 300.9 | 294.1 | 285.2 | 274.8 | 263.5 | 251.7 | 239.8 |
| AR(0.7,-0.9,0.1,-0.2) | 292.3 | 299.3 | 303.1 | 303.7 | 301.2 | 295.9 | 288.4 | 279.3 | 269.0 | 258.2 | 247.2 |

First of all, we can notice that the growth of the ARL0 value is not present in the case of independent observations (i.e., the AR(0) process), and not significant in the case of processes with weak or moderate autocorrelations (e.g., AR(-0.5) or AR(0.5) processes). In the case of strongly autocorrelated processes, and processes of complicated autocorrelation structure, the dependence of ARL0 upon the value of the weight $w$ assigned to the alternative model is more visible. This property indicates that for significantly correlated process observations in the case of the sXWAM chart we can expect lower false alarm rates than in the case of usual X charts for residuals. However, for processes without strong correlations the profits from the usage of the sXWAM chart may not compensate the costs related to its complicated design. Analyzing the data presented in Tables 7 - 8 we can also see somewhat unexpected finding. The most advantageous effect of the usage of the XWAM chart, i.e., the decrease of the false alarm rate (i.e., the increase of the ARL0 values), in the case of the sXWAM chart seems to be more significant for larger sample sizes.

When the knowledge about a monitored process is limited or when a practitioner is advised to use a certain (e.g., the sXWAM) chart it is interesting to know if the practically unnecessary choice of the sXWAM chart may decrease abilities of the chart to detect unwanted shifts of the process level. In Tables 9 - 10 we present the values of the ARL for differently shifted process levels when consecutive observations are independent. The shifts are given in units of the standard deviation in the distribution of innovations.

**Table 9** Average ARL for different weights assigned to the estimated model and different shifts of the process level, $AR(0)$, n=20

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 4.92 | 4.92 | 4.91 | 4.91 | 4.91 | 4.91 | 4.92 | 4.93 | 4.95 | 4.96 | 4.98 |
| -2 | 9.8 | 9.8 | 9.7 | 9.7 | 9.7 | 9.7 | 9.7 | 9.8 | 9.8 | 9.8 | 9.8 |
| -1 | 46.2 | 46.1 | 45.9 | 45.8 | 45.7 | 45.6 | 45.5 | 45.4 | 45.4 | 45.4 | 45.3 |
| 0 | 169.5 | 169.8 | 170.0 | 170.1 | 170.0 | 169.8 | 169.6 | 169.2 | 168.8 | 168.4 | 167.9 |
| 1 | 45.0 | 44.8 | 44.6 | 44.5 | 44.4 | 44.3 | 44.2 | 44.2 | 44.1 | 44.1 | 44.1 |
| 2 | 9.6 | 9.5 | 9.5 | 9.5 | 9.5 | 9.5 | 9.5 | 9.5 | 9.6 | 9.6 | 9.6 |
| 3 | 4.92 | 4.91 | 4.91 | 4.91 | 4.91 | 4.91 | 4.92 | 4.93 | 4.95 | 4.96 | 4.99 |

**Table 10** Average ARL for different weights assigned to the estimated model and different shifts of the process level, $AR(0)$, n=100

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 4.81 | 4.80 | 4.79 | 4.79 | 4.79 | 4.79 | 4.80 | 4.81 | 4.82 | 4.83 | 4.85 |
| -2 | 9.2 | 9.2 | 9.1 | 9.1 | 9.1 | 9.1 | 9.1 | 9.2 | 9.2 | 9.2 | 9.3 |
| -1 | 46.1 | 46.0 | 45.9 | 45.8 | 45.7 | 45.6 | 45.5 | 45.4 | 45.4 | 45.3 | 45.2 |
| 0 | 288.1 | 288.5 | 288.7 | 286.6 | 288.2 | 287.7 | 286.9 | 285.2 | 284.6 | 283.2 | 281.5 |
| 1 | 46.4 | 46.3 | 46.2 | 46.1 | 46.0 | 45.9 | 45.8 | 45.7 | 45.7 | 45.6 | 45.5 |
| 2 | 9.2 | 9.2 | 9.2 | 9.1 | 9.1 | 9.1 | 9.2 | 9.2 | 9.2 | 9.2 | 9.3 |
| 3 | 4.81 | 4.80 | 4.79 | 4.79 | 4.79 | 4.80 | 4.80 | 4.81 | 4.82 | 4.84 | 4.85 |

The results presented in Tables 9 - 10 show that when we use the sXWAM chart for uncorrelated processes the detection abilities have not been deteriorated, and in some cases are even very slightly improved. Similar results have been obtained for weakly and moderately autocorrelated processes (not presented in this paper). Therefore, we can say that even in cases when the sXWAM chart is not advisable for process monitoring purposes, its usage does not lead to worse detection abilities than in the case of the X chart for residuals.

Now, let us consider cases when the usage of the sXWAM chart seems to be profitable, as by increasing the value of the weight assigned to the alternative model, i.e, by decreasing the value of $w$ (but not too much), we can decrease the false alarm rate. In practice we can profit from this property only in cases when such increasing does not lead to significant worsening of the detection abilities. Let us start from the case of strongly negative autocorrelation, i.e., the $AR(-0.9)$ process. The respective values of the ARL are given in Tables 11 - 12.

In the considered case of the $AR(-0.9)$ process by decreasing the value of $w$ (i.e., by increasing the weight of the alternative model) we also increase the value of ARL for shifted processes. However, when the value of $w$ is in the range of $(0.5-0.8)$ we

**Table 11** Average ARL for different weights assigned to the estimated model and different shifts of the process level, $AR(-0.9)$, n=20

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 2.21 | 2.21 | 2.21 | 2.22 | 2.22 | 2.23 | 2.24 | 2.26 | 2.28 | 2.31 | 2.36 |
| -2 | 2.7 | 2.7 | 2.7 | 2.8 | 2.8 | 2.9 | 3.0 | 3.1 | 3.2 | 3.4 | 3.6 |
| -1 | 13.8 | 14.8 | 15.9 | 17.1 | 18.3 | 19.5 | 20.6 | 21.6 | 22.6 | 23.7 | 24.7 |
| 0 | 258.3 | 270.2 | 277.6 | 280.8 | 280.6 | 278.0 | 274.0 | 269.5 | 264.8 | 260.4 | 256.4 |
| 1 | 13.6 | 14.6 | 15.7 | 16.9 | 18.0 | 19.2 | 20.2 | 21.3 | 22.3 | 23.4 | 24.4 |
| 2 | 2.7 | 2,7 | 2,7 | 2.8 | 2.8 | 2.9 | 3.0 | 3.1 | 3.2 | 3.4 | 3.6 |
| 3 | 2.21 | 2.21 | 2.21 | 2.22 | 2.22 | 2.23 | 2.24 | 2.26 | 2.28 | 2.31 | 2.36 |

**Table 12** Average ARL for different weights assigned to the estimated model and different shifts of the process level, $AR(-0.9)$, n=100

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 2.05 | 2.05 | 2.05 | 2.05 | 2.06 | 2.07 | 2.08 | 2.10 | 2.11 | 2.14 | 2.17 |
| -2 | 2.3 | 2.4 | 2.4 | 2.5 | 2.5 | 2.6 | 2.6 | 2.7 | 2.8 | 2.9 | 3.0 |
| -1 | 9.1 | 9.7 | 10.3 | 10.9 | 11.5 | 12.1 | 12.7 | 13.3 | 13.8 | 14.4 | 14.9 |
| 0 | 323.6 | 341.5 | 351.0 | 352.7 | 349.0 | 342.3 | 335.2 | 329.0 | 324.3 | 321.2 | 319.4 |
| 1 | 9.1 | 9.7 | 10.3 | 10.9 | 11.5 | 12.2 | 12.7 | 13.3 | 13.9 | 14.4 | 14.9 |
| 2 | 2.3 | 2.4 | 2.4 | 2.5 | 2.5 | 2.6 | 2.6 | 2.7 | 2.8 | 2.8 | 3.0 |
| 3 | 2.05 | 2.05 | 2.05 | 2.05 | 2.06 | 2.07 | 2.08 | 2.10 | 2.11 | 2.14 | 2.15 |

observe the increased values of the ARL0. Thus the positive effect of the decreased false alarm rate may overweight the negative effect of the increased alarm rate for shifted processes. The analysis of the discrimination rate, defined as the quotient of the ARL value for a shifted process and the ARL0, shows that for small sample sizes (e.g., when $n = 20$) the 'optimal' value of $w$ is close to 0.5. On the other hand, for larger sample sizes (e.g., when $n = 100$) this value may be even smaller. However, despite higher discrimination rate for low values of $w$ the values of the ARL for shifted processes may be regarded as too high. Thus, the value of $w$ equal to 0.5 may be recommended also in this case.

An interesting case of strongly positively autocorrelated process, $AR(0.9)$, is presented in Tables 13 - 14. It is quite clear that for considered sample sizes the detection abilities of the chart are very poor. The false alarm rates for a small sample size ($n = 20$) are high, and alarm rates for shifted processes are prohibitively low (large values of ARL's). Other experiments show that this bad property is related to the autocorrelation of sample residuals, and vanishes only in the case of very large sample sizes. However, when we have only small sample sizes by the usage of the sXWAM chart we can decrease the false alarm rate and increase the discrimination rate, thus making the monitoring process more effective. However, this positive effect is not significant, so the usage of the sXWAM chart in this case may be questionable.

Finally, let us consider the case of a process with a complicated autocorrelation structure, namely the $AR(0.7,-0.9,0.1,-0.2)$ process. In the case of this process the autocorrelation function has a shape of a slowly damped sinusoid, and thus the effects of the autocorrelations are not easily evaluated using small samples. The results of the respective simulation experiment are presented in Tables 15 - 16.

**Table 13** Average ARL for different weights assigned to the estimated model and different shifts of the process level, $AR(0.9)$, n=20

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 100.5 | 100.4 | 99.0 | 98.6 | 98.1 | 98.2 | 98.9 | 100.1 | 101.7 | 103.4 | 105.1 |
| -2 | 122.9 | 124.2 | 124.5 | 124.3 | 123.8 | 123.5 | 123.4 | 123.5 | 123.9 | 124.4 | 124.9 |
| -1 | 140.0 | 142.9 | 143.8 | 144.3 | 144.2 | 143.9 | 143.4 | 143.0 | 142.6 | 142.2 | 141.9 |
| 0 | 146.5 | 149.3 | 151.0 | 151.8 | 151.8 | 151.4 | 150.9 | 150.2 | 149.6 | 149.0 | 148.4 |
| 1 | 140.0 | 142.3 | 143.6 | 144.0 | 143.7 | 143.2 | 142.6 | 142.1 | 141.7 | 141.4 | 141.1 |
| 2 | 122.7 | 123.9 | 124.1 | 123.7 | 123.1 | 122.6 | 122.5 | 122.6 | 123.1 | 123.6 | 124.2 |
| 3 | 100.1 | 100.0 | 99.0 | 98.0 | 97.4 | 97.6 | 98.4 | 99.6 | 101.2 | 102.9 | 104.6 |

**Table 14** Average ARL for different weights assigned to the estimated model and different shifts of the process level, $AR(0.9)$, n=100

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 214.0 | 196.8 | 199.9 | 188.6 | 188.2 | 190.5 | 194.8 | 200.5 | 207.0 | 214.0 | 221.1 |
| -2 | 247.6 | 250.1 | 248.5 | 245.3 | 242.8 | 242.1 | 243.3 | 246.0 | 249.7 | 254.0 | 258.6 |
| -1 | 286.7 | 295.5 | 299.0 | 298.5 | 296.1 | 293.4 | 291.5 | 290.8 | 291.1 | 292.3 | 294.1 |
| 0 | 302.6 | 314.2 | 320.3 | 322.0 | 320.6 | 317.7 | 314.8 | 312.6 | 311.2 | 310.8 | 311.1 |
| 1 | 287.5 | 296.3 | 299.8 | 299.3 | 296.9 | 294.1 | 292.2 | 291.3 | 291.1 | 292.7 | 294.5 |
| 2 | 248.8 | 251.4 | 249.8 | 246.5 | 243.9 | 243.1 | 244.2 | 246.8 | 250.4 | 254.6 | 259.1 |
| 3 | 214.6 | 198.0 | 193.0 | 189.7 | 189.2 | 191.4 | 195.6 | 201.3 | 207.7 | 214.6 | 221.6 |

**Table 15** Average ARL for different weights assigned to the estimated model and different shifts of the process level, process $AR(0.7, -0.9, 0.1, -0.2$, n=20

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 4.31 | 4.33 | 4.36 | 4.39 | 4.43 | 4.47 | 4.52 | 4.58 | 4.65 | 4.74 | 4.84 |
| -2 | 9.2 | 9.3 | 9.5 | 9.7 | 9.9 | 10.1 | 10.3 | 10.5 | 10.7 | 10.9 | 11.1 |
| -1 | 54.2 | 55.8 | 57.2 | 58.3 | 58.9 | 59.2 | 59.2 | 58.9 | 58.4 | 57.8 | 57.0 |
| 0 | 209.5 | 214.2 | 217.4 | 219.0 | 219.0 | 217.5 | 214.8 | 211.1 | 206.7 | 201.9 | 196.8 |
| 1 | 53.4 | 55.1 | 56.5 | 57.6 | 58.3 | 58.7 | 58.7 | 58.5 | 58.0 | 57.4 | 56.7 |
| 2 | 9.1 | 9.3 | 9.4 | 9.6 | 9.8 | 10.0 | 10.2 | 10.4 | 10.6 | 10.8 | 11.0 |
| 3 | 4.34 | 4.36 | 4.39 | 4.42 | 4.46 | 4.51 | 4.56 | 4.62 | 4.69 | 4.77 | 4.86 |

**Table 16** Average ARL for different weights assigned to the estimated model and different shifts of the process level, process $AR(0.7, -0.9, 0.1, -0.2$, n=100

| Shift/w: | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -3 | 4.16 | 4.17 | 4.19 | 4.21 | 4.23 | 4.25 | 4.28 | 4.31 | 4.35 | 4.39 | 4.44 |
| -2 | 7.3 | 7.4 | 7.4 | 7.5 | 7.6 | 7.7 | 7.8 | 7.9 | 8.0 | 8.1 | 8.2 |
| -1 | 38.8 | 39.7 | 40.4 | 40.8 | 41.0 | 41.0 | 40.8 | 40.5 | 40.0 | 39.5 | 38.8 |
| 0 | 292.3 | 299.3 | 303.1 | 303.7 | 301.2 | 295.9 | 288.4 | 279.3 | 269.0 | 258.2 | 247.2 |
| 1 | 39.1 | 39.9 | 40.6 | 41.1 | 41.3 | 41.3 | 41.1 | 40.7 | 40.2 | 39.7 | 39.0 |
| 2 | 7.3 | 7.4 | 7.5 | 7.5 | 7.6 | 7.7 | 7.8 | 7.9 | 8.0 | 8.1 | 8.2 |
| 3 | 4.16 | 4.18 | 4.19 | 4.21 | 4.23 | 4.26 | 4.28 | 4.31 | 4.35 | 4.39 | 4.44 |

Similarly to the case of the $AR(0.9)$ process, the best discrimination rate can be obtained for the values of $w$ close to 0.7. However, the final choice of the value of $w$ has to be made taking into account a balance between a required high value of the ARL0, and possibly low values of ARL's for shifted processes. Thus, the value of $w$ in the range $(0.7-0.9)$ may by preferable.

## 5 Conclusions

In the paper we have proposed a simplified version of the XWAM control chart for residuals. In contrast to the original XWAM control chart, described in the papers by Hryniewicz and Kaczmarek, [15] and [16], there is only one alternative model that is taken into account for model averaging purposes. Moreover, the choice of the alternative model is not based on the comparison of the observed sample time series and the template time series of greater length, but on the comparison of the sample autocorrelation function with autocorrelation functions derived theoretically for all stationary autoregression processes of the maximal order of four.

In order to evaluate the proposed methodology we have performed many simulation experiments. From the results of these experiments we can see that the proposed sXWAM control chart has good properties only for relatively strongly negatively autocorrelated processes. For positively correlated processes and for processes with weak or moderate autocorrelations of both types the original XWAM chart performs much better, and the performance of our new simpler chart is only slightly better than the performance of the classical X chart for residuals. A probable explanation of this effect is the following. By choosing only one alternative model that is well fitted to the observed data we, in some sense, replicate the estimated model. Therefore, we do averaging using two similar models. In contrast, in the case of the original XWAM chart we use several alternative models, thus making the final model more flexible, and possibly closer to the original one. It is interesting, however, how the performance of the sXWAM chart will change if we take into consideration more alternative models chosen according to their proximity to the estimated one in terms of the autocorrelation function. Moreover, in future investigations the methodology used for the construction of the sXWAM chart may be used for the construction of other control charts for residuals, such as EWMA or CUSUM. These have been proved to perform better than the classical Shewhart X chart for individual observations, so it is interesting if this superiority will be preserved for similar control charts based on the model averaging principles.

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