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The explicit form of the membership function for the multiplication of two L-L fuzzy numbers

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The explicit form of the membership function for the multiplication of two $L-L$ fuzzy numbers.

by
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Abstract.

In this paper we deal with $L-L$ fuzzy numbers with membership functions of the following form $\mu_A(z) = L(\frac{|z-a|}{c})$, where $L()$ is a reference function defined on $[0, +\infty)$ that is continuous and strictly monotone decreasing with $L(0) = 1$ and $c > 0$. It is well known that the membership function for the sum of two $L-L$ fuzzy numbers A_1 and A_2 is equal to

$$\mu_{A_1+A_2}(z) = L(\frac{|z-(a_1+a_2)|}{c_1+c_2})$$

[1].

One can also find some approximations for the membership function for the multiplication of two $L-L$ fuzzy numbers. For example[3], if $a_1 > 0$ and $a_2 > 0$ then

$$\mu_{A_1A_2}(z) \simeq L(\frac{|z-(a_1a_2)|}{a_1c_2+a_2c_1}).$$

It was emphasized in [3, page 73] that the explicit form of the mumbership function for the multiplication can not directly be obtained.

In this paper we calculate the membership function for the multiplication of two $L-L$ fuzzy numbers. Of course, the multiplication is no longer $L-L$ fuzzy number but its membership function is quite simple and as is shown in this paper is equal to

$$\mu_{A_1A_2}(z) = \begin{cases} L(\frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2}}), & \text{for } z < 0; \\ L(\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) - \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}), & \text{for } z \in [0, a_1a_2]; \\ L(-\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}), & \text{for } z > a_1a_2 \end{cases}$$

,where $a_1, a_2 > 0$ and $\frac{a_1}{c_1} \leq \frac{a_2}{c_2}$.

If we have the explicit form of the membership function $\mu_{A_1A_2}(z)$ we can see how good is the approximation we use. For example we will see how good is the above approximation $\mu_{A_1A_2}(z) \simeq L(\frac{|z-(a_1a_2)|}{a_1c_2+a_2c_1})$.

As an application of the above formula we calculate the exact formula for possibility index $Pos(A_3, A_1A_2)$ for $L-L$ fuzzy numbers A_1, A_2, A_3. The possiblity index is very useful in fuzzy linear regression [3, page 61].

1.Introduction.

A fuzzy number is defined as a set of ordered pairs $\{(z, \mu_A(z))\}$, where $z \in A$ and $\mu_A :\rightarrow [0,1]$ is the membership function of A. The $\alpha-$level set of a fuzzy number A is $(A)_\alpha = \{z : \mu_A(z) \geq \alpha\}$. For the fuzzy numbers one define, using Zadeh extension principle, the binary arithmetic operations as follows

addition $A_1 + A_2$: $\mu_{A_1+A_2}(z) = \max_{z=x+y}\{\mu_{A_1}(x) \wedge \mu_{A_2}(y)\}$;
multiplication A_1A_2:$\mu_{A_1A_2}(z) = \max_{z=xy}\{\mu_{A_1}(x) \wedge \mu_{A_2}(y)\}$;
opposite number $-A$:$\mu_{-A}(z) = \mu_A(-z)$.

A fuzzy number A is called $L-L$ fuzzy number if its membership function is of the following form

$$\mu_A(z) = L(\frac{|z-a|}{c})$$

, where $L()$ is a reference function defined on $[0, +\infty)$ that is continuous and strictly monotone decreasing with $L(0) = 1$ and $c > 0$. It is well known that the membership function for the sum of two $L-L$ fuzzy numbers A_1 and A_2 is equal to $\mu_{A_1+A_2}(z) = L(\frac{|z-(a_1+a_2)|}{c_1+c_2})$[1]. From the result of Nguyen (1978) [1] the $\alpha-$level set

$$(A_1A_2)_\alpha = \{z : \mu_{A_1A_2}(z) \geq \alpha\}$$

for the multiplication of A_1A_2 is $(A_1A_2)_\alpha = [a_\alpha, b_\alpha]$, where

$$\begin{aligned} a_\alpha &= \min\{(a_1 - L^{-1}(\alpha)c_1)(a_2 - L^{-1}(\alpha)c_2), (a_1 - L^{-1}(\alpha)c_1)(a_2 + L^{-1}(\alpha)c_2)\}, \\ b_\alpha &= \max\{(a_1 + L^{-1}(\alpha)c_1)(a_2 + L^{-1}(\alpha)c_2), (a_1 + L^{-1}(\alpha)c_1)(a_2 - L^{-1}(\alpha)c_2)\}. \end{aligned}$$

We have $\mu_{-A}(z) = L(\frac{|-z-a|}{c}) = L(\frac{|z+a|}{c}) = L(\frac{|z-(-a)|}{c})$, for a fuzzy number A with a membership function $\mu_A(z) = L(\frac{|z-a|}{c})$.

We will also consider the possibility index for the relation between two fuzzy numbers proposed by Dubois (1978)[2]

$Pos(A_1 = A_2) = \sup_{z \in R} \min\{\mu_{A_1}(z), \mu_{A_2}(z)\}$.

In the following we will assume that we have two $L-L$ fuzzy numbers A_1, A_2 with membership functions $\mu_{A_1}(z) = L(\frac{|z-a_1|}{c_1}), \mu_{A_2}(z) = L(\frac{|z-a_2|}{c_2})$.

2.Main results.

We start with the following lemma.

Lemma 1.

Let A_1, A_2 be two $L-L$ fuzzy numbers. If $\mu_{A_1A_2}(z) = w$, $z = xy$ and $\mu_{A_1}(x) = w$ then $\mu_{A_2}(y) = w$.

Proof.

Assume that $\mu_{A_1}(x) < \mu_{A_2}(y)$. From continuity and monotonicity of L function there exists a number $h \approx 1$ such that $\mu_{A_1}(x) < \mu_{A_1}(xh) < \mu_{A_2}(y\frac{1}{h})$. Hence $z = xhy\frac{1}{h}$ and $\mu_{A_1A_2}(z) \geq \mu_{A_1}(xh) \wedge \mu_{A_2}(y\frac{1}{h}) > \mu_{A_1}(x) = w$. This result contradicts with the assumption that $\mu_{A_1A_2}(z) = w$. This ends the proof of the lemma.

It follows from the above lemma that when we calculate $\mu_{A_1A_2}(z)$ it is enough to consider $z = xy$ and $\mu_{A_1}(x) = \mu_{A_2}(y)$.

Lemma 2.

Let $\mu_{A_1}(z) = L(\frac{|z-a_1|}{c_1}), \mu_{A_2}(z) = L(\frac{|z-a_2|}{c_2}), w \in [0, L(0)], \mu_{A_1}(x_0) = w = \mu_{A_2}(y_0)$ and $a_1, a_2 > 0, \frac{a_1}{c_1} \leq \frac{a_2}{c_2}$.

If

$$\begin{aligned}(x_0 &\in (0,a_1) \text{ and } y_0 \in (0,a_2)) \text{ or } (x_0 > a_1 \text{and } y_0 \in (a_2, 2a_2)) \\ \text{or } (x_0 &< 0 \text{ and } y_0 > a_2)\end{aligned}$$

then $\min\{\mu_{A_1}(x_0h), \mu_{A_2}(y_0\frac{1}{h})\} \leq w$ for any $h \neq 0$.

Proof.

Assume that $x_0 \in (0, a_1)$ and $y_0 \in (0, a_2)$. If $h > 0$, then one of the numbers $\mu_{A_1}(x_0h), \mu_{A_2}(y_0\frac{1}{h})$ will decrease since the functions $\mu_{A_1}(z), \mu_{A_2}(z)$ are increasing on $(0, a_1)$ and $(0, a_2)$ respectively. If $h < 0$, then

$$\mu_{A_1}(x_0h) < \mu_{A_1}(0) < \mu_{A_1}(x_0) \text{ and } \mu_{A_2}(y_0\frac{1}{h}) < \mu_{A_2}(0) < \mu_{A_2}(x_0).$$

Hence

$$\min\{\mu_{A_1}(x_0h), \mu_{A_2}(y_0\frac{1}{h})\} \leq \mu_{A_1}(x_0) = w.$$

Assume now that $x_0 > a_1$ and $y_0 > a_2$. If $h > 0$, then one of the numbers $\mu_{A_1}(x_0h), \mu_{A_2}(y_0\frac{1}{h})$ will decrease since the functions $\mu_{A_1}(z), \mu_{A_2}(z)$ are decreasing on $(a_1, +\infty)$ and $(a_2, +\infty)$ respectively. If $h < 0$, then

$$\mu_{A_1}(x_0h) < \mu_{A_1}(-x_0(-h)) \text{ and } \mu_{A_2}(y_0\frac{1}{h}) < \mu_{A_2}(-y_0\frac{1}{(-h)}).$$

Hence from the symmetry of the functions $\mu_{A_1}(z), \mu_{A_2}(z)$ we have that

$$\mu_{A_1}(-x_0) < \mu_{A_1}(x_0) \text{ and } \mu_{A_2}(-y_0) < \mu_{A_2}(y_0).$$

Now, we have that $(-h) > 0$ and again one of the numbers

$$\mu_{A_1}(-x_0(-h)), \mu_{A_2}(-y_0\frac{1}{(-h)})$$

will decrease since the functions $\mu_{A_1}(z), \mu_{A_2}(z)$ are both increasing on$(-\infty, 0)$
.

Hence

$$\min\{\mu_{A_1}(x_0 h), \mu_{A_2}(y_0\frac{1}{h})\} = \min\{\mu_{A_1}(-x_0(-h)), \mu_{A_2}(-y_0\frac{1}{(-h)})\} \leq w.$$

Lastly, assume that $x_0 < 0$ and $a_2 < y_0 \leq 2a_2$. If $h < 0$ then from the symmetry of the function $\mu_{A_2}(z)$ about a_2 we have that

$$\mu_{A_2}(y_0\frac{1}{h}) < \mu_{A_2}(0) \leq \mu_{A_2}(y_0).$$

If $h > 0$ then for $h < 1$ we have $\mu_{A_2}(y_0\frac{1}{h}) < \mu_{A_2}(y_0)$ and for $h > 1$ we have $\mu_{A_1}(x_0 h) < \mu_{A_1}(x_0)$. Hence

$$\min\{\mu_{A_1}(x_0 h), \mu_{A_2}(y_0\frac{1}{h})\}\mu_{A_1}(x_0) = w.$$

This ends the proof of the lemma.

Theorem.

Let $\mu_{A_1}(z) = L(\frac{|z-a_1|}{c_1}), \mu_{A_2}(z) = L(\frac{|z-a_2|}{c_2})$ where L is a continuous and stricly decreasing function on $[0, +\infty)$ and $a_1, a_2 > 0$, $\frac{a_1}{c_1} \leq \frac{a_2}{c_2}$.

Then the membership function of the multiplication $A_1 A_2$ is equal to

$$\mu_{A_1 A_2}(z) = \begin{cases} L(\frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1 c_2}}), & \text{for } z < 0; \\ L(\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) - \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1 c_2}}), & \text{for } z \in [0, a_1 a_2]; \\ L(-\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1 c_2}}), & \text{for } z > a_1 a_2 \end{cases}$$

Proof.

First we shall calculate the numbers $\mu_{A_1 A_2}(0)$ and $\mu_{A_1 A_2}(a_1 a_2)$.

If $z = xy = 0$, then $x = 0$ or $y = 0$. We have

$$\mu_{A_2}(0) = L(\frac{a_2}{c_2}) \leq L(\frac{a_1}{c_1}) = \mu_{A_1}(0) \leq L(0) = \mu_{A_1}(a_1) = \mu_{A_2}(a_2).$$

Hence

$$\mu_{A_1 A_2}(0) = \max\{\mu_{A_2}(0), \mu_{A_1}(0)\} = \mu_{A_1}(0) = L(\frac{a_1}{c_1}).$$

Lastly $\mu_{A_1 A_2}(a_1 a_2) = L(0)$.

Assume now that

$$x_0 = a_1 - L^{-1}(w)c_1 < 0 \text{ and } y_0 = a_2 + L^{-1}(w)c_2 \in (a_2, 2a_2)$$

and take

$$0 > z = x_0 y_0 = (a_1 - L^{-1}(w)c_1)(a_2 + L^{-1}(w)c_2).$$

It follows that $\frac{a_2}{c_2} > L^{-1}(w) > \frac{a_1}{c_1}$ or equivalently that

$$\mu_{A_2}(0) = L(\frac{a_2}{c_2}) > w < L(\frac{a_1}{c_1}) = \mu_{A_1}(0).$$

For each $w \in (L(\frac{a_2}{c_2}), L(\frac{a_1}{c_1}))$ there exists $z = x_0 y_0 < 0$ since the parabola $(a_1 - L^{-1}(w)c_1)(a_2 + L^{-1}(w)c_2)$ has two roots $L^{-1}(w) = -\frac{a_2}{c_2}$ or $L^{-1}(w) = \frac{a_1}{c_1}$. Now we can calculate $L^{-1}(w)$ from the quadratic equation

$$-c_1 c_2 [L^{-1}(w)]^2 + (a_1 c_2 - a_2 c_1) L^{-1}(w) + a_1 a_2 - z = 0.$$

We have

$$\Delta = (a_1 c_2 - a_2 c_1)^2 + 4 c_1 c_2 (a_1 a_2 - z) = (a_1 c_2 + a_2 c_1)^2 - 4 c_1 c_2 z.$$

Hence

$$L^{-1}(w) = \frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) \pm \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1 c_2}}.$$

We have $L^{-1}(w) = \frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1 c_2}}$ since $L^{-1}(w) > \frac{a_1}{c_1}$. So we have from the Lemma 2 that

$$\mu_{A_1 A_2}(z) = w = L(\frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1 c_2}}).$$

If $L^{-1}(w) > \frac{a_2}{c_2}$, then $a_2 - L^{-1}(w)c_2 < 0$ and $a_1 - L^{-1}(w)c_1 < 0$. Using the result of Nguyen[1] we have that

$$\begin{aligned} a_w &= \min\{(a_1 - L^{-1}(w)c_1)(a_2 - L^{-1}(w)c_2), (a_1 - L^{-1}(w)c_1)(a_2 + L^{-1}(w)c_2)\} = \\ &= (a_1 - L^{-1}(w)c_1)(a_2 + L^{-1}(w)c_2). \end{aligned}$$

Hence for $z = (a_1 - L^{-1}(w)c_1)(a_2 + L^{-1}(w)c_2)$ we have

$$\mu_{A_1A_2}(z) = w = L(\frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2}}).$$

Lastly we have for $z < 0$ that

$$\mu_{A_1A_2}(z) = L(\frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2}}).$$

Now take

$$\begin{aligned} x_0 &= a_1 - L^{-1}(w)c_1 > 0, y_0 = a_2 - L^{-1}(w)c_2 > 0 \\ \text{and } 0 &< z = x_0y_0, \text{where } 0 < L^{-1}(w) < \frac{a_1}{c_1}. \end{aligned}$$

Hence $0 < z < a_1a_2$ since the parabola $(a_1 - L^{-1}(w)c_1)(a_2 - L^{-1}(w)c_2)$ has two roots $L^{-1}(w) = \frac{a_1}{c_1}$ and $L^{-1}(w) = \frac{a_2}{c_2}$. For the quadratic equation

$$c_1c_2[L^{-1}(w)]^2 - (a_1c_2 + a_2c_1)L^{-1}(w) + a_1a_2 - z = 0$$

we have

$$\Delta = (a_1c_2 + a_2c_1)^2 - 4c_1c_2(a_1a_2 - z) = (a_1c_2 - a_2c_1)^2 + 4c_1c_2z.$$

Hence $L^{-1}(w) = \frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) - \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}$ since $L^{-1}(w) < \frac{a_1}{c_1}$. So we have from the Lemma 2 that

$$\mu_{A_1A_2}(z) = w = L(\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) - \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}).$$

Assume now that

$$\begin{aligned} x_0 &= a_1 + L^{-1}(w)c_1, y_0 = a_2 + L^{-1}(w)c_2 \\ \text{and } a_1a_2 &< z = x_0y_0 = (a_1 + L^{-1}(w)c_1)(a_2 + L^{-1}(w)c_2). \end{aligned}$$

We have $L^{-1}(w) \geq 0$ since the parabola $(a_1 + L^{-1}(w)c_1)(a_2 + L^{-1}(w)c_2)$ has roots $L^{-1}(w) = -\frac{a_2}{c_2}$ or $L^{-1}(w) = -\frac{a_1}{c_1}$. We calculate w form the equation

$$c_1c_2[L^{-1}(w)]^2 + (a_1c_2 + a_2c_1)L^{-1}(w) + a_1a_2 - z = 0.$$

We have

$$\Delta = (a_1c_2 + a_2c_1)^2 - 4c_1c_2(a_1a_2 - z) = (a_1c_2 - a_2c_1)^2 + 4c_1c_2z.$$

Hence $L^{-1}(w) = -\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}$ since $L^{-1}(w) \geq 0$. So we have from the Lemma 2 that

$$\mu_{A_1A_2}(z) = w = L(-\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}).$$

What ends the proof of the theorem.

Remark.

To check the validity of the formula from the above theorem one can plot the following two functions $f_1(\alpha) = \mu_{A_1A_2}(a_\alpha)$ and $f_2(\alpha) = \mu_{A_1A_2}(b_\alpha)$ for $\alpha \in (0, 1]$. We take

$$\begin{aligned} a_\alpha &= \min\{(a_1 - L^{-1}(\alpha)c_1)(a_2 - L^{-1}(\alpha)c_2), (a_1 - L^{-1}(\alpha)c_1)(a_2 + L^{-1}(\alpha)c_2)\}, \\ b_\alpha &= \max\{(a_1 + L^{-1}(\alpha)c_1)(a_2 + L^{-1}(\alpha)c_2), (a_1 + L^{-1}(\alpha)c_1)(a_2 - L^{-1}(\alpha)c_2)\}. \end{aligned}$$

As a result one should obtain the strait line passing throught points $(0, 0)$ and $(1, 1)$.

Lemma 3.

$\mu_{A_1A_2}(z) = \mu_{-A_1A_2}(-z) = \mu_{A_1(-A_2)}(-z)$.

Proof.

$\mu_{A_1A_2}(z) = \max_{z=xy}\{\mu_{A_1}(x) \wedge \mu_{A_2}(y)\} = \max_{-z=(-x)y}\{\mu_{-A_1}(-x) \wedge \mu_{A_2}(y)\} =$
$= \mu_{-A_1A_2}(-z) = \max_{-z=x(-y)}\{\mu_{A_1}(x) \wedge \mu_{-A_2}(-y)\} = \mu_{A_1(-A_2)}(-z)$.

Remark.

From the above lemma we see that it suffices to find out the formula for the membership function $\mu_{A_1A_2}(z)$ for fuzzy numbers A_1, A_2 with $\mu_{A_1}(z) = L(\frac{|z-a_1|}{c_1}), \mu_{A_2}(z) = L(\frac{|z-a_2|}{c_2})$,where $a_1, a_2 > 0$ and $\frac{a_1}{c_1} \leq \frac{a_2}{c_2}$. If for example $a_1 < 0$ and $a_2 > 0$, then $\mu_{A_1A_2}(z) = \mu_{-A_1A_2}(-z)$, where $-a_1 > 0$ and $a_2 > 0$. (if $\frac{a_1}{c_1} > \frac{a_2}{c_2}$ we replace a_1 with a_2 and c_1 with c_2).

Thus we have for $0 < \frac{-a_1}{c_1} \leq \frac{a_2}{c_2}$ that

$$\mu_{A_1A_2}(z) = \begin{cases} L(\frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2}}), & \text{for } z < a_1a_2; \\ L(\frac{1}{2}(\frac{a_2}{c_2} - \frac{a_1}{c_1}) - \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2}}), & \text{for } z \in [a_1a_2, 0]; \\ L(-\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}), & \text{for } z > 0 \end{cases} .$$

Example

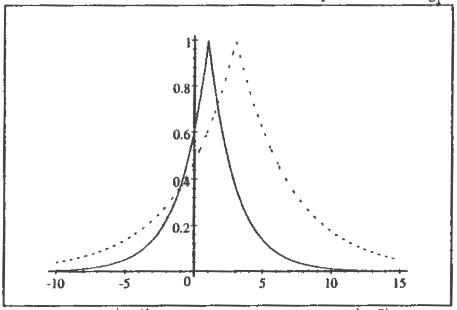
Let us take $a_1 = 1, c_1 = 2, a_2 = 3, c_2 = 4, \frac{a_1}{c_1} = \frac{1}{2} < \frac{a_2}{c_2} = \frac{3}{4}$,

$L(x) = e^{-x}$,

$\mu_{A_1}(z) = L(\frac{|z-a_1|}{c_1}), \mu_{A_2}(z) = L(\frac{|z-a_2|}{c_2})$,

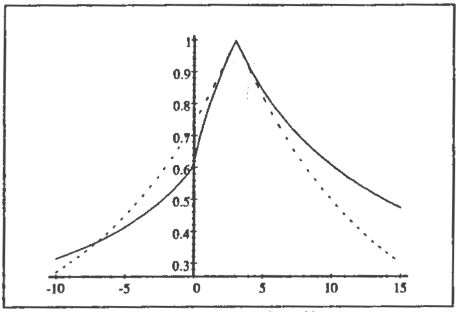
$$\mu_{A_1A_2}(z) = \begin{cases} L(\frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2}}) & \text{, for } \quad z < 0 \\ L(\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) - \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}) & \text{, for } \quad 0 \le z \le a_1a_2 \\ L(-\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}}) & \text{, for } \quad a_1a_2 < z \end{cases}$$

First we plot the functions $\mu_{A_1}(z) = L(\frac{|z-a_1|}{c_1}), \mu_{A_2}(z) = L(\frac{|z-a_2|}{c_2})$.



$\mu_{A_1}(z) = e^{-(\frac{|z-1|}{2})}$(solid), $\mu_{A_2}(z) = e^{-(\frac{|z-3|}{4})}$(dots)

The next plot shows the membership function $\mu_{A_1A_2}(z)$ and its approximation $L(\frac{|z-(a_1a_2)|}{a_1c_2+a_2c_1})$



$\mu_{A_1A_2}(z)$(solid) $L(\frac{|z-(a_1a_2)|}{a_1c_2+a_2c_1})$(dots)

3.Possibility index

We will consider the possibility index for the relation between two fuzzy numbers: $Pos(A_1 = A_2) = \sup_{z\in R}\min\{\mu_{A_1}(z), \mu_{A_2}(z)\}$. In the following theorem we calculate $Pos(A_3 = A_1A_2)$.

Theorem

Assume that $\mu_{A_1}(z) = L(\frac{|z-a_1|}{c_1}), \mu_{A_2}(z) = L(\frac{|z-a_2|}{c_2}), \mu_{A_3}(z) = L(\frac{|z-a_3|}{c_3})$, where $a_1, a_2 > 0$ and $\frac{a_1}{c_1} \leq \frac{a_2}{c_2}$. Then

$Pos(A_3 = A_1A_2) =$

$$= \begin{cases} L(g_1(a_1, c_1, a_2, c_2, a_3, c_3)) & \text{if} \quad a_1a_2 < a_3 \\ L(g_2(a_1, c_1, a_2, c_2, a_3, c_3)) & \text{if} \quad (0 \leq a_3 \leq a_1a_2) \text{ or } (a_3 < 0, \frac{|a_3|}{c_3} \leq \frac{a_1}{c_1}) \\ L(g_3(a_1, c_1, a_2, c_2, a_3, c_3)) & \text{if} \quad a_3 < 0, \frac{|a_3|}{c_3} > \frac{a_1}{c_1} \end{cases}$$

,where

$g_1(a_1, c_1, a_2, c_2, a_3, c_3) = \frac{1}{2}(-\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2} + \sqrt{(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2})^2 + 4\frac{a_3-a_1a_2}{c_1c_2}}),$

$g_2(a_1, c_1, a_2, c_2, a_3, c_3) = \frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2} - \sqrt{(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2})^2 + 4\frac{a_3-a_1a_2}{c_1c_2}}),$

$g_3(a_1, c_1, a_2, c_2, a_3, c_3) = \frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2} + \sqrt{(\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2})^2 + 4\frac{a_1a_2-a_3}{c_1c_2}}).$

Proof.

Let g be a function as follows

$$g(z) = \begin{cases} \frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2}} & \text{, for} \quad z < 0 \\ \frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) - \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}} & \text{, for} \quad 0 \leq z \leq a_1a_2 \\ -\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}} & \text{, for} \quad a_1a_2 < z \end{cases}$$

We have that

$$Pos(A_3 = A_1A_2) = \sup_{z\in R}\min\{\mu_{A_3}(z), \mu_{A_1A_2}(z)\} =$$

$$\sup_{z\in R}\min\{L(\frac{|z-a_3|}{c_3}), L(g(z))\} = \sup_{z\in R} L(\{\max\{\frac{|z-a_3|}{c_3}, g(z)\}) =$$

$$= L(\inf_{z\in R}\max\{\frac{|z-a_3|}{c_3}, g(z)\}).$$

To calculate $\inf_{z\in R}\max\{\frac{|z-a_3|}{c_3}, g(z)\}$ we assume first that $a_1a_2 < a_3$. We obtain the optimal point z from the equation

$$-\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}} = \frac{a_3 - z}{c_3}.$$

Thus we have that

$\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2} = (\frac{a_3-z}{c_3} + \frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}))^2;$

$\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2} = (\frac{a_3-z}{c_3})^2 + \frac{a_3-z}{c_3}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \frac{1}{4}(\frac{a_1}{c_1} + \frac{a_2}{c_2})^2;$

$0 = \frac{(a_3-z)^2}{c_3^2} + \frac{a_3-z}{c_3}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) + \frac{1}{4}(\frac{a_1}{c_1} + \frac{a_2}{c_2})^2 - \frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2} + \frac{a_3}{c_1c_2} - \frac{a_3}{c_1c_2};$

$0 = (\frac{a_3-z}{c_3})^2 + (\frac{a_3-z}{c_3})(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2}) - \frac{a_3-a_1a_2}{c_1c_2};$

$\Delta = (\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2})^2 + 4\frac{a_3-a_1a_2}{c_1c_2};$

Hence $\frac{a_3-z}{c_3} = \frac{1}{2}(-\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2} + \sqrt{(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2})^2 + 4\frac{a_3-a_1a_2}{c_1c_2}}).$

Assume now that $0 \le a_3 \le a_1a_2$ or ($a_3 < 0$ and $\frac{|a_3|}{c_3} \le \frac{a_1}{c_1}$). We obtain the optimal point z from the equation $\frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2}) - \sqrt{\frac{1}{4}(\frac{a_2}{c_2} - \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2}} = \frac{z-a_3}{c_3}$. Thus we have the same equation. This time we have that

$\frac{z-a_3}{c_3} = \frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2} - \sqrt{(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2})^2 + 4\frac{a_3-a_1a_2}{c_1c_2}}).$

Assume now that $a_3 < 0$ and $\frac{|a_3|}{c_3} > \frac{a_1}{c_1}$. We obtain the optimal point z from the equation $\frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \sqrt{\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2}} = \frac{z-a_3}{c_3}$. Thus we have that $\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2} = (\frac{z-a_3}{c_3} - \frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2}))^2;$

$\frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 - \frac{z}{c_1c_2} = (\frac{z-a_3}{c_3})^2 - \frac{z-a_3}{c_3}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \frac{1}{4}(\frac{a_1}{c_1} - \frac{a_2}{c_2})^2;$

$0 = \frac{(z-a_3)^2}{c_3^2} - \frac{z-a_3}{c_3}(\frac{a_1}{c_1} - \frac{a_2}{c_2}) + \frac{1}{4}(\frac{a_1}{c_1} - \frac{a_2}{c_2})^2 - \frac{1}{4}(\frac{a_2}{c_2} + \frac{a_1}{c_1})^2 + \frac{z}{c_1c_2} - \frac{a_3}{c_1c_2} + \frac{a_3}{c_1c_2};$

$0 = \frac{(z-a_3)^2}{c_3^2} - \frac{(z-a_3)}{c_3}(\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2}) + \frac{a_3-a_1a_2}{c_1c_2};$

$\Delta = (\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2})^2 + 4\frac{a_1a_2-a_3}{c_1c_2};$

Hence $\frac{z-a_3}{c_3} = \frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2} + \sqrt{(\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2})^2 + 4\frac{a_1a_2-a_3}{c_1c_2}}).$
Finally we can conclude that
$\mathrm{Pos}(A_3 = A_1A_2) = \ L(\inf_{z\in R}\max\{g(z), \frac{|z-a_3|}{c_3}\}) =$

$$= \begin{cases} L(g_1(a_1,c_1,a_2,c_2,a_3,c_3)) & \text{, for } \quad a_1a_2 < a_3 \\ L(g_2(a_1,c_1,a_2,c_2,a_3,c_3)) & \text{, for } \quad (0 \leq a_3 \leq a_1a_2) \text{ or } (a_3 < 0, \frac{|a_3|}{c_3} \leq \frac{a_1}{c_1}) \\ L(g_3(a_1,c_1,a_2,c_2,a_3,c_3)) & \text{, for } \quad a_3 < 0, \frac{|a_3|}{c_3} > \frac{a_1}{c_1} \end{cases}$$

,where
$g_1(a_1,c_1,a_2,c_2,a_3,c_3) = \frac{1}{2}(-\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2} + \sqrt{(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2})^2 + 4\frac{a_3-a_1a_2}{c_1c_2}}),$
$g_2(a_1,c_1,a_2,c_2,a_3,c_3) = \frac{1}{2}(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2} - \sqrt{(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{c_3}{c_1c_2})^2 + 4\frac{a_3-a_1a_2}{c_1c_2}}),$
$g_3(a_1,c_1,a_2,c_2,a_3,c_3) = \frac{1}{2}(\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2} + \sqrt{(\frac{a_1}{c_1} - \frac{a_2}{c_2} - \frac{c_3}{c_1c_2})^2 + 4\frac{a_1a_2-a_3}{c_1c_2}}).$
What ends the proof.

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