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1. Introduction.

Reliability indices of complex systems can be estimated either by using the results of specially designed lifetime tests or from reliability field data. There exist two general types of reliability data. The data of a first type consist of observed times to failures, either to a first failure or between consecutive failures. For the data of this type, when a system is treated as a one entity we can distinguish two different types of reliability tests. In the first one, we observe consecutive failures of a system, and after each of them a failed system is completely renewed. Therefore, each observed time to failure can be regarded as the time to a first failure, and observed random times between consecutive failures may be described by random variables having independent and identical probability distributions. If this assumption is true, we can estimate a required reliability characteristic using a sample of observed lifetimes. In the second case, we have to observe several identical systems working in the same conditions. Times to first failures of these systems constitute a sample which may be used for the estimation of the considered reliability characteristic. One can think, of course, about the combination of these two types of tests when we have the lifetime data coming from the test of several renewable systems. In all these cases, however, we need to have either sufficiently long time of test or sufficiently large number of observed systems. Both these requirements are seldom met in practice. Thus, such methods of the reliability estimation are rarely used in practice despite the fact that from a statistical point of view the required estimators are obtained in the simplest possible way. Moreover, we cannot profit from the information about the structure of the considered system, and from the knowledge of times to failure of its particular elements.

Another type of reliability data is mainly typical for field data, especially for data coming from the analysis of warranty claims. For this type of data we have information about the number of observed failures of a system during a certain time interval. A special case of this type of data is that coming from mission tests where the only available information regards the fact whether the system failed or not failed during a pre-specified time of the mission. For example, we may consider a warranty time as the "mission time", and record only the number of systems which have failed during that time. In all such cases our reliability data has discrete form, as we record discrete numbers of failures.

In practice of reliability we are frequently faced with a different problem: how to evaluate reliability characteristics of a system on its design stage. There exist many methods for the prediction of reliability using available statistical data. In this paper we consider the simplest one, when we can utilize the results of reliability tests of system's elements performed in presumably the same conditions as the conditions of work of the designed complex system. Thus, we call this type of data discrete.

Research studies on statistical methods aimed at the estimation of system's reliability using the results of reliability tests of its elements were initiated independently in the 1950s in the United States and the Soviet Union, where they were performed by prominent mathematicians and statisticians. Some strong mathematical results were obtained, and these results can be used for both point and interval estimation of system's reliability using the data obtained for its elements or subsystems. In this paper we will focus our attention on the interval estimation. The reason for the importance of the results of this type stems from practice. Usually we can use scarce reliability data, and thus the obtained point estimators are not very precise. Therefore, we need to know some lower bounds for the predicted reliability characteristics.

Preliminary analysis of the theoretical results shows undoubtedly that even in the cases of simple systems analytical methods for the calculation of exact confidence intervals require utilization of complex mathematical tools such as nonlinear mathematical programming. On the other hand, interesting approximate results, obtained mainly by American researchers, can be used in practice only when a sufficiently large number of failures have been observed. For this reasons already in the 1980s the reliability theoreticians lost their interest in further research on those problems. However, the problem is still interesting for practitioners who need approximate, or even heuristic, methods which may be used for the prediction of reliability using existing statistical data.

In this paper we consider the case of confidence intervals when reliability data come from independent tests of subsystems and is available in the discrete form. In the following five sections we give an overview of different methods for the construction of confidence intervals for the reliability of systems using discrete data. In all these methods it has been assumed that the elements of a system are independent. However, this assumption is frequently either not valid in practice or, to be more precise, not verified. The problem arises then, how possible dependencies between lifetimes of subsystems influence the accuracy of interval estimates. In this paper we consider this problem only for the case of simple heuristic interval estimates introduced in Hryniewicz (2009). Using Monte Carlo simulation experiments we have investigated the robustness of these simple interval estimates to the departure from the assumption of independence. In our experiments we still assume that the subsystem reliability data come from independent tests, but the lifetimes of these subsystems, when observed in whole systems, are dependent. To model such dependencies we use the concept of copulas which seems to be very convenient for the reliability data of the considered type. The results of our experiments, described in the second part of this paper, show that in certain cases, but not in every case, the newly proposed heuristic interval estimators are more robust to departures from the assumption of independence.

2. General methodology for the evaluation of system's reliability

Evaluation of reliability of complex systems became the subject of intensive theoretical investigation in the beginning of 1960s. Fundamental results were summarized in the famous book by Barlow and Proschan (1965). In the developed mathematical models we assume that both the system as a whole, and system's elements at any time instant $t>0$ are either in the state of functioning (or failure-free state), when the random variable $X(t)$ describing the reliability state adopts the value 1, or in the state of failure, when this random variable adopts the value 0. When the considered system consists of m elements, then its reliability state is described by the random vector $\mathbf{X} = (X_1, X_2, \ldots X_m)$, and the probability of the observation of any reliability state is given by

$$P(\mathbf{X}) = \prod_{i=1}^{m} p_i^{X_i} (1 - p_i)^{1-X_i}, \tag{1}$$

where

$$p_i = P(X_i = 1) = E(X_i), i = 1, \ldots, n. \tag{2}$$

In the above formulae we have omitted time t assuming that in case of specific calculations it adopts the same value for all components of the random vector.

Reliability state of the whole system depends on the states of all individual system's elements. Denote by Ω the set of all 2^m possible states of system's elements. We can divide this set into two exclusive subsets: the subset of all functioning states of the system G, and the system of all failure states of this system $\overline{G}$ ($G \cup \overline{G} = \Omega$). The function

$$\Psi(\mathbf{X}) = \begin{cases} 1 & \mathbf{X} \in G \\ 0 & \mathbf{X} \in \overline{G} \end{cases} \tag{3}$$

is called the *structure function*, and it describes the relation between reliability state of the whole system and reliability states of its elements. The effective construction of this function is the subject of numerous research works. Particular results may be found in all classical textbooks on reliability, e.g. Barlow & Proschan (1965, 1975).

Probability that the considered system is in the failure-free state depends on the vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ that describes the probabilities of failure-free functioning of system's elements, and system's reliability structure function. It is given by the function called the reliability function which is given by the following formula

$$R(\mathbf{p}) = P(\mathbf{X} \in G) = E[\Psi(\mathbf{X})] = \sum_{\mathbf{X} \in G} \Psi(\mathbf{X}) \prod_{i=1}^{m} p_i^{X_i} (1 - p_i)^{1 - X_i} \ . \tag{4}$$

Below, we present the respective formulae for the reliability structures which are most frequently met in practice.

a) In case of a system with series reliability structures which consists of m groups of identical $n_i, i = 1, \ldots, m$ elements we have:

$$R(\mathbf{p}) = \prod_{i=1}^{m} p_i^{n_i} \ . \tag{5}$$

b) For the system with a parallel reliability structure which consists of m elements the respective formula is given by

$$R(\mathbf{p}) = 1 - \prod_{i=1}^{m} (1 - p_i) = 1 - \prod_{i=1}^{m} q_i \ . \tag{6}$$

c) In case of a series-parallel reliability system which consists of m connected in series groups, where each of these groups consists of n_i connected in parallel identical elements, the reliability function is given by the formula:

$$R(\mathbf{p}) = \prod_{i=1}^{m} \left[1 - (1 - p_i)^{n_i}\right]. \tag{7}$$

d) For a parallel-series system consisting of m connected in parallel groups, where each of these groups consists of n_i identical elements connected in series, the reliability function is given by the formula:

$$R(\mathbf{p}) = 1 - \prod_{i=1}^{m} \left[1 - \prod_{j=1}^{n_i} p_{ij}\right]. \tag{8}$$

In formulae (5) – (8) p_{ij} denotes the probability that the j-th element in the i-th subsystem is in a failure-free state.

The systems with structures described above belong to a more general class of systems called coherent systems, or systems with monotonic structure. The system has monotonic structure if

$$\Psi(\mathbf{X}) \geq \Psi(\mathbf{Y}) \tag{9}$$

holds when $X_i \geq Y_i, i = 1,\ldots,m$, and when

$$\Psi(\mathbf{0}) = 0, \ldots \Psi(\mathbf{1}) = 1, \tag{10}$$

with $\mathbf{0}$=(0,...,0) and $\mathbf{1}$=(1,...,1). For systems with a monotonic structure the reliability function can be always computed. However, for large and complex systems this can be a hard computational task.

In order to compute the probability that the system is in the failure-free state we need to know the estimates of the elements of the vector $\mathbf{p}$. These estimates can be obtained from the results of reliability tests. We assume that for each of system's elements we have the results of *independent* reliability tests. From these tests we obtain the vector of estimates

$\mathbf{p}^* = (p_1^*, p_2^*, \ldots, p_m^*)$. The estimators p_i^* are unbiased estimators of unknown probabilities p_i only in certain particular cases. However, in the majority of practical cases, when we apply the maximum likelihood method of estimation, these estimators are asymptotically unbiased, but in practice the conditions of required for asymptotic distributions usually do not hold due to the limited number of the pertinent statistical data. The knowledge of estimates $\mathbf{p}^* = (p_1^*, p_2^*, \ldots, p_m^*)$ allows for simple estimation of the reliability $R(\mathbf{p})$. In such a case we apply the method of substitution. We substitute in (4) unknown probabilities $\mathbf{p}$ with their estimates $\mathbf{p}^*$. The estimator of the reliability of the whole system $R(\mathbf{p}^*)$ is unbiased only in a particular case of systems with a series reliability structure and unbiased estimators of p_i. In all other cases $R(\mathbf{p}^*)$ is biased or at best asymptotically unbiased. Therefore, in practical situations the estimates of the system's reliability are very uncertain, and we need to have methods for the computation of lower bounds for its possible value. Such bounds may be obtained by the calculation of confidence intervals for $\boldsymbol{R(p)}$.

Let us now consider a system consisting of element s of m different types. Suppose that the reliability of the element of the i-th type, $i=1,\ldots,m$, is a certain function of a parameter θ_i whose value is unknown. Thus, we may assume that the reliability of the whole system is described by a function $R(\boldsymbol{\theta})$ which depends on the vector $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ of parameters describing the reliability of system's elements. Moreover, we assume that the information from reliability tests of system's elements is denoted by x_i, $i=1,\ldots,m$. Thus, the results of the tests are described by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. We have to note that the values of θ_i and x_i only in special cases are represented by single numbers. In a general case they are represented by vectors of numbers. The interval $(\underline{R}, \overline{R})$, where $\underline{R} = \underline{R}(\mathbf{x})$ and $\overline{R} = \overline{R}(\mathbf{x})$ is the two-sided confidence interval, calculated on the confidence level γ, for the unknown value of $R(\boldsymbol{\theta})$ if the following condition is fulfilled

$$P_\theta\left(\underline{R} \le R(\theta) \le \overline{R}\right) \ge \gamma . \quad (11)$$

In an analogical way we can define one-sided lower and upper confidence intervals for the reliability function $R(\theta)$. In the sections which follow we present methods for the calculation of such confidence intervals. In this presentation we use notation given in the book by Gnedenko *et al.* (1999).

3. Confidence intervals for system's reliability in the case of discrete reliability data

Let us consider the problem of reliability estimation when the results of reliability tests of system's elements are available in a discrete form. Let us assume that the elements of all types are independently tested in exactly the same conditions as the work conditions of the considered system. In the simplest case we test samples of size N_i, $i=1,\ldots,m$, for all m types of elements, and the duration of all tests is the same, and is equal to t. In this simplest case we assume that we know the reliability state of each tested element at the end of the test. Thus, we assume that we know the numbers of elements d_i, $i=1,\ldots,m$, which failed during the test. The test result is described, therefore, by pairs of integer numbers (d_i,N_i), $i=1,\ldots,M$. In such a case we say that the reliability tests, also known as pass-fail tests, are performed according to a *binomial scheme*. In this simple case there exists an unbiased estimator of the reliability of a tested element given by a simple formula

$$\hat{p}_i = 1 - \frac{d_i}{N_i}, i = 1,\ldots,m \quad (12)$$

The random number of the observed failures is thus described by the binomial distribution

$$P\left(d_i = d_i^*\right) = \binom{N_i}{d_i^*}(1 - p_i)^{d_i^*} p_i^{N_i - d_i^*}, i = 1,\ldots,m \quad (13)$$

Calculation of the confidence interval for the reliability p_i is not simple. For a given confidence level γ one can calculate the confidence interval using a so called fiducial approach. The respective formulae are known as the Clopper-Pearson formulae, and in the considered case of reliability estimation they have the form given in Gnedenko *et al.* (1999). The lower bound $\underline{p}$ of the one-sided confidence interval for the reliability p is given as the solution of the following equation

$$\sum_{k=0}^{d} \binom{N}{k} (1-\underline{p})^k (\underline{p})^{N-k} = 1-\gamma, \tag{14}$$

and the upper bound $\overline{p}$ of the one-sided confidence interval for the reliability p is given as the solution of the equation

$$\sum_{k=0}^{N-d} \binom{N}{k} \overline{p}^k (1-\overline{p})^{N-k} = 1-\gamma. \tag{15}$$

In case of $d=N$ we have $\overline{p}=1$, and when $d=0$ we have $\underline{p}=0$. It is worth noticing that if we replace $1-\gamma$ in (14)-(15) with $0,5<\alpha<1$ and $0,5<\beta<1$, respectively, we can use these formulae for the calculation of a two-sided confidence interval for the reliability p on the confidence level equal to $1-\alpha-\beta$.

When the probability of a failure is low, or when reliability is high, i.e. when the strong inequality $q_i = 1-p_i << 1, i=1,\ldots,m$ holds, and when the number of tested elements N_i, $i=1,\ldots,m$ is large, the probability distribution of the number of failed elements d_i, $i=1,\ldots,m$ can be approximated by the Poisson distribution with the parameter $\Lambda_i = q_i N_i$, and the probability mass function given by the formula

$$P(d_i = d_i^*) = \frac{\Lambda_i^{d_i^*}}{d_i^*!} e^{-\Lambda_i}, i=1,\ldots,m \tag{16}$$

This approximation is valid when in case of $q \to 0$ and $N \to \infty$ the condition Nq=const holds. One-sided confidence intervals for the parameter Λ of the Poisson distribution can be found by solving the following equations

$$e^{-\underline{\Lambda}} \sum_{j=0}^{d-1} \frac{\Lambda^j}{j!} = 1 - \beta \tag{17}$$

$$e^{-\overline{\Lambda}} \sum_{j=0}^{d} \frac{\Lambda^j}{j!} = \alpha \tag{18}$$

When $d = 0$ we have $\underline{\Lambda} = 0$. For further calculation we can use the connection between the Poisson distribution and the special case of the gamma distribution, namely the chi-square distribution. The confidence intervals can be thus calculated from the formulae:

$$\underline{\Lambda} = \frac{1}{2} \chi^2_{\beta}(2d), \tag{19}$$

$$\overline{\Lambda} = \frac{1}{2} \chi^2_{1-\alpha}(2d + 2), \tag{20}$$

where $\chi^2_{\gamma}(n)$ is the quantile of order γ of the chi-square distribution with n degrees of freedom. Similarly, as in the case of the binomial distribution, for $0{,}5<\alpha<1$ and $0{,}5<\beta<1$ we can use (19) – (20) for the calculation of the two-sided confidence interval for the parameter Λ on the confidence level $1 - \alpha - \beta$.

The Poisson distribution can be also used when the times to failure are described by the exponential distribution. When all elements failed during the test are replaced by new ones, and the duration of the test is equal to T, the observed number of failures is described by the Poisson distribution with the parameter $\Lambda = \lambda NT$, where λ is the failure (hazard) rate in the exponential distribution, and N is the number of simultaneously tested elements. Confidence intervals for the parameter Λ (and for the failure rate λ) are in this case calculated from the formulae (19) – (20).

4. Confidence intervals in the absence of observed failures

Contemporary technical systems are built of very reliable elements. For such elements we usually do not observe failures during reliability tests. In such a case, the point estimation of system's reliability is trivial, and is equal to 1. However, we are interested in the lower bound for this characteristic which may be interpreted as kind of guaranteed reliability. Suppose, that for each of the m types of elements the system is built of we test N_i, $i=1,\ldots,m$, elements, and in every case the number of observed failures is $d_i = 0$, $i=1,\ldots,m$. For such results tests the upper bound for the confidence interval is always equal to $\overline{R} = 1$. On the other hand, it is possible to calculate the lower bound $\underline{R}$ of the confidence interval for the reliability of the considered system. In the book by Gnedenko *et al.* (1999), where results of many works were summarized, it has been shown that the computation of this bound is equivalent to solving the following optimization problem

$$\underline{R} = \min_{\mathbf{p}\in H_0} R(\mathbf{p}), \tag{21}$$

where the set H_0 contains all values of the vector $\mathbf{p}=(p_1,p_2,\ldots,p_m)$ such that

$$\prod_{i=1}^{m} p_i^{N_i} \geq 1-\gamma \tag{22}$$

and

$$0 \leq p_i \leq 1, i = 1,\ldots,m. \tag{23}$$

In many interesting cases there exist closed solutions to this optimization problem. In case of a series system such solution was given by Mirnyi and Solovev (1964). They showed that the lower bound of the confidence interval for system's reliability is given by a simple formula

$$\underline{R} = \min_i \underline{p}_i \tag{24}$$

where $\underline{p}_i$ is the lower bund of the one-sided confidence interval, calculated according to the Clopper-Pearson method (14). It is easy to show that this bound can be calculated from an equivalent formula

$$\underline{R} = (1-\gamma)^{1/N^*} \tag{25}$$

where

$$N^* = \min_i N_i . \tag{26}$$

For systems with a more complicated structure very strong theoretical results were obtained by Pavlov (1982) who considered systems with a convex cumulative risk function defined as follows

$$R(t) = e^{-H(t)} . \tag{27}$$

He has shown that for such systems

$$\underline{R} = \min_i R(1,\ldots,1,\underline{p}_i,1,\ldots,1) \tag{28}$$

where

$$\underline{p}_i = (1-\gamma)^{1/N_i}, i = 1,\ldots,m . \tag{29}$$

The solutions of this problem for parallel, series-parallel, parallel-series, and k-out-of-n systems have been presented in the book by Gnedenko *et al.* (1999). For example, in the case of a system with a parallel reliability structure, consisting of n different elements the lower bound of the one-sided confidence interval for system's reliability is given by:

$$\underline{R} = 1 - \prod_{j=1}^{n} \frac{t}{t+N_j} \tag{30}$$

where t is the solution of the following equation

$$\sum_{j=1}^{n} N_j \, ln\left(1 + \frac{t}{N_j}\right) = -ln(1-\gamma) \tag{31}$$

In a particular case, when $N_1 = \cdots = N_n = N$ Tyoskin and Kurskiy obtained a simple analytic solution (see Gnedenko *et al.* (1999))

$$\underline{R} = 1 - \left[1 - (1-\gamma)^{1/nN}\right]^n . \tag{32}$$

For systems with a more general coherent structure such simple solutions do not exist. However, in the book by Gnedenko *et al.* (1999) two boundaries for the lower bound of the confidence interval have been proposed. Consider the set of all minimal cuts of the system, and assume that the minimal cut with the smallest number of elements consists of *b* elements. Then, consider the set of all possible minimal paths. For this set consider its all possible subsets consisting of independent, i.e. having no common elements, paths. Let *a* be the number of such paths in the subset with the largest number of independent paths. Assume additionally, that for each type of system elements exactly *N* elements have been tested. The boundaries for the lower bound for the system's reliability are the given by

$$1 - \left[1 - (1-\gamma)^{1/Na}\right]^a \le \underline{R} \le 1 - \left[1 - (1-\gamma)^{1/Nb}\right]^b \tag{33}$$

In a particular case of $a = b$ we have

$$\underline{R} = 1 - \left[1 - (1-\gamma)^{1/Nb}\right]^b \tag{34}$$

The authors of Gnedenko *et al.* (1999) notice, that this case is typical for many reliability structures such as lattice or radial structures which are typical for large network systems.

Another very interesting method for the calculation of the lower bound of the confidence interval for system's reliability was presented in Gnedenko *et al.* (1999). Let us assume that the same vector of reliabilities $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ is used for the calculation of reliability of two systems: the reliability $R(\mathbf{p})$ of the considered complex system, and the reliability $R'(\mathbf{p})$ of a simple (e.g. series) auxillary system. For this auxillary system we must know the lower bound of the respective confidence interval $\underline{R}'(\mathbf{p})$. In order to find the lower bound of the confidence interval for the reliability of the considered system we have to solve the following optimization problem:

$$\underline{R} = \min_{\mathbf{p}} R(\mathbf{p}) \tag{35}$$

where the element of the vector **p** must fulfill the following constraint

$$\prod_{i=1}^{m} p_i \geq \underline{R}^{'},\ 0 \leq p_i \leq 1,\ i = 1,\ldots,m\,. \tag{36}$$

The lower bound calculated in this way fulfills all the requirements for a lower bound of a confidence interval, but the length of such interval is usually not the shortest possible.

5. Confidence intervals in the presence of observed failures

When failures are observed during reliability tests of system's elements the problem of building confidence intervals for the reliability of the whole system becomes much more complicated. Comprehensive information about available methods can be found in the fundamental book by Gnedenko *et al.* (1999). Below, we present only some basic results considered in this book and related literature.

Let us assume that the considered system consists of elements of m different types. For each of these types we test a sample of N_i elements, and for each sample we observe $d_i \geq 0, 1 = 1,\ldots,m$ failures. Let

$$S = R(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m) \tag{37}$$

be the point estimator of system's reliability, where $\hat{p}_i, i = 1,\ldots,m$ are the estimators of the reliability of systems elements calculated according to (12). Now, denote by $\mathbf{d}^* = (d_1^*, d_2^*, \ldots, d_m^*)$ the vector of numbers of observed failures. Moreover, denote by $S^* = S(\mathbf{d}^*)$ the observed value of the estimator of system's reliability presented as the function of the vector $\mathbf{d}^*$. The lower bound of the confidence interval for the system's reliability is now calculated from the formula

$$\max_{\mathbf{p}\in A_R} \sum_{S(\mathbf{d})\leq S(\mathbf{d}^*)} \prod_{i=1}^{m} \binom{N_i}{d_i} p_i^{N_i-d_i} (1-p_i)^{d_i} = 1-\gamma, \tag{38}$$

where maximum is calculated over the set A_R of vectors $(p_1, p_2, \ldots, p_m)$, such that

$$R(p_1, p_2, \ldots, p_m) = R, 0 \leq p_i \leq 1, 1 = 1, \ldots, m. \tag{39}$$

The sum in (38) is calculated over all possible values of the vector $\mathbf{d} = (d_1, d_2, \ldots, d_m)$ that fulfill the condition given for this sum in (38). In certain cases other formulation of this optimization problem is more suitable for computations. According to this formulation we denote by $n(\mathbf{d}) = n(d_1, d_2, \ldots, d_m)$ a non-decreasing, with respect to all components, series of vectors. The first element of this series is the vector (0,0,…,0), and then we have the vectors of the type (0,..,0,1,0,…,0), etc. The lower bound of the confidence interval for system's reliability can be calculated from

$$\underline{R} = \min R(p_1, p_2, \ldots, p_m), \tag{40}$$

where minimum is taken over the set of all values of the vector $(p_1, p_2, \ldots, p_m)$ such that

$$\sum_{n(\mathbf{d})\leq n(\mathbf{d}^*)} \prod_{i=1}^{m} \binom{N_i}{d_i} p_i^{N_i-d_i} (1-p_i)^{d_i} \geq 1-\gamma, \qquad 0 \leq p_i \leq 1, i = 1, \ldots, m \tag{41}$$

The optimization problem given by (40) – (41) was formulated first time by Buehler [6] who considered a system consisted of two elements. This was the first result of the calculation of the confidence interval for system's reliability.

Let us now consider the series system consisted of m different elements. The optimization problem is now the following:

$$\underline{R} = \min \prod_{i=1}^{m} p_i, \tag{42}$$

where minimum is taken over all vectors $(p_1, p_2, \ldots, p_m)$ such that

$$\sum_{R(\mathbf{d}) \geq R(\mathbf{d}^*)} \prod_{i=1}^{m} \binom{N_i}{d_i} p_i^{N_i - d_i} (1 - p_i)^{d_i} \geq 1 - \gamma, 0 \leq p_i \leq 1, i = 1, \ldots, m \tag{43}$$

The calculation of the lower bound of the confidence interval for system's reliability $\underline{R}$ can be simplified when the probabilities of failures are small, i.e. when the inequality $q_i = 1 - p_i << 1, i = 1, \ldots, m$ holds. In such a case we can assume that the number of failures is described by the Poisson distribution with the parameter $\Lambda_i = q_1 N_i, i = 1, \ldots, m$. It has been shown in the book by Gnedenko *et al.* (1999) that in this case we have

$$\underline{R} = e^{-\bar{f}} \tag{44}$$

where

$$\bar{f} = max\left(\sum_{i=1}^{m} \frac{\Lambda_i}{N_i}\right), \tag{45}$$

and the maximum in (45) is taken over all vectors $\Lambda = (\Lambda_1, \Lambda_2, \ldots \Lambda_m)$ such that

$$\sum_{R(\mathbf{d}) \geq R(\mathbf{d}^*)} \prod_{i=1}^{m} e^{-\Lambda_i} \left(\frac{\Lambda_i^{d_i}}{d_i!}\right) \geq 1 - \gamma, \lambda_i \geq 0, i = 1, \ldots, m \tag{46}$$

This practical result was obtained first time by Bol'shev and Loginov (1966) for the case of equal values of N_i, and, independently, by Pavlow (1973) and Sudakov (1974) for any values of these numbers.

6. Approximate confidence intervals for system's reliability

Computation of exact bounds of confidence intervals for system's reliability requires, with only few exceptions, solving difficult optimization problems. Therefore, its practical applicability is somewhat limited unless specialized software is available. For this reason several authors, mainly American, have tried to obtain approximate, but relatively easy for computation, solutions. Different approximate solutions have been proposed by such authors like Madansky (1965), Myhre and Saunders (1968), Easterling (1972), Mann (1974), or Mann and Grubbs (1974). Comprehensive review of such results can be found in a well known book by Mann, Shaefer, and Singpurwalla (1974). However, probably the most interesting from a practical point of view result was presented in one of the first textbooks on reliability written by Lloyd and Lipow (1962). These authors presented a heuristic method, attributed to Lindstrom and Madden, for the calculation of the approximate confidence interval for the system with a series reliability structure. This method utilizes the concept of so called equivalent tests. To present this method we consider, following the book by Gnedenko *et al.* (1999), a system with a series-parallel structure which has the same elements in its parallel subsystems. Let R^* be the estimated value of the reliability function for the considered system, and N_i, $i=1,\dots,m$ be the number of tested items for the element of the i-th type. The equivalent number of failures D_i^* for the element of this type is then calculated from the equation

$$R\left(1,\dots,1,1-D_i^*/N_i,1,\dots,1\right)=R^* \tag{47}$$

At the next stage of the computation procedure, for each equivalent test $\left(N_i,D_i^*\right)$ we calculate the lower bound of the confidence interval $\underline{P}_i\left(N_i,D_i^*\right)$ by solving the equation

$$B_p\left(N_i-D_i^*,D_i^*+1\right)=1-\gamma, \tag{48}$$

where

$$B_p(a,b)=\frac{\int_0^p x^{a-1}(1-x)^{b-1}dx}{\int_0^1 x^{a-1}(1-x)^{b-1}dx} \tag{49}$$

is the incomplete beta function whose values can be computed using an available numerical procedure. The lower bound of the confidence interval is now calculated from a simple formula

$$\underline{R} = \min_{1 \le i \le m} R\left(1,\ldots,1,\underline{P}_i\left(N_i, D_i^*\right),1,\ldots,1\right). \tag{50}$$

The Lindstrom-Madden method was proposed as an approximate heuristic method. However, it has been proved (see the book by Gnedenko *et al.* (1999) for additional information) that for many simple reliability structures it produces exact confidence intervals.

Another method which uses the concept of equivalent tests, and which can be used for the analysis of complex systems consisted of many simple subsystems, was proposed by Martz and Duran (1985). In this method it is assumed that for each simple subsystem we are able to calculate the value of its reliability estimator R_i, and the lower bound for the respective confidence interval $\underline{R}_i$. Next, from a set of equations

$$1 - \frac{r_i}{M_i} = R_i \tag{51}$$

and

$$\underline{R}_i = \underline{P}_i\left(M_i, r_i\right) \tag{52}$$

we calculate the parameters (M_i, r_i) of the equivalent binomial reliability tests. In further analysis the considered subsystem is treated as a single element described by the equivalent test. Note, that for the application of this method it is not important how we have found the values of R_i and $\underline{R}_i$.

7. Approximate lower bounds for system's reliability based on minimum values of the reliability of system's elements

7.1 Modeling of dependence in reliability computations

Computation of optimal (i.e. the shortest) and exact confidence intervals is, with a few exceptions, a very difficult task. Moreover, in all published results it is assumed that the elements in a system are mutually independent. Additional problems arise from a fact that confidence intervals used for the description of test results may be conservative, as in the case of intervals based on the Clopper-Pearson formula. In this section we present approximate bounds for system's reliability which, under certain conditions, may replace lower bounds of confidence intervals.

In order to investigate the robustness of the confidence intervals for system's reliability against the departure from the assumption of independence of system's elements let us introduce the notion of a *copula*. According to a famous theorem of Sklar (see e.g. Nelsen (2006)) any two-dimensional probability distribution function $H(x,y)$ with marginal distributions $F(x)$ and $G(y)$ is represented using a function C called a *copula* in the following way:

$$H(x,y) = C(F(x), G(y)) \tag{53}$$

for all $x, y \in R$. Conversely, for any distribution functions F and G and any copula C, the function H defined by (53) is a two-dimensional distribution function with marginal distributions F and G. Moreover, if F and G are continuous, then the copula C is unique.

In our investigation we will consider three types of copulas:

a) Clayton copula defined as

$$H(x,y) = \left[F^{-\theta}(x) + G^{-\theta} - 1\right]^{-1/\theta}, \theta > 0 \tag{54}$$

b) Gumbel copula defined as

$$H(x,y) = \exp\left(-\left[(-\ln F(x))^{\theta} + (-\ln G(y))^{\theta}\right]^{1/\theta}\right), \quad \theta > 0 \tag{55}$$

c) Frank copula defined as

$$H(x,y) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta F(x)} - 1\right)\left(e^{-\theta G(y)} - 1\right)}{e^{-\theta} - 1}\right), \theta \in \{(-\infty, \infty) \setminus \{0\}\}, \tag{56}$$

d) Fairlie-Gumbel-Morgenstern (FGM) copula defined as

$$H(x,y) = F(x)G(y)(1+\theta(1-F(x))(1-G(y))),$$
$$-1 \le \theta \le 1 \tag{57}$$

The Clayton and Gumbel copulas can be used for modeling a positive stochastic dependence. The FGM copula can be used for modeling both negative (θ<1) and positive (θ>1) dependence, but the strength of dependence is limited (the absolute value of Kendall's τ is not greater than 2/9). The Frank copula can be used for modeling of both types of dependence, but without such limitations.

The Clayton copula is especially interesting in reliability applications as it describes stronger dependence for smaller lifetimes than for larger ones. If this type of dependence exists the reliability of a series system with dependent elements is greater than in the case of independence. On the other hand, for a parallel system the reliability of a system with dependent elements is smaller.

7.2 Series systems

In the majority practical cases the reliability of tested elements is high, and even for moderate sample sizes the number of observed failures is small. This suggests utilization of the result obtained for the case of zero-failure tests for the calculation of the lower bounds for reliability of a series system given by the expression (24). To analyze the properties of this approximation let us consider a two-element series system whose elements are equally reliable. We also assume that the sample sizes for both elements are the same. On Figure 1 we present the comparison of the values of our simple approximate bound with the bounds calculated for this system using a substitution method. For obtaining the presented results we performed a Monte Carlo simulation experiments, and in each of them we generated 500 000 test cases, Our approximate bound, plotted against the expected number of observed failures in a sample (for a probability of failure equal to 0,01), is represented by a continuous upper curve. The middle curve represents the bound calculated by the insertion into (5) the respective lower bound of the confidence intervals for the reliability of elements, calculated for the same confidence level (γ=0,9).The lower curve is a similar to the previous one, but calculated for the confidence level equal to $\sqrt{\gamma}$, as it is suggested in statistical literature. Then, we calculated the coverage probability of the considered confidence intervals. The results of the comparison are presented on Figure 2 for our approximate bound, and the bound represented by a middle curve on Figure 1.

As we can see, our simple bound fulfills requirements for a confidence interval not only for zero-failure reliability tests, but for all tests with the expected number of failures not greater than 1,95. The classical and much wider confidence intervals have the probability of coverage close to 1, i.e. much greater than the designed value of 0,9.

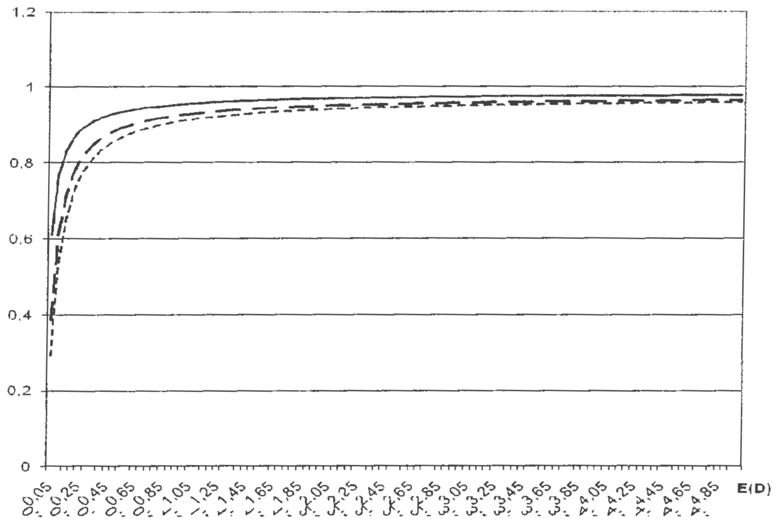

Figure 1. Lower bounds for a series system

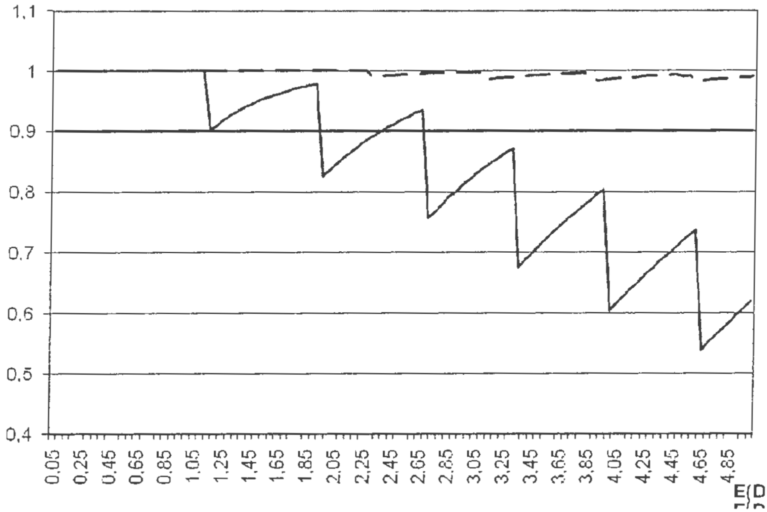

Figure 2. Coverage probabilities for a series system in case of independence (τ=0)

Now, let us consider the case when the elements of the system are dependent. On Figure 3 we show the coverage probability when this dependence is described by the Clayton copula with dependence parameter θ=2, and the Gumbel copula, with dependence parameter θ=2. For this value of the parameter the Kendall measure of dependence τ for both copulas is equal to 0,5. It means that the dependence is positive and fairly strong.

The coverage probability in the case of the Clayton copula (solid line) is greater than the designed value for tests with the expected value of observed failures greater than 5. However, in the case of the dependence described by the Gumbel copula (dashed line) this feature is guaranteed only for this value not greater than 2. It shows, how the type of dependence influences the results despite the fact that the popular measure of dependence, such as Kendall τ in both cases gives exactly the same value.

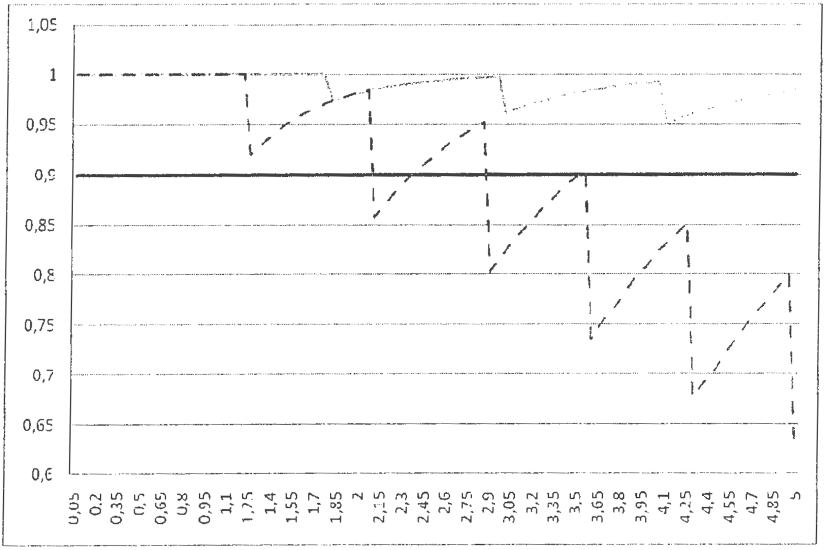

Figure 3. Coverage probabilities for a series system in case of dependence (τ=0,222222)

Let us now consider the influence of the type of dependence, and the method of the evaluation of the lower bound for system's reliability, on statistical properties of the calculated bounds. As previously, let us consider a two-element series system whose elements are characterized by the same reliability (the same probability of failure p). In Table 1 we show how the type of dependence may influence the reliability of the whole system.

Type of dependence	Dependence parameter	$p = 0{,}001$	$p = 0{,}01$	$p = 0{,}1$
independent	-	0,998001	0,9801	0,81
Clayton	0,571428	0,9983024	0,98317	0,838252
Gumbel	1,385714	0,9980072	0,980372	0,819298
Frank	2,08436	0,998002	0,980233	0,819799
FGM	1	0,998002	0,980198	0,8181

Table 1. Reliability of a 2-element series system for different types of dependence ($\tau = 0{,}2222$).

The dependence parameter of each copula has been chosen in order to have the same value of the Kendall measure of dependence, $\tau = 0{,}2222$. As we can see, in the considered case of moderate positive dependence the reliability of the whole system is better than in the case of independence, and depends upon the type of dependence. When lifetimes of system's elements are dependent according to the Gumbel, Frank and FGM copulas the reliability of the system is nearly the same as the reliability of the system with independent elements. However, when the dependence is described by the Clayton copula, the difference between thic case and that of independence is rather significant. This feature influences statistical properties of the bounds for system's reliability, calculated according to different methodologies.

The dependence of the properties of the proposed approximate lower bound upon the type of dependence and the expected number of elements which failed during the tests of subsystems is illustrated on Figures 4 to 6 for the case of confidence intervals with the nominal probability of coverage (confidence level) equal to 0,5. On each of these figures the dependence of the coverage probability for different type of dependence is presented as the function of the expected number of failures in a sample of tested subsystems. Each point on the graphs has been obtained using 500 000 simulation runs.

On Figure 4 we present such dependence when the actual probability of failure is equal to 0,001, and the lower bound for system's reliability is calculated using the proposed approximation. In this case the expected number of failed subsystems during the lifetime tests is not greater than 0,5.

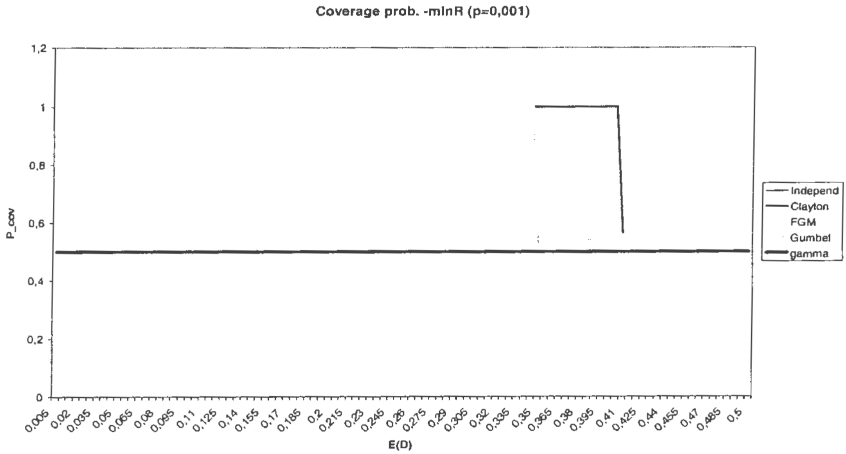

Figure 4. Coverage probabilities for a two-element series system in case of dependence (τ=0,222222) presented as the function of the number of tested subsystems (p=0,001). Approximate confidence bounds.

The result of the simulation experiment shows that in the case of reliable subsystems the proposed approximate method gives the lower bound for system's reliability which is characterized by actual coverage probability greater than the designed one (equal to 0,5). Coverage probabilities in the cases of independence and Gumbel or FGM copulas are the same. A difference has been noted only in the case of the Clayton copula. When the lower bound is calculated using the substitution method the coverage probability is in all considered cases equal to one. It shows that in the case of reliable subsystems the confidence interval calculated according to this method is really very conservative.

On Figure 5 we present the result of similar experiment when the subsystems are less reliable (p=0,01). In the case of the proposed approximate bound the coverage probability is greater than the designed one only in such cases when the expected number of failed elements in the sample is smaller than one. Only in the case of dependence described by the Clayton copula the number of expected failures in the sample may be slightly greater.

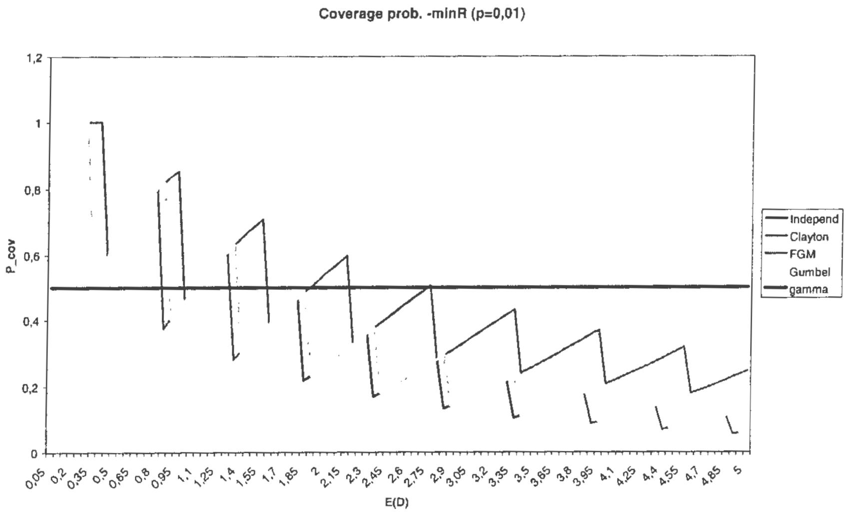

Figure 5. Coverage probabilities for a two-element series system in case of dependence (τ=0,222222) presented as the function of the expected number of failed subsystems during tests (*p*=0,01). Approximate confidence bounds.

In contrast to the previous case of highly reliable subsystems (*p*=0,001), the lower bounds of the confidence intervals obtained by the substitution method yield coverage probabilities closer to the designed value. The dependence of the coverage probability upon the expected number of failures, and thus upon the sample size, is in this case presented on Figure 6. The graph for the case of independent elements is practically the same as the graph obtained in the case of dependence described by the FGM copula. Moreover, in all considered cases the actual coverage probability is significantly greater than the designed one. This indicates that the confidence intervals calculated according to the substitution method are still too wide.

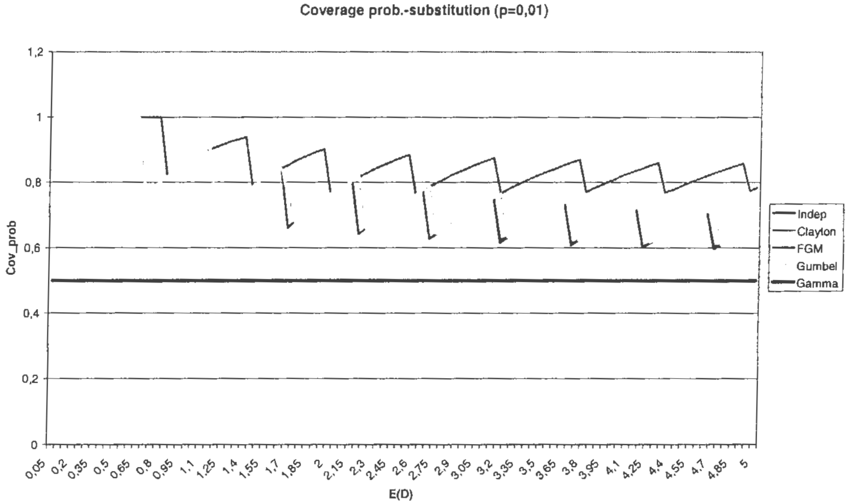

Figure 6. Coverage probabilities for a two-element series system in case of dependence (τ=0,222222) presented as the function of the number of tested subsystems (p=0,01). Bounds calculated using the substitution method.

Finally, let us consider the case when the subsystems are rather unreliable (p=0,1). The results of the simulation experiment for the case of the approximate bounds are presented on Figure 7. One can easily see that in this case the approximate bound is totally unacceptable. This hardly unexpected, as according to theoretical results, it is optimal in the case of no-failure tests.

When in this case we apply the substitution method the results will be quite different. They are presented on Figure 8. In all considered cases the coverage probabilities are greater than the designed value. However, in the case of independent elements this probability approaches the designed value when the sample size increases. When lifetimes of elements are positively dependent this feature seems to be not present. Especially in the case of dependence described by the Clayton copula the behaviour of the respective graph seems to be somewhat strange. This phenomenon definitely requires further investigations.

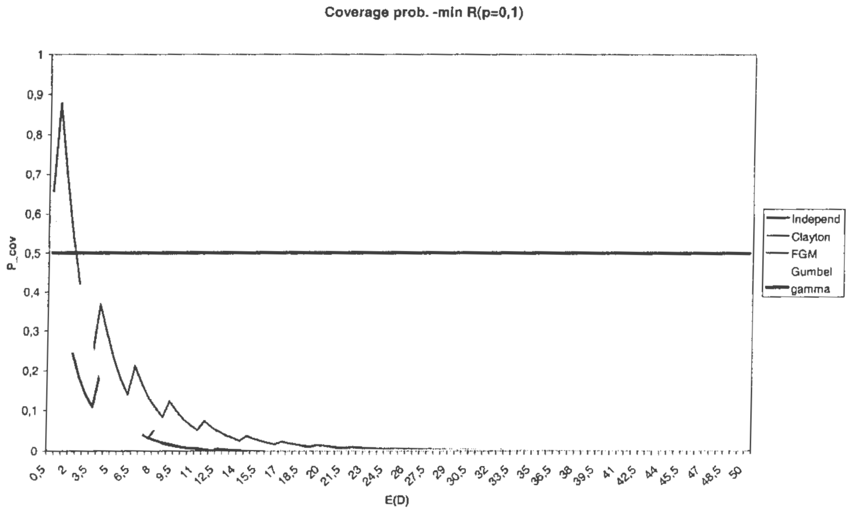

Figure 7. Coverage probabilities for a two-element series system in case of dependence (τ=0,222222) presented as the function of the number of tested subsystems (p=0,1). Approximate confidence bounds.

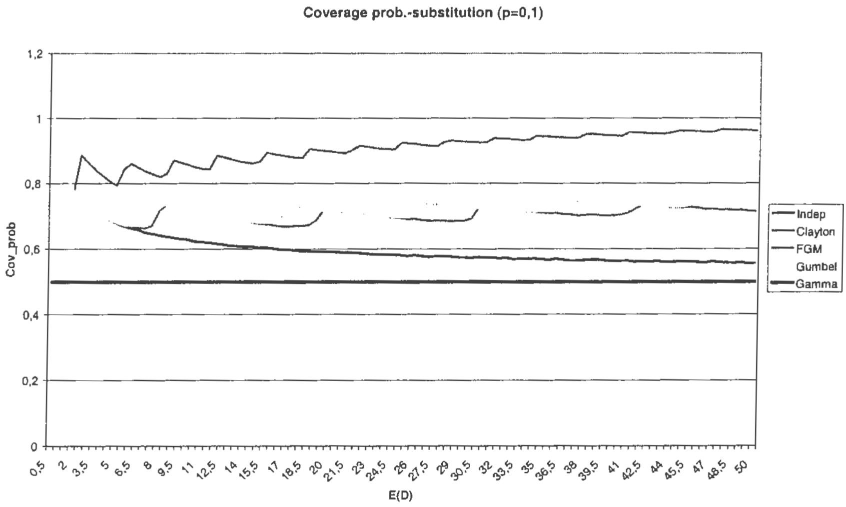

Figure 8. Coverage probabilities for a two-element series system in case of dependence (τ=0,222222) presented as the function of the number of tested subsystems (p=0,1). Bounds calculated using the substitution method.

Consider now the influence of the type of dependence on the properties of confidence bounds when this dependence is much stronger. In Table 2 we show how the type of dependence may influence the reliability of the whole system when the value of Kendall's τ is equal to 0,8.

Type of dependence	Dependence parameter	$p = 0{,}001$	$p = 0{,}01$	$p = 0{,}1$
independent	-	0,998001	0,9801	0,81
Clayton	8	0,998917	0,98917	0,891700
Gumbel	5	0,998358	0,985042	0,871007
Frank	13,815511	0,998014	0,981215	0,859543
FGM	x	x	x	x

Table 2. Reliability of a 2-element series system for different types of dependence (τ=0,8) .

The influence of the type of dependence on the value of reliability is now very significant, especially for the case of elements with low reliability (p=0,1).

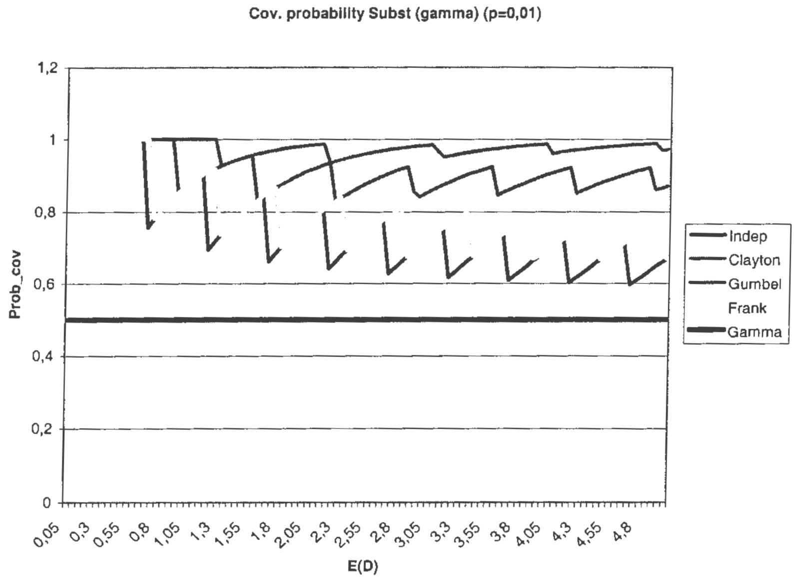

Figure 9. Coverage probabilities for a two-element series system in case of dependence (τ=0,8) presented as the function of the expected number of failed subsystems in the test (p=0,01). Bounds calculated using the substitution method (confidence level for elements the same as for the whole system).

As we can easily see on Figure 9, the application of the substitution method in case of strong positive dependence yields very conservative lower bounds for reliability, especially when the dependence is described by the Clayton copula. When we calculate the γ100% lower bound for the reliability of the whole system using for individual elements the confidence intervals calculated at the confidence level equal to $\sqrt{\gamma}$, as it is frequently advised in statistical textbooks our lower bounds will be extremely conservative. This phenomenon is presented on Figure 10.

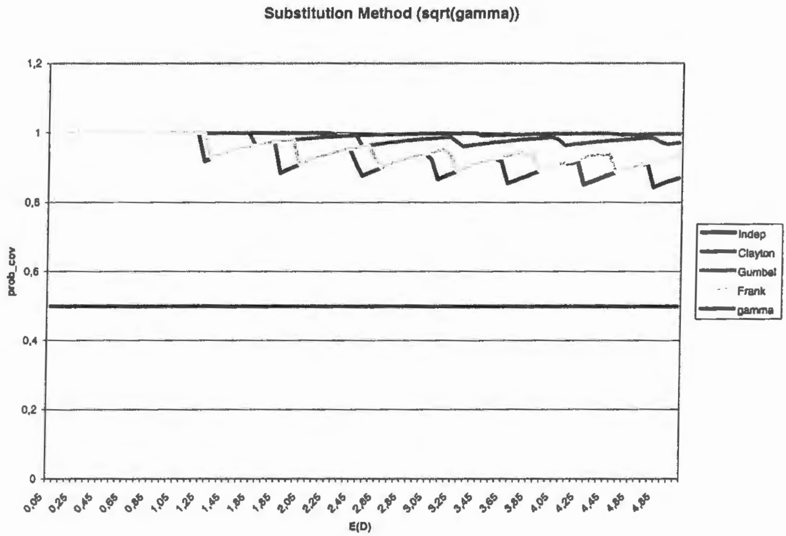

Figure 10. Coverage probabilities for a two-element series system in case of dependence (τ=0,8) presented as the function of the expected number of failed subsystems in the test (p=0,01). Bounds calculated using the substitution method (confidence level for elements equal to the square root of the confidence level for the whole system).

When we apply in the considered case of strong dependence our approximate method for the construction of the lower bound for the reliability of a series system our results will be significantly different from those presented on Figure 5 for the case of relatively weak dependence. The properties of the obtained bound strongly depend on the type of dependence as it can be seen on Figure 11. In case of dependence described by the Clayton copula the proposed approximate method gives satisfactory result even in cases when several failures are observed in tests of subsystems. On Figure 12 we present the comparison of the coverage probabilities of confidence intervals calculated according to three considered methods when the dependence is strong (τ=0,8) and described by the Clayton copula. The bounds obtained using our approximate method yield these probabilities much closer to the designed value γ. Therefore, these bounds are closer to the actual value of system's reliability.

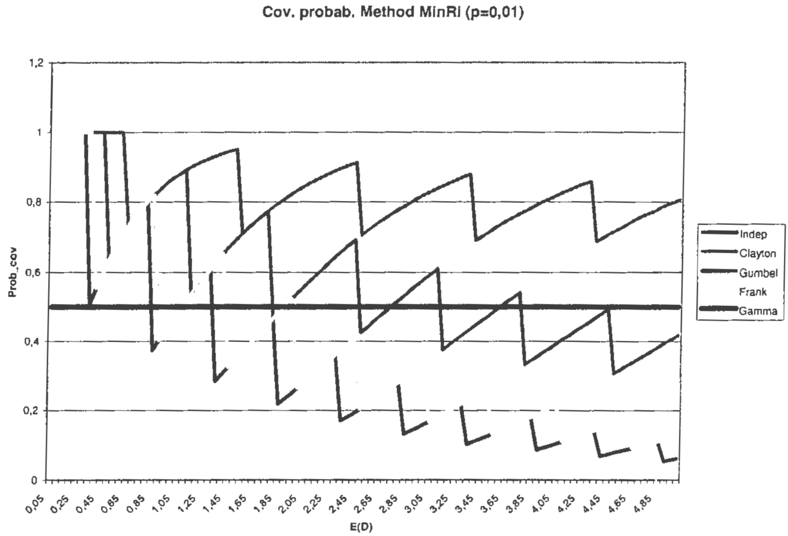

Figure 11. Coverage probabilities for a two-element series system in case of strong dependence (τ=0,8) presented as the function of the expected number of failed subsystems during tests (p=0,01). Approximate confidence bounds.

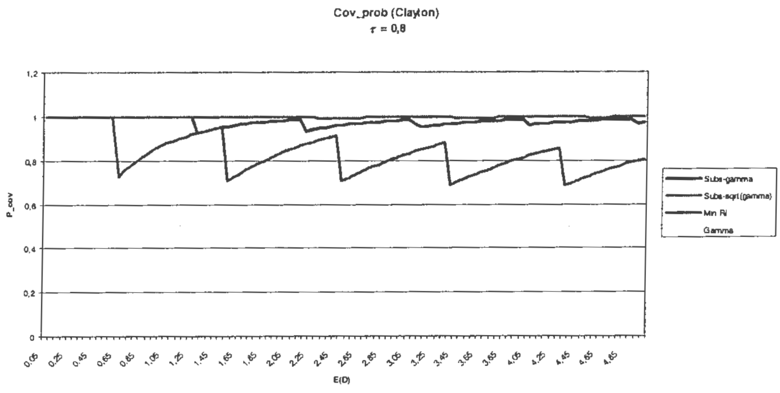

Figure 12 Coverage probabilities for a two-element series system in case of strong dependence (τ=0,8) presented as the function of the expected number of failed subsystems during tests (p=0,01). Dependence described by the Clayton copula. Comparison of different methods used for the calculation of the lower bound.

The advantage of the approximate method when the dependence is strong and modeled according to the Clayton copula is even more visible on Figure 13. On this figure we present the comparison of statistical properties (coverage probabilities) of confidence bounds calculated according to different methods when reliability of subsystems is low. In case of independence, and when this dependence is weak or moderate, our approximate bound is applicable only in case of low expected number of observed failures. However, when dependence is strong our approximate bound performs better (i.e. has the coverage probability closer to the designed one) than bounds obtained by other considered methods.

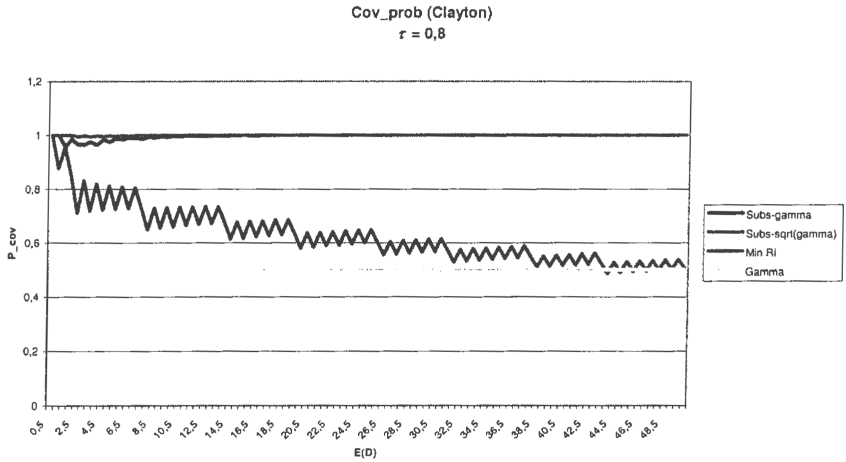

Figure 13 Coverage probabilities for a two-element series system in case of strong dependence (τ=0,8) presented as the function of the expected number of failed subsystems during tests (p=0,1). Dependence described by the Clayton copula. Comparison of different methods used for the calculation of the lower bound.

In all considered above cases we have considered positive type of dependence between system's elements. This type of dependence seems to be natural in reliability applications. Therefore, the properties of the confidence bounds for reliability of a series system in case of negative dependence we present only on few examples.

In Table 3 we show how the type of negative dependence may influence the reliability of the whole system.

Type of dependence	Dependence parameter	$p = 0{,}001$	$p = 0{,}01$	$p = 0{,}1$
independent	-	0,998001	0,9801	0,81
Frank	-2,08436	0,998	0,98003	0,803647
FGM	-1	0,998	0,980002	0,8019

Table 3. Reliability of a 2-element series system for different types of negative dependence ($\tau = -0{,}2222$).

One can immediately see that in the case of existing negative dependence the reliability of a series system is worse than in the case of a system with independent elements. Therefore, in case of negative dependence coverage probabilities should be always greater than in the case of independence. In case of relatively weak negative dependence ($\tau = -0{,}2222$) the properties of confidence bounds are similar to those in the case of independence. Therefore, our approximate bounds are applicable only in case of a small number of expected observed failures. Confirmation of this fact is seen on Figure 14. The similar behavior of the coverage probabilities has been observes also in the case of strong negative dependence. Therefore, the approximation method cannot be recommended for the computation of the lower bound for the reliability of a series system is in the case of negative dependence between system's elements.

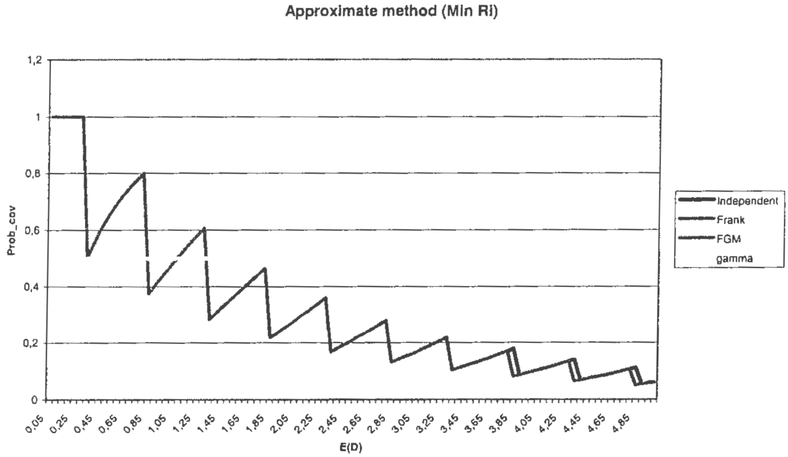

Figure 14. Coverage probabilities for a two-element series system in case of negative dependence (τ=-0,222222) presented as the function of the expected number of failed subsystems during tests (p=0,01). Approximate confidence bounds.

7.3 Parallel systems

Now, let us consider the case of the system with elements connected in parallel. For such systems a simple for computation bound which is similar to that for a series system does not exist. Instead we propose the following approximation

$$\underline{R} = 1 - \min_i \overline{q}_i \,, \qquad (58)$$

where $\overline{q}_i$ is the upper bound of the confidence interval for the probability of failure. The lower bound calculated according to (58) is *always* smaller than the bound obtained by substitution of the probabilities of failures q_i with their respective upper bounds $\overline{q}_i$. Thus, the coverage probability in case of independent elements of the system, calculated according to (58), is always greater than the respective confidence level. It can be seen at Figure 15, where this probability is always equal to 1. On Figure 15 we also show the coverage probability in case of the bound obtained by substitution which is also much greater than the confidence level which in the considered case is equal to 0,9.

The situation changes dramatically when the elements of the system are dependent, and when their dependence is described either by the Clayton copula or by the Gumbel copula. In Table 4 we show how the type of dependence may influence the reliability of the whole system.

Type of dependence	Dependence parameter	$p = 0{,}001$	$p = 0{,}01$	$p = 0{,}1$
independent	-	0,999999	0,9999	0,99
Clayton	0,571428	0,999698	0,99683	0,961748
Gumbel	1,385714	0,999993	0,999628	0,980702
Frank	2,08436	0,999998	0,999767	0,980201
FGM	1	0,999998	0,999802	0,9819

Table 4. Reliability of a 2-element parallel system for different types of dependence (τ= 0,2222).

In all considered case the reliability of a parallel system in case of positive dependence between systems elements is worse than in the case of independence. Therefore, the coverage probabilities for the confidence bounds calculated under the assumption of independence by the substitution method should be lower than in the case of actual independence. On Figure 16 we present the coverage probabilities in such cases when the confidence intervals are calculated using the substitution method.

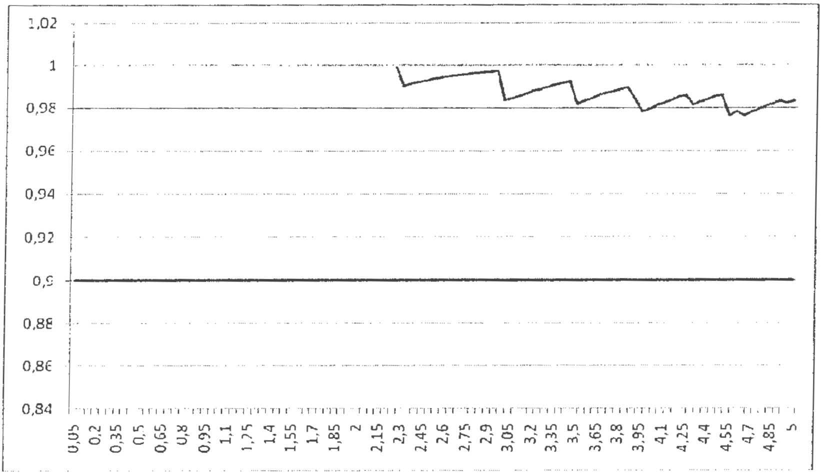

Figure 15. Coverage probabilities for a parallel system in case of independence (approximate bounds)

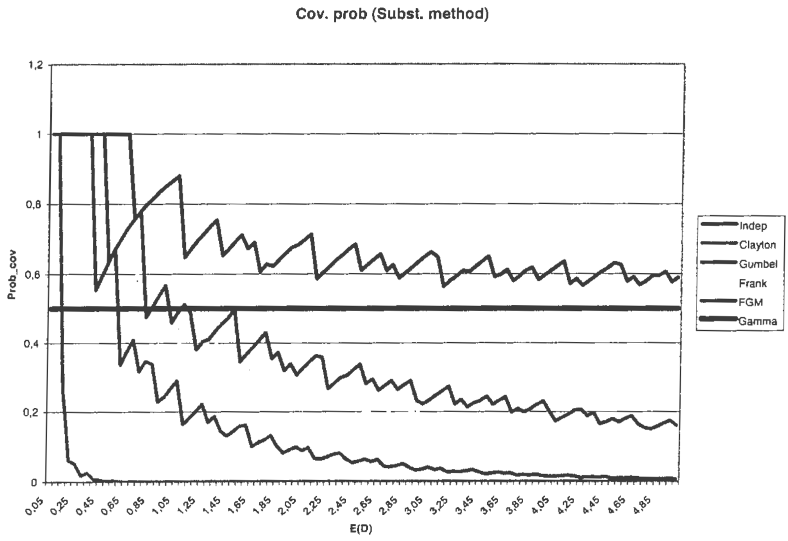

Figure 16. Coverage probabilities for a two-element parallel system in case of dependence (τ=0,222222) presented as the function of the number of tested subsystems (p=0,01). Bounds calculated using the substitution method.

The coverage probabilities in case of dependence (especially the left-most curve for the Clayton copula, and the curve next to it for the Gumbel copula) show dramatically that the confidence intervals obtained by substitution under the assumption of independence are too narrow. On the other hand, the interval calculated according to (58) has the coverage probability greater than the designed confidence level. This property is presented on Figure 17.

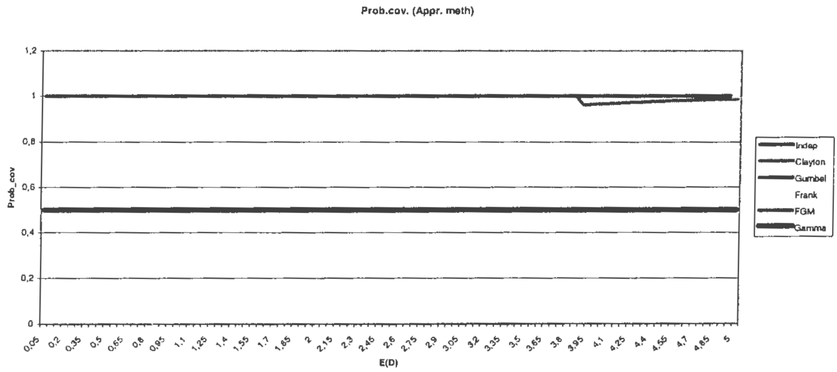

Figure 17 Coverage probabilities for a two-element parallel system in case of dependence (τ=0,222222) presented as the function of the number of tested subsystems (p=0,01). Bounds calculated using the approximated method.

In case of stronger dependence the influence of the type of dependence on the reliability of a parallel system is – similarly to the case of a series system – more visible. In Table 5 we present the results of computations of system's reliability for a 2-element parallel system with identical, but possibly dependent, elements

Type of dependence	Dependence parameter	$p = 0,001$	$p = 0,01$	$p = 0,1$
independent	-	0,999999	0,9999	0,99
Clayton	8	0,999083	0,99083	0,9083
Gumbel	5	0,999642	0,994958	0,928993
Frank	13,815511	0,999986	0,998785	0,940457

Table 5. Reliability of a 2-element parallel system for different types of dependence (τ=0,8) .

The existence of strong positive dependence between the elements, and the knowledge of its strength, is not sufficient for a correct evaluation of the actual value of system's reliability.

When this dependence is described by the Clayton copula the actual value of system's reliability is in the considered case much lower not only than in the case of independence, but in the case of other types of dependence as well. This phenomenon is reflected by the values of coverage probabilities presented on Figure 18 for the case of bounds calculated according to the substitution method for reliable elements (p=0,001).

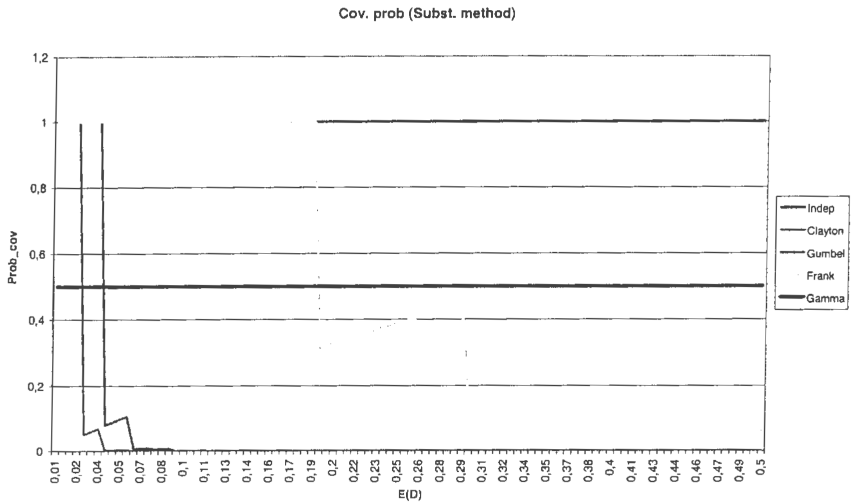

Figure 18. Coverage probabilities for a two-element parallel system in case of dependence (τ=0,8) presented as the function of the number of tested subsystems (p=0,001). Bounds calculated using the substitution method.

In case of independence the coverage probability of such bound is equal to 1 in the whole analyzed range of values of the expected number of failures in the test sample (sample sizes from 5 to 500). However, in case of dependence the bounds obtained by the substitution method are acceptable only for small and very small sample. However, when we apply the approximate method described by (58) the coverage probability in the considered range of sample sizes will be equal to 1 for all considered types of dependence. Therefore, the proposed approximate bound is very conservative, but robust to strong dependencies between systems elements.

In all cases considered above we assumed that reliability data about systems elements comes from different tests. This assumption is reasonable only in some cases when subsystems connected in parallel are different. However, in practice we frequently have redundant systems with identical elements connected in parallel. Thus, in such cases we use the same test results for the calculation of confidence bounds for individual subsystems. This may influence the coverage probability of the confidence bound for the whole system. On Figure 19 we present coverage probabilities in the case of the weak dependence described by the Clayton copula when a parallel system consists of identical or different (but equally reliable) elements, and we use the substitution method for the calculation of the lower bound for reliability. The similar analysis for the lower bound obtained by our approximate method is given on Figure 20.

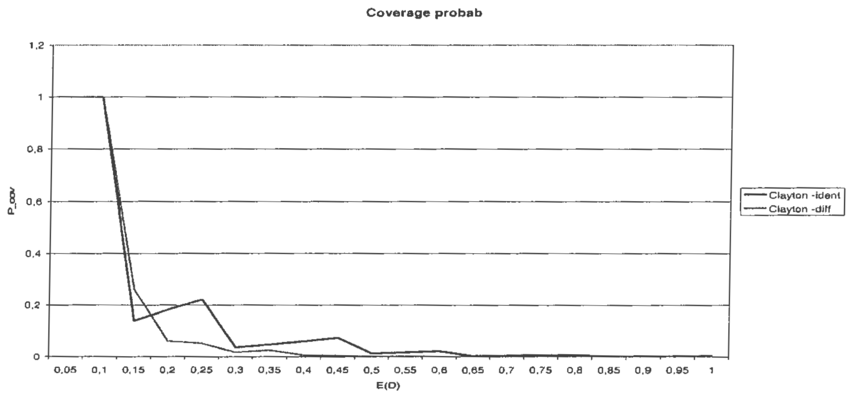

Figure 19. Coverage probabilities for a two-element parallel system in case of dependence described by the Clayton copula (τ=0,222222) presented as the function of the number of tested subsystems (p=0,01). Bounds calculated using the substitution method when subsystems are either identical or different.

In both considered cases the coverage probabilities calculated for systems with identical elements are usually larger than those calculated for systems with different elements. Slight differences from this behaviour are probably due to non-monotonic character of the dependence of the values of lower bounds upon the sample size.

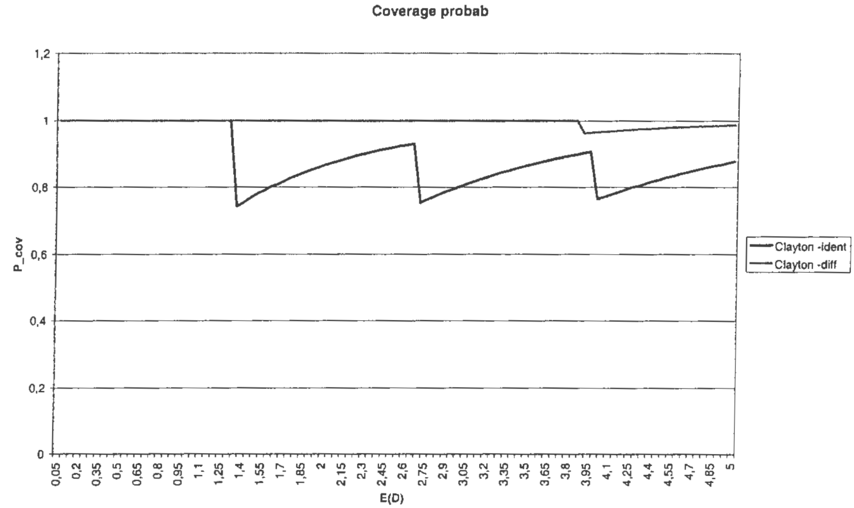

Figure 20. Coverage probabilities for a two-element parallel system in case of dependence described by the Clayton copula (τ=0,222222) presented as the function of the number of tested subsystems (p=0,01). Bounds calculated using the approximate method when subsystems are either identical or different.

Similarly to the case of a series system negative dependence between subsystems connected in parallel is rather unexpected in practice. On Figure 21 we show how this type of dependence influences the coverage probability of the confidence interval calculated according to the substitution method. In general, when such negative dependence exists the coverage probabilities are larger than in the case of independence, and much larger than their designed values. Thus, the confidence intervals are in this case very conservative. On figure 22 we present the same characteristic of the confidence interval when this interval is calculated according to the approximate method. These properties are exactly the same as in the case of the intervals calculated according to the substitution method. The influence of the type of negative dependence described by the Frank and the FGM copulas is even less significant.

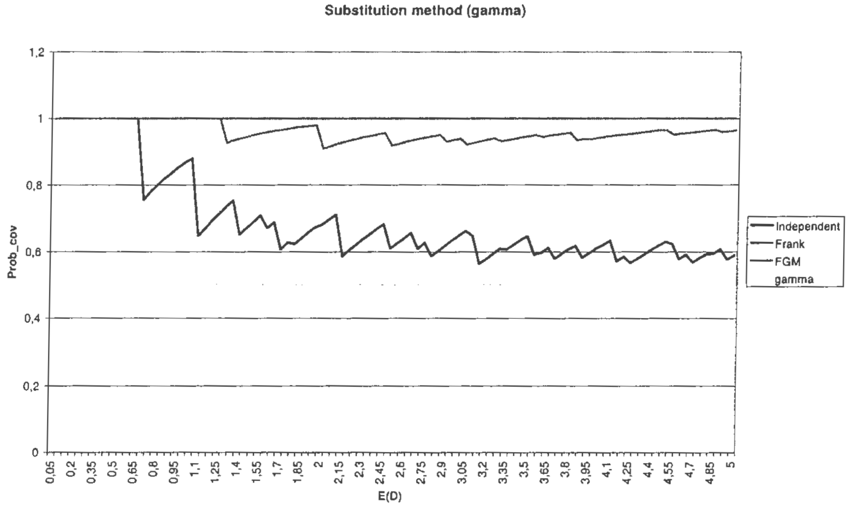

Figure 21. Coverage probabilities for a two-element parallel system in case of negative dependence (τ=-0,222222) presented as the function of the expected number of failed subsystems during tests (p=0,01). Confidence bounds calculated according to the substitution method.

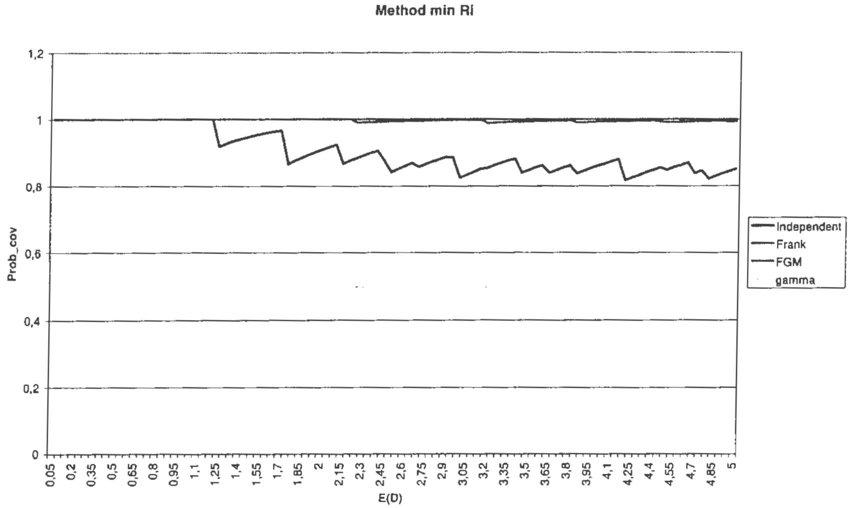

Figure 22. Coverage probabilities for a two-element parallel system in case of negative dependence (τ=-0,222222) presented as the function of the expected number of failed subsystems during tests (p=0,01). Confidence bounds calculated according to the approximate method.

8. Conclusions

Many prominent authors, mainly from USA and the Soviet Union, contributed to the problem of computing the lower confidence bounds for system's reliability using the data from tests of separate elements or subsystems. The proposed exact bounds are usually difficult to compute. Good approximations exist, but they are usually obtained under the assumption that failures of all elements or subsystems are observed during the tests. In the paper we have shown using Monte Carlo simulation that in case when elements working together in a system are dependent these bounds are inaccurate or even useless, as it is the case of parallel (redundant) systems. In the paper, we have proposed very simple bounds characterized by satisfactory performance, at least for highly reliable system elements, which are robust against the presence of positive dependence of the elements of a system.

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