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CONTROL PROBLEM FOR A NONLINEAR THERMOELASTICITY SYSTEM

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Abstract

The control problem for a three–dimensional nonlinear thermoelasticity system is considered. The system may represent, among others, the dynamical model of shape memory materials. As controls we take distributed heat sources and body forces. The goal functional refers to the desired evolution of displacement, strain and temperature.

The continuity and differentiability of solutions with respect to controls is studied. The existence of optimal controls is proved and the necessary optimality conditions are formulated. The existence of adjoint state variables is proved under additional regularity of data.

Keywords: *nonlinear thermoelasticity, stability, control, shape memory materials.*

1 Introduction

The main objective of the paper consists in proving the existence and characterizing the control laws for optimization problems concerning fairly general nonlinear three–dimensional (3–D) thermoelastic systems. The main representative of such systems describes the behaviour of shape memory materials (SMM) and its study was the primary motivation of this work.

The shape memory materials have a peculiar property that their free energy functions posess, depending on temperature, variable number of stable minima in terms of strain. Above certain temperature there is only one minimum corresponding to the strain–free state, and below it the minima occur also for several nonzero strains.

Thus, at a temperature below critical, an external force may cause shift of the state from the strain–free configuration to another stable shape, and the subsequent heating causes the appearence of elastic forces striving to restore the initial configuration. This property, known as shape memory effect, is a consequence of structural phase transitions between low–temperature martensitic phases and high–temperature austenitic phase. It is used in many applications, see e.g. [6],[11].

As we see, the choice of control variables is natural, namely the intensity and location of external heat sources and forces. The goal functional should refer to a desired evolution of a structure made of SMM. Therefore it can depend in particular on the variable configuration (displacement) and strain, which in turn is related to the material phases, as well as on temperature distribution.

The generality of the problem statement is due to the fact that the system under consideration expresses balance laws of linear momentum and energy with constitutive relations characteristic for a broad class of materials. In particular, we admit governing elastic energy function corresponding to several types of SMM models, like 3-D Falk–Konopka model for metallic alloys [5] and 3–D "averaged" model for a polymer material [20].

In 1–D case the problem is identical to the well–known Falk's model for martensitic phase transitions of the shear type [4].

Questions related to thermodynamical background of thermoelastic systems under consideration, the existence and uniqueness of global in time solutions have been addressed in the previous papers [14],[15],[16]. Here we study the stability and differentiability properties of these solutions with respect to control variables. Furthermore, we prove the existence result for an optimal control problem and formulate the neccessary optimality conditions. We note that our analysis of the stability and differentiability properties is based on the technique developed in [16] for the global in time existence.

Similar control problems but for special kinds of 2–D systems, namely plates activated by shape memory reinforcements, have been treated in [8],[9],[20].
For control problems related to 1–D Falk's model we refer e.g. to [1],[2],[7],[17], [19].

2 State equations

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded domain with a smooth boundary $\partial\Omega$, occupied by an elastic body in a reference configuration. Let also $I = (0,T)$, $Q_t = (0,t) \times \Omega$, $\Omega_t = \{t\} \times \Omega$, $S_t = (0,t) \times \partial\Omega$, and n stands for the unit outward normal to $\partial\Omega$.
Let $\mathrm{u} : Q_T \to \mathbb{R}^n$ be the displacement vector, and $\theta : Q_T \to \mathbb{R}_+$ the absolute temperature. We denote by $\epsilon = (\epsilon_{ij})$, with $\epsilon_{ij}(\mathrm{u}) = \frac{1}{2}(u_{i/j} + u_{j/i})$, the linearized strain tensor, and by $\epsilon_t = \epsilon(\mathrm{u}_t)$ the strain rate tensor.
Throughout the paper we use the notation $f_{/i} = \partial f/\partial x_i$, $f_t = \partial f/\partial t$.

The state equations to be considered express balances of linear momentum and energy which, under simplifying assumption of constant material density $\rho \equiv 1$, are given by

$$\mathrm{u}_{tt} - \nu \mathrm{Q}\mathrm{u}_t + \frac{\kappa}{4}\mathrm{Q}\mathrm{Q}\mathrm{u} = \nabla \cdot F_{/\epsilon}(\epsilon,\theta) + \mathrm{b}, \tag{2.1}$$

$$c(\epsilon,\theta)\theta_t - k\Delta\theta = \theta F_{/\theta\epsilon}(\epsilon,\theta) : \epsilon_t + \nu(\mathrm{A}\epsilon_t) : \epsilon_t + g \quad \text{in} \quad Q_T, \tag{2.2}$$

with initial

$$\mathrm{u}(0,\mathrm{x}) = \mathrm{u}_0(\mathrm{x}), \quad \mathrm{u}_t(0,\mathrm{x}) = \mathrm{u}_1(\mathrm{x}), \tag{2.3}$$

$$\theta(0,\mathrm{x}) = \theta_0(\mathrm{x}) \quad \text{in} \quad \Omega, \tag{2.4}$$

and boundary conditions

$$\mathbf{u} = 0, \quad \mathbf{Q}\mathbf{u} = 0, \tag{2.5}$$

$$\nabla\theta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T, \tag{2.6}$$

where

$$c(\epsilon, \theta) = c_v - \theta F_{/\theta\theta}(\epsilon, \theta). \tag{2.7}$$

We shall refer to (2.1)–(2.7) as problem (P).

The quantities in (P) have the following meaning: $F(\epsilon, \theta)$ – elastic energy, $c(\epsilon, \theta)$ – specific heat coefficient, c_v, k, ν and κ – positive constants corresponding to thermal specific heat, heat conductivity, viscosity and interface energy.

The vector $\mathbf{b}$ is a distributed external force and g a distributed heat source which represent possible mechanical and thermal controls.

The linear map

$$\mathbf{u} \mapsto \mathbf{A}\epsilon(\mathbf{u}) = \lambda \operatorname{trace} \epsilon(\mathbf{u}) \mathbf{I} + 2\mu\epsilon(\mathbf{u}), \tag{2.8}$$

where λ, μ are the Lamé constants and $\mathbf{I} = (\delta_{ij})$ is the unit matrix, represents Hooke's law for the homogeneous isotropic material. Here $\mathbf{A} = (A_{ijkl})$ with

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

is the fourth order elasticity tensor.

The second order differential operator $\mathbf{Q}$ defined by

$$\mathbf{u} \mapsto \mathbf{Q}\mathbf{u} = \nabla \cdot (\mathbf{A}\epsilon(\mathbf{u})) = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u})$$

is known as operator of linearized elasticity.

In the divergence operator $\nabla\cdot$ we use the convention of the contraction over the last index, i.e.,

$$\nabla \cdot (\mathbf{A}\epsilon(\mathbf{u})) = \partial_j(\, A_{ijkl}\epsilon_{kl}(\mathbf{u})\,) = A_{ijkl}\partial_j\epsilon_{kl}(\mathbf{u}) = \mathbf{A}\nabla\epsilon(\mathbf{u}).$$

Moreover, the summation convention over repeated indices is used, and the following notation: for vectors $\mathbf{a} = (a_i)$, $\mathbf{b} = (b_i)$ and tensors $\mathbf{B} = (B_{ij})$, $\mathbf{C} = (C_{ij})$, $\mathbf{A} = (A_{ijkl})$ we write $\mathbf{a} \cdot \mathbf{b} = a_i b_i$, $\mathbf{B} : \mathbf{C} = B_{ij}C_{ij}$, $\mathbf{aB} = (a_i B_{ij})$, $\mathbf{Ba} = (B_{ij}a_j)$, $\mathbf{BA} = (B_{ij}A_{ijkl})$, etc.

Problem (P) is associated with the free energy functional of the Ginzburg–Landau form

$$f(\epsilon(\mathbf{u}), \nabla\epsilon(\mathbf{u}), \theta) = -c_v\theta \log\theta + F(\epsilon(\mathbf{u}), \theta) + \frac{\kappa}{8} \mid \mathbf{Q}\mathbf{u} \mid^2 \tag{2.9}$$

with the three terms representing thermal, elastic and interfacial energy.

The main characteristic feature of (2.9) as a model of shape memory materials is the nonlinearity of the elastic energy. Namely, $F(\epsilon, \theta)$ is a multiple–well in ϵ with the shape changing qualitatively with θ. The second characteristic feature is the presence of strain-gradient term which accounts for interaction effects on phase interfaces.

A typical example of the elastic energy is the Falk–Konopka model [5] in the form of sixth order polynomial in terms of ϵ_{ij}:

$$F(\epsilon, \theta) = \sum_{i=1}^{3} F_i^2(\theta)J_i^2(\epsilon) + \sum_{i=1}^{5} F_i^4(\theta)J_i^4(\epsilon) + \sum_{i=1}^{2} F_i^6(\theta)J_i^6(\epsilon), \tag{2.10}$$

where $J_i^k(\epsilon)$, $i = 1, \dots, i^k$, are k-th order crystallographical invariants, that is appropriate combinations of the strain tensor components ϵ_{ij}, and

$$F_i^2(\theta) = \alpha_i^2(\theta - \theta_c), \qquad F_i^4(\theta) = \alpha_i^4(\theta - \theta_c), \qquad F_i^6(\theta) = \alpha_i^6,$$

with constant parameters α_i^k, θ_c.

The form (2.10) represents a generalization of the well known 1–D Landau–Devonshire energy proposed for shape memory alloys by Falk [4],

$$F(\epsilon, \theta) = \alpha_1(\theta - \theta_c)\epsilon^2 - \alpha_2\epsilon^4 + \alpha_3\epsilon^6,$$

where $\alpha_i > 0$ are constant parameters, and $\theta_c > 0$ is a critical temperature. Our formulation (2.1)–(2.7) constitutes an analog of 1–D dynamical Falk's model [4].

The problem (P) is studied under several conditions concerning data and constitutive functions. We assume that

(A1) Domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with the boundary $\partial\Omega$ of the class C^3.

(A2) The coefficients of the operator **Q** satisfy conditions

$$\mu > 0, \qquad n\lambda + 2\mu > 0.$$

This ensures the following properties:

(i) Coercivity and boundedness of the algebraic operator **A**, that is

$$a_*|\epsilon|^2 \leq (\mathbf{A}\epsilon) : \epsilon \leq a^*|\epsilon|^2,$$

where

$$a_* = \min[n\lambda + 2\mu, 2\mu], \qquad a^* = \max[n\lambda + 2\mu, 2\mu].$$

(ii) Strong ellipticity of the operator **Q** (see Section 7.2 [16]). This, due to Nečas [13], implies the estimate:

$$c\|\mathbf{u}\|_{\mathbf{W}_2^2(\Omega)} \leq \|\mathbf{Q}\mathbf{u}\|_{\mathbf{L}_2(\Omega)} \qquad \text{for all} \quad \mathbf{u} \in \{\mathbf{u} \in \mathbf{W}_2^2(\Omega) \mid \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega\}.$$

(iii) Parabolicity in general Solonnikov sense of systems (2.11), (2.12) (see [16] Sect. 7).

(A3) The function $F(\epsilon, \theta)$ is of class C^3 on $S^2 \times [0, \infty)$, where S^2 denotes the set of symmetric tensors of second order in $\mathbb{R}^n$. We assume the splitting

$$F(\epsilon, \theta) = F_1(\epsilon, \theta) + F_2(\epsilon),$$

where $F_1(\epsilon, \theta)$ is a concave function with respect to θ,

$$F_{1/\theta\theta}(\epsilon, \theta) \leq 0 \quad \text{for} \quad (\epsilon, \theta) \in S^2 \times [0, \infty),$$

such that $F_1(\epsilon, \theta)$ is linear in θ over a certain interval $[0, \theta_1)$, $\theta_1 = const$, and has the polynomial growth θ^r for $\theta \geq \theta_1$.

(A4) Growth conditions: There exists a positive constant Λ such that for $\theta \geq \theta_1$ and large values of ϵ_{ij} the following conditions are satisfied:

$$|F_{1/\epsilon\epsilon}(\epsilon,\theta)| \leq \Lambda\theta^r|\epsilon|^{q-1}, \qquad |F_{2/\epsilon\epsilon}(\epsilon)| \leq \Lambda|\epsilon|^{\bar{q}-1},$$
$$|F_{1/\epsilon\theta}(\epsilon,\theta)| \leq \Lambda\theta^{r-1}|\epsilon|^{q}, \qquad |F_{1/\theta\theta}(\epsilon,\theta)| \leq \Lambda\theta^{r-2}|\epsilon|^{q+1},$$
$$|F_{1/\epsilon}(\epsilon,\theta)| \leq \Lambda\theta^r|\epsilon|^{q}, \qquad |F_{2/\epsilon}(\epsilon)| \leq \Lambda|\epsilon|^{\bar{q}},$$

with

$$0 < r < \frac{1}{2}, \qquad 1 < \bar{q} \leq \frac{q_n p_n}{4n}, \qquad 0 < q \leq (\bar{q}+1)\left(\frac{1}{2} - r\right),$$

where $p_n = n+2$, and q_n is the Sobolev exponent for which the imbedding of $W_2^1(\Omega)$ into $L_{q_n}(\Omega)$ is continuous, that is, $q_n = 2n/(n-2)$ for $n \geq 3$ and q_n is any finite number for $n = 2$. We note that

$$0 < q \leq \frac{q_n p_n}{2n}\left(\frac{1}{2} - r\right).$$

The above conditions imply the following growth of $F(\epsilon,\theta)$:

$$|F_1(\epsilon,\theta)| \leq \Lambda + \Lambda\theta^r|\epsilon|^{q+1}, \qquad |F_2(\epsilon)| \leq \Lambda + \Lambda|\epsilon|^{\bar{q}+1}.$$

We add some comments on the above conditions. The restrictions concern θ–growth exponent of F_1, ϵ–growth exponent of F_2 and the condition relating ϵ–growth of F_1 with its θ–growth and ϵ–growth of F_2.

The most restrictive is the condition $r < 1/2$, and $\bar{q} \leq 5/2$ in 3–D. In 2–D, since q_n is any finite number, arbitrary polynomial growth is admissible.
In particular, in 3–D the above conditions are satisfied for

$$\bar{q} = \frac{5}{2}, \quad q = 1, \quad r = \frac{3}{14}.$$

Moreover, we assume the structural lower bound for the part $F_2(\epsilon)$ of the free energy.

(A5) There exist positive constants c, Λ such that

$$c|\epsilon|^{\bar{q}+1} - \Lambda \leq F_2(\epsilon).$$

This is satisfied by the model example (2.10) with the growth restriction (A4).

The next assumption concerns the structural simplification of the energy equation by neglecting the nonlinear elastic contribution $-\theta F_{1/\theta\theta}(\epsilon,\theta)$ in the specific heat coefficient. This allows to apply the classical parabolic theory in the existence proof.

We point out that because of the applied technique we were unable either to allow $F_1(\epsilon,\theta)$ linear in θ or, assuming θ–growth condition, to incorporate the arising nonlinearity in the specific heat coefficient.

(A6) The elastic energy contribution $-\theta F_{1/\theta\theta}(\epsilon,\theta)$ to the specific heat coefficient due to the nonlinearity of F_1 in θ is neglected, that is, we set

$$c(\epsilon,\theta) = c_v = const > 0.$$

We are looking for the solution in the anisotropic Sobolev space

$$V(p) = \{ (\mathbf{u}, \theta) \in \mathbf{W}_p^{4,2}(Q_T) \times W_p^{2,1}(Q_T) \},$$

with a parameter p related to L_p–integrability. The assumptions on the initial data and the source terms correspond to this space.

(A7) The initial conditions satisfy for $1 < p < \infty$ the inclusions

$$\mathbf{u}_0 \in \mathbf{W}_p^{4-2/p}(\Omega), \quad \mathbf{u}_1 \in \mathbf{W}_p^{2-2/p}(\Omega),$$

$$0 \le \theta_0 \in W_p^{2-2/p}(\Omega),$$

and the compatibility relations. The source terms satisfy

$$\mathbf{b} \in \mathbf{L}_p(Q_T), \quad g \in L_p(Q_T), \quad g \ge 0 \text{ a.e. in } Q_T.$$

We recall here the existence and uniqueness results for problem (P) proved in [16].

Theorem 2.1 *Under assumptions (A1) – (A7) and the condition*

$$0 < \sqrt{\kappa} \le \nu,$$

there exists for $p_n \le p < \infty$ a solution $(\mathbf{u}, \theta) \in V(p)$ to problem (P) for any $T > 0$. Moreover, $\theta \ge 0$ in Q_T, and the following a priori estimates hold:

$$\| \mathbf{u} \|_{\mathbf{W}_p^{4,2}(Q_T)} \le \Lambda, \quad \| \theta \|_{W_p^{2,1}(Q_T)} \le \Lambda$$

with a constant Λ depending on the data of the problem, Ω and time T.

We note some properties of the solution which follow directly from the classical imbeddings.

Corollary 2.1 *For a solution to problem (P) the following holds:* $\mathbf{u}$, $\nabla\mathbf{u}$, $\nabla^2\mathbf{u}$, $\mathbf{u}_t$, θ *are Hölder continuous in Q_T, $\nabla^3\mathbf{u}, \nabla\mathbf{u}_t, \nabla\theta \in \mathbf{L}_p(Q_T)$, $p_n \le p < \infty$, and*

$$|\mathbf{u}|, \ |\nabla\mathbf{u}|, \ |\nabla^2\mathbf{u}|, \ |\mathbf{u}_t| \ \le \Lambda, \quad 0 \le \theta \le \Lambda \quad \text{in } Q_T,$$

$$\| \nabla^3\mathbf{u} \|_{\mathbf{L}_p(Q_T)}, \ \| \nabla\mathbf{u}_t \|_{\mathbf{L}_p(Q_T)}, \ \| \nabla\theta \|_{\mathbf{L}_p(Q_T)} \le \Lambda.$$

The proof of uniqueness requires the continuity of $\nabla\mathbf{u}_t$ in Q_T, which holds provided $p > p_n$.

Theorem 2.2 *Let the assumptions of Theorem 2.1 be satisfied for*

$$p_n < p < \infty.$$

Then the solution to the problem (P) is unique for any $T > 0$.

We collect now a priori bounds which follow from the imbeddings.

Corollary 2.2 *The solution to problem (P) has in case $p_n < p < \infty$ the following properties :* $\nabla^3\mathbf{u}$, $\nabla\mathbf{u}_t$, $\nabla\theta$ *are Hölder continuous in Q_T and satisfy the bounds*

$$|\nabla^3\mathbf{u}|, \ |\nabla\mathbf{u}_t|, \ |\nabla\theta| \le \Lambda \quad \text{in } Q_T.$$

The existence proof in [16] is based on the parabolic decomposition (see [20]) of the problem (P). The same decomposition is used here for the proof of the stability and differentiability results. Choosing numbers α, β so that

$$\alpha + \beta = \nu, \qquad \alpha\beta = \frac{\kappa}{4},$$

the system (2.1) with initial conditions (2.3) and boundary conditions (2.5) is equivalent to the following two sets of BVP's for a vector field $\mathbf{w}$:

$$\begin{aligned} \mathbf{w}_t - \beta \mathbf{Q}\mathbf{w} &= \nabla \cdot F_{/\epsilon}(\epsilon, \theta) + \mathbf{b}, \quad \text{in } Q_T, \\ \mathbf{w}(0, \mathbf{x}) &= \mathbf{u}_1(\mathbf{x}) - \alpha \mathbf{Q}\mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \\ \mathbf{w} &= 0 \quad \text{on } S_T, \end{aligned} \tag{2.11}$$

and the displacement $\mathbf{u}$:

$$\begin{aligned} \mathbf{u}_t - \alpha \mathbf{Q}\mathbf{u} &= \mathbf{w}, \quad \text{in } Q_T, \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } S_T, \end{aligned} \tag{2.12}$$

The condition between parameters κ and ν, required by Theorem 2.1, assures that $\alpha,\ \beta > 0$.

3 Stability

In this section we prove the stability of solutions $(\mathbf{u}, \theta)$ of problem (P) with respect to control parameters $(\mathbf{b}, g)$. Let $(\mathbf{u}^1, \theta^1)$ and $(\mathbf{u}^2, \theta^2)$ be the solutions corresponding to $(\mathbf{b}^1, g^1)$ and $(\mathbf{b}^2, g^2)$, respectively. We have the following

Theorem 3.1 *Under the assumptions of Theorem 2.2 the solutions $(\mathbf{u}^i, \theta^i)$ satisfy the inequality*

$$||(\mathbf{u}^2 - \mathbf{u}^1, \theta^2 - \theta^1)||_{V(p)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)}) \quad \text{for } p_n < p < \infty, \tag{3.1}$$

where Λ is a constant depending on the data of the problem, Ω and time T.

Proof. To simplify notation we set

$$\begin{aligned} &\mathbf{v} = \mathbf{u}^2 - \mathbf{u}^1 \qquad && \eta = \theta^2 - \theta^1 \qquad && \epsilon^i = \epsilon(\mathbf{u}^i) \\ &\epsilon^i_t = \epsilon(\mathbf{u}^i_t) && F^i_{/\epsilon} = F_{/\epsilon}(\epsilon^i, \theta^i) && F^i_{/\theta\epsilon} = F_{/\theta\epsilon}(\epsilon^i, \theta^i). \end{aligned}$$

The difference $(\mathbf{v}, \eta)$ satisfies the following BVP:

$$\mathbf{v}_{tt} - \nu \mathbf{Q}\mathbf{v}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{v} = \nabla \cdot (F^2_{/\epsilon} - F^1_{/\epsilon}) + \mathbf{b}^2 - \mathbf{b}^1, \tag{3.2}$$

$$\begin{aligned} c_v \eta_t - k\Delta\eta &= \theta^2 F^2_{/\theta\epsilon} : \epsilon^2_t - \theta^1 F^1_{/\theta\epsilon} : \epsilon^1_t \\ &\quad + \nu(\mathbf{A}\epsilon^2_t) : \epsilon^2_t - \nu(\mathbf{A}\epsilon^1_t) : \epsilon^1_t \\ &\quad + g^2 - g^1 \\ &=: R_1 + R_2 + R_3 \quad \text{in } Q_T, \end{aligned} \tag{3.3}$$

$$\mathbf{v}(0, x) = 0, \quad \mathbf{v}_t(0, x) = 0, \quad \eta(0, x) = 0 \quad \text{in } \Omega, \tag{3.4}$$

$$\mathbf{v} = \mathbf{Q}\mathbf{v} = 0, \quad \nabla\eta \cdot \mathbf{n} = 0 \quad \text{on } S_T. \tag{3.5}$$

In the first step we obtain energy estimates for $\mathbf{v}$. To this purpose we multiply (3.2) by $\mathbf{v}_t$ and integrate over Q_t to get

$$\frac{1}{2}\int_{Q_t}\frac{d}{dt}\mid \mathbf{v}_t\mid^2\,dxdt' - \nu\int_{Q_t}(\mathbf{Q}\mathbf{v}_t)\cdot\mathbf{v}_t\,dxdt' + \frac{\kappa}{4}\int_{Q_t}(\mathbf{Q}\mathbf{Q}\mathbf{v})\cdot\mathbf{v}_t\,dxdt'$$
$$= \int_{Q_t}(\nabla\cdot(F^2_{/\epsilon} - F^1_{/\epsilon}))\cdot\mathbf{v}_t\,dxdt' + \int_{Q_t}(\mathbf{b}^2 - \mathbf{b}^1)\cdot\mathbf{v}_t\,dxdt'. \quad (3.6)$$

Integrating by parts the second integral gives

$$-\nu\int_{Q_t}(\mathbf{Q}\mathbf{v}_t)\cdot\mathbf{v}_t\,dxdt' = \nu\int_{Q_t}(\mathbf{A}\epsilon(\mathbf{v}_t)) : \epsilon(\mathbf{v}_t)\,dxdt'. \quad (3.7)$$

Similarly, for the third integral, after applying twice integration by parts and using symmetry property for $\mathbf{A}$, we get

$$\frac{\kappa}{4}\int_{Q_t}(\mathbf{Q}\mathbf{Q}\mathbf{v})\cdot\mathbf{v}_t\,dxdt' = -\frac{\kappa}{4}\int_{Q_t}(\mathbf{A}\epsilon(\mathbf{Q}\mathbf{v})) : \epsilon(\mathbf{v}_t)\,dxdt'$$
$$= -\frac{\kappa}{4}\int_{Q_t}\epsilon(\mathbf{Q}\mathbf{v}) : (\mathbf{A}\epsilon(\mathbf{v}_t))\,dxdt'$$
$$= \frac{\kappa}{4}\int_{Q_t}(\mathbf{Q}\mathbf{v}) : (\mathbf{Q}\mathbf{v}_t)\,dxdt' = \frac{\kappa}{8}\int_{Q_t}\frac{d}{dt}\mid \mathbf{Q}\mathbf{v}\mid^2\,dxdt'. \quad (3.8)$$

Finally, after integrating by parts, the fourth integral in (3.6) is

$$\int_{Q_t}(\nabla\cdot(F^2_{/\epsilon} - F^1_{/\epsilon}))\cdot\mathbf{v}_t\,dxdt' = -\int_{Q_t}(F^2_{/\epsilon} - F^1_{/\epsilon}) : \epsilon(\mathbf{v}_t)\,dxdt'. \quad (3.9)$$

Combining (3.6)–(3.9) and using initial conditions (3.4) yields

$$\int_{\Omega_t}\left(\frac{1}{2}\mid \mathbf{v}_t\mid^2 + \frac{\kappa}{8}\mid \mathbf{Q}\mathbf{v}\mid^2\right)dx + \nu\int_{Q_t}(\mathbf{A}\epsilon(\mathbf{v}_t)) : \epsilon(\mathbf{v}_t)\,dxdt'$$
$$= -\int_{Q_t}(F^2_{/\epsilon} - F^1_{/\epsilon}) : \epsilon(\mathbf{v}_t)\,dxdt' + \int_{Q_t}(\mathbf{b}^2 - \mathbf{b}^1)\cdot\mathbf{v}_t\,dxdt'. \quad (3.10)$$

Moreover, in view of (3.4), we have

$$\frac{1}{2}\int_{\Omega_t}|\epsilon(\mathbf{v})|^2\,dx = \frac{1}{2}\int_{Q_t}\frac{d}{dt}|\epsilon(\mathbf{v})|^2\,dxdt' = \int_{Q_t}\epsilon(\mathbf{v}) : \epsilon(\mathbf{v}_t)\,dxdt'. \quad (3.11)$$

Adding (3.10) and (3.11) and using estimate

$$\mid F^2_{/\epsilon} - F^1_{/\epsilon}\mid \leq \Lambda(\mid \epsilon(\mathbf{v})\mid + \mid \eta\mid), \quad (3.12)$$

which follows from the regularity assumption for F and the uniform bounds on ϵ^i, θ^i in Q_T, by Young's inequality we arrive at

$$\int_{\Omega_t}(\frac{1}{2}|\mathbf{v}_t|^2 + |\epsilon(\mathbf{v})|^2 + \frac{\kappa}{8}|\mathbf{Q}\mathbf{v}|^2)\,dx + a_*\int_{Q_t}|\epsilon(\mathbf{v}_t)|^2\,dxdt'$$
$$\leq \delta\int_{Q_t}|\epsilon(\mathbf{v}_t)|^2\,dxdt' + \Lambda c(\delta)\int_{Q_t}(|\epsilon(\mathbf{v})|^2 + |\eta|^2)\,dxdt'$$
$$+ \frac{1}{2}\int_{Q_t}|\mathbf{v}_t|^2\,dxdt' + \frac{1}{2}\int_{Q_t}|\mathbf{b}^2 - \mathbf{b}^1|^2\,dxdt'.$$

Choosing $\delta = a_*/2$, the use of Gronwall's inequality implies

$$||\mathbf{v}_t||_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + ||\epsilon(\mathbf{v})||_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + ||\mathbf{Q}\mathbf{v}||_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + ||\epsilon(\mathbf{v}_t)||_{\mathbf{L}_2(Q_T)} \\ \leq \Lambda(||\eta||_{L_2(Q_T)} + ||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_2(Q_T)}). \quad (3.13)$$

Hence, recalling the ellipticity property of the operator $\mathbf{Q}$,

$$||\mathbf{v}||_{L_\infty(0,T;\mathbf{W}_2^2(\Omega))} \leq \Lambda(||\eta||_{L_2(Q_T)} + ||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_2(Q_T)}).$$

The energy estimates for η follow by multiplying equation (3.3) by η and integrating over Q_t:

$$\frac{c_v}{2}\int_{\Omega_t} \eta^2\,dx + k\int_{Q_t} |\nabla\eta|\,dxdt' = \sum_{i=1}^{3}\int_{Q_t} R_i\eta\,dxdt'. \quad (3.14)$$

In view of the estimate

$$|R_1| + |R_2| \leq \Lambda(|\eta| + |\epsilon(\mathbf{v})| + |\epsilon(\mathbf{v}_t)|), \quad (3.15)$$

which follows from uniform estimates on $\epsilon^i, \theta^i, \epsilon^i_t$ and using (3.13), we get

$$\int_{Q_t} (R_1\eta + R_2\eta + R_3\eta)\,dxdt' \leq \Lambda\int_{Q_t} (\eta^2 + |\mathbf{b}^2 - \mathbf{b}^1|^2 + |g^2 - g^1|^2)\,dxdt'.$$

Hence, by Gronwall's inequality,

$$||\eta||_{L_\infty(0,T;L_2(\Omega))} + ||\nabla\eta||_{\mathbf{L}_2(Q_T)} \leq \Lambda D(2), \quad (3.16)$$

where for simplicity we use the following abbreviation

$$D(p) = (||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)}).$$

Now, combining (3.16) and (3.13), yields

$$||\mathbf{v}_t||_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + ||\mathbf{v}||_{L_\infty(0,T;\mathbf{W}_2^2(\Omega))} + ||\epsilon(\mathbf{v}_t)||_{\mathbf{L}_2(Q_T)} + \\ + ||\eta||_{L_\infty(0,T;L_2(\Omega))} + ||\nabla\eta||_{\mathbf{L}_2(Q_T)} \leq \Lambda D(2). \quad (3.17)$$

We note the following consequences of (3.17). By the imbedding $W_2^1(\Omega) \subset L_{q_n}(\Omega)$,

$$||\epsilon(\mathbf{v})||_{L_\infty(0,T;\mathbf{L}_{q_n}(\Omega))} \leq \Lambda D(2), \quad (3.18)$$

and by the imbedding (see e.g. [3])

$$L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;W_2^1(\Omega)) \subset L_{2p_n/n}(Q_T), \qquad 2 < \frac{2p_n}{n} = \frac{2(n+2)}{n} < q_n,$$

we have

$$||\eta||_{L_{2p_n/n}(Q_T)} \leq D(2). \quad (3.19)$$

Now we make use of the parabolic decomposition of (3.2). Let $(\mathbf{w}^1, \mathbf{u}^1)$ and $(\mathbf{w}^2, \mathbf{u}^2)$ denote the corresponding solutions of the decomposed problems (2.11), (2.12), and let

$$\mathbf{y} = \mathbf{w}^2 - \mathbf{w}^1.$$

The functions $(\mathbf{y}, \mathbf{v})$ satisfy the following BVP's:

$$\begin{aligned} \mathbf{y}_t - \beta \mathbf{Q}\mathbf{y} &= \nabla \cdot (F^2_{/\epsilon} - F^1_{/\epsilon}) + \mathbf{b}^2 - \mathbf{b}^1 \quad \text{in} \quad Q_T, \\ \mathbf{y}(0,x) &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{y} &= 0 \quad \text{on} \quad S_T, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \mathbf{v}_t - \alpha \mathbf{Q}\mathbf{v} &= \mathbf{y} \quad \text{in} \quad Q_T, \\ \mathbf{v}(0,x) &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{v} &= 0 \quad \text{on} \quad S_T, \end{aligned} \tag{3.21}$$

Thanks to the regularity properties of parabolic systems (see [16], Lemma 7.2) the following estimate holds

$$||\mathbf{y}||_{\mathbf{W}^{1,1/2}_p(Q_T)} \leq \Lambda(||F^2_{/\epsilon} - F^1_{/\epsilon}||_{\mathbf{L}_p(Q_T)} + ||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)}) \quad \text{for} \quad 1 < p < \infty,$$

where Λ is a constant depending on $\Omega, \lambda + 2\mu, \mu, T$ and p. Hence, in view of (3.12),

$$||\mathbf{y}||_{\mathbf{W}^{1,1/2}_p(Q_T)} \leq \Lambda(||\epsilon(\mathbf{v})||_{\mathbf{L}_p(Q_T)} + ||\eta||_{L_p(Q_T)} + ||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)}) \quad \text{for} \quad 1 < p < \infty.$$

Consequently, using estimates (3.18) and (3.19) it follows

$$||\mathbf{y}||_{\mathbf{W}^{1,1/2}_{2p_n/n}(Q_T)} \leq \Lambda(D(2) + ||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{2p_n/n}(Q_T)}) \leq \Lambda D(\frac{2p_n}{n}).$$

Applying another regularity property of parabolic systems (see [16], Lemma 7.3), we conclude that

$$||\epsilon(\mathbf{v})||_{\mathbf{W}^{2,1}_{2p_n/n}(Q_T)} \leq \Lambda ||\nabla \mathbf{v}||_{\mathbf{W}^{2,1}_{2p_n/n}(Q_T)} \leq \Lambda D(\frac{2p_n}{n}). \tag{3.22}$$

Hence, by imbedding

$$||\epsilon(\mathbf{v})||_{\mathbf{L}_p(Q_T)} \leq \Lambda D(\frac{2p_n}{n}) \qquad \text{for} \quad 1 < p < \infty, \tag{3.23}$$

and

$$||\nabla \epsilon(\mathbf{v})||_{\mathbf{L}_p(Q_T)} \leq \Lambda D(\frac{2p_n}{n}) \qquad \text{for} \quad 1 < p \leq \frac{q_n p_n}{n}. \tag{3.24}$$

Now, with the help of classical parabolic theory [10], we can obtain additional bounds on η. To this purpose using (3.19), (3.22) and (3.15) we estimate the right–hand side of (3.3).

$$\begin{aligned} &||R_1 + R_2 + R_3||_{L_p(Q_T)} \\ &\quad \leq \Lambda\left(||\eta||_{L_p(Q_T)} + ||\epsilon(\mathbf{v})||_{\mathbf{L}_p(Q_T)} + ||\epsilon(\mathbf{v}_t)||_{\mathbf{L}_p(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)}\right) \\ &\quad \leq \Lambda D(\frac{2p_n}{n}) \qquad \text{for} \quad 1 < p \leq \frac{2p_n}{n}. \end{aligned} \tag{3.25}$$

Hence,

$$||\eta||_{W^{2,1}_{2p_n/n}(Q_T)} \leq \Lambda D(\frac{2p_n}{n}).$$

Consequently, by imbedding,

$$\|\eta\|_{L_p(Q_T)} \leq \Lambda D(\frac{2p_n}{n}) \qquad \text{for} \quad 1 < p < \infty, \tag{3.26}$$

and

$$\|\nabla\eta\|_{L_p(Q_T)} \leq \Lambda D(\frac{2p_n}{n}) \qquad \text{for} \quad 1 < p \leq \frac{q_n p_n}{n}. \tag{3.27}$$

Now, returning to the system (3.20), (3.21), in view of (3.24) and (3.27) we can obtain further improvement of estimates. Namely, since thanks to the continuity of $\epsilon^i, \theta^i, \nabla\epsilon^i$ and $\nabla\theta^i$ in Q_T,

$$|\nabla \cdot (F^2_{/\epsilon} - F^1_{/\epsilon})| \leq \Lambda(|\epsilon(\mathbf{v})| + |\eta| + |\nabla\epsilon(\mathbf{v})| + |\nabla\eta|),$$

it follows that

$$\begin{aligned}
&\|\nabla \cdot (F^2_{/\epsilon} - F^1_{/\epsilon})\|_{\mathbf{L}_p(Q_T)} \\
&\quad \leq \Lambda(\|\epsilon(\mathbf{v})\|_{\mathbf{L}_p(Q_T)} + \|\eta\|_{L_p(Q_T)} + \|\nabla\epsilon(\mathbf{v})\|_{\mathbf{L}_p(Q_T)} + \|\nabla\eta\|_{\mathbf{L}_p(Q_T)}) \\
&\quad \leq \Lambda D(\frac{2p_n}{n}) \quad \text{for} \quad 1 < p \leq \frac{q_n p_n}{n}.
\end{aligned} \tag{3.28}$$

Now, with the help of Solonnikov theory of parabolic systems [18] (see also [16], Corollary 7.1), since $q_n p_n / n > p_n$, we obtain

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{W}^{4,2}_{p_n}(Q_T)} &\leq \Lambda\|\mathbf{y}\|_{\mathbf{W}^{2,1}_{p_n}(Q_T)} \\
&\leq \Lambda(D(\frac{2p_n}{n}) + \|\mathbf{b}^2 - \mathbf{b}^1\|_{\mathbf{L}_{p_n}(Q_T)}) \leq \Lambda D(p_n).
\end{aligned}$$

Hence, by imbedding,

$$\|\nabla\epsilon(\mathbf{v})\|_{\mathbf{L}_p(Q_T)} + \|\epsilon(\mathbf{v}_t)\|_{\mathbf{L}_p(Q_T)} \leq \Lambda D(p_n) \qquad \text{for} \quad 1 < p < \infty. \tag{3.29}$$

Repeating estimation (3.25), in view of (3.23), (3.26) and (3.29), it follows that

$$\begin{aligned}
&\|R_1 + R_2 + R_3\|_{L_p(Q_T)} \\
&\quad \leq \Lambda\left(D(\frac{2p_n}{n}) + D(p_n) + \|g^2 - g^1\|_{L_p(Q_T)}\right) \\
&\quad \leq \Lambda(D(p_n) + D(p)) \qquad \text{for} \quad 1 < p < \infty.
\end{aligned}$$

Consequently, by the classical parabolic theory,

$$\begin{aligned}
\|\eta\|_{W^{2,1}_p(Q_T)} &\leq \Lambda(D(p_n) + D(p)) \\
&\leq \Lambda D(p) \qquad \text{for} \quad p_n \leq p < \infty.
\end{aligned}$$

Hence, by imbedding,

$$\|\nabla\eta\|_{\mathbf{L}_q(Q_T)} \leq \Lambda D(p) \qquad \text{for} \quad 1 < q < \infty. \tag{3.30}$$

Finally, repeating estimation (3.28), in view of (3.23), (3.26), (3.29) and (3.30) we get

$$\begin{aligned}
\|\nabla \cdot (F^2_{/\epsilon} - F^1_{/\epsilon})\|_{\mathbf{L}_p(Q_T)} &\leq \Lambda(D(\frac{2p_n}{n}) + D(p_n) + D(p)) \\
&\leq \Lambda D(p) \qquad \text{for} \quad p_n \leq p < \infty.
\end{aligned}$$

Thus, by the Solonnikov theory,

$$\begin{aligned}||\mathbf{v}||_{W_p^{4,2}(Q_T)} &\le \Lambda ||\mathbf{y}||_{W_p^{2,1}(Q_T)}\\ &\le \Lambda D(p) \qquad \text{for } \; p_n \le p < \infty.\end{aligned}$$

This completes the proof. □

4 Differentiability

Let us consider two control pairs $(\mathbf{b}^i, g^i) \in \mathbf{L}_p(Q_T) \times L_p(Q_T)$, $g^i \ge 0$ a.e. in Q_T, $i = 1, 2$, such that

$$\mathbf{b}^2 = \mathbf{b}^1 + \tau\phi, \qquad g^2 = g^1 + \tau\psi, \tag{4.1}$$

where $0 \le \tau \le \tau_0$.

Let $(\mathbf{u}^i, \theta^i) \in V(p)$, $p > p_n$, be the unique solutions of problem (P) corresponding to $(\mathbf{b}^i, g^i)$. According to Theorem 3.1, we have the following stability estimate

$$\|(\mathbf{u}^2 - \mathbf{u}^1, \theta^2 - \theta^1)\|_{V(p)} \le \Lambda(\|\tau\phi\|_{\mathbf{L}_p(Q_T)} + \|\tau\psi\|_{L_p(Q_T)}) \le \Lambda\tau \tag{4.2}$$

for $p > p_n$. Consequently, by the imbedding theorem, similar bounds hold pointwise in Q_T for the differences $\mathbf{u}^2 - \mathbf{u}^1$, $\theta^2 - \theta^1$, $\nabla(\mathbf{u}_t^2 - \mathbf{u}_t^1)$, $\nabla^k(\mathbf{u}^2 - \mathbf{u}^1)$, $k = 1, 2, 3$, and $\nabla(\theta^2 - \theta^1)$. Our goal is to find a pair $(\mathbf{v}, \eta) \in V(p)$ such, that

$$\mathbf{u}^2 = \mathbf{u}^1 + \tau\mathbf{v} + o(\tau), \qquad \theta^2 = \theta^1 + \tau\eta + o(\tau)$$

in the sense of the space $V(p)$. For simplicity we introduce the notation:

$$\begin{aligned}G(\epsilon, \epsilon_t, \theta) &= \theta F_{/\theta\epsilon}(\epsilon, \theta) : \epsilon_t + \nu(\mathbf{A}\epsilon_t) : \epsilon_t,\\ \mathbf{H}_1 &= G_{/\epsilon}(\epsilon^1, \epsilon_t^1, \theta^1),\\ \mathbf{H}_2 &= G_{/\epsilon_t}(\epsilon^1, \epsilon_t^1, \theta^1),\\ H_3 &= G_{/\theta}(\epsilon^1, \epsilon_t^1, \theta^1).\end{aligned}$$

Using formal approximation by Taylor series we obtain the following system of equations for the pair $(\mathbf{v}, \eta)$:

$$\mathbf{v}_{tt} - \nu\mathbf{Q}\mathbf{v}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{v} = \nabla \cdot (F^1_{/\epsilon\epsilon}\epsilon(\mathbf{v}) + F^1_{/\epsilon\theta}\eta) + \phi, \tag{4.3}$$

$$c_v\eta_t - k\Delta\eta = \mathbf{H}_1 : \epsilon(\mathbf{v}) + \mathbf{H}_2 : \epsilon(\mathbf{v}_t) + H_3\eta + \psi \quad \text{in} \quad Q_T, \tag{4.4}$$

with initial and boundary conditions

$$\mathbf{v}(0, \mathbf{x}) = 0, \quad \mathbf{v}_t(0, \mathbf{x}) = 0, \quad \eta(0, \mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{4.5}$$

$$\mathbf{v} = \mathbf{Q}\mathbf{v} = 0, \quad \nabla\eta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T. \tag{4.6}$$

We note that thanks to regularity of the solutions $(\mathbf{u}^i, \theta^i)$, $\mathbf{H}_1, \mathbf{H}_2, H_3$ are continuous in Q_T. Clearly, there exists the unique solution $(\mathbf{v}, \eta) \in V(p)$ to problem (4.3)–(4.6) for any $T > 0$. This claim follows from the fact that we can adapt the arguments of Theorem 3.1 to prove a priori bound for a fixed point of the solution map.
We shall prove here the following differentiability result:

Theorem 4.1 *Let the assumptions of Theorem 2.2 hold with the data* $(\mathbf{b}^i, g^i)$ *given by (4.1). Then* $(\mathbf{u}^i, \theta^i)$ *and* $(\mathbf{v}, \eta)$ *satisfy the following estimate*

$$\|(\mathbf{u}^2 - \mathbf{u}^1 - \tau\mathbf{v}, \theta^2 - \theta^1 - \tau\eta)\|_{V(p)} \leq \Lambda\tau^2 \qquad \text{for } p_n < p < \infty,$$

where Λ *is a constant depending on the data of the problem,* Ω *and time* T. *Hence*

$$\lim_{\tau\to 0+} \frac{1}{\tau} \|(\mathbf{u}^2 - \mathbf{u}^1 - \tau\mathbf{v}, \theta^2 - \theta^1 - \tau\eta)\|_{V(p)} = 0,$$

what means that the pair $(\mathbf{v}, \eta)$ *is a Gateaux derivative of the solution with respect to the parameters* $(\mathbf{b}, g)$.

Proof. Let

$$\mathbf{z} = \mathbf{u}^2 - \mathbf{u}^1 - \tau\mathbf{v}, \qquad \varphi = \theta^2 - \theta^1 - \tau\eta.$$

By definition, $(\mathbf{z}, \varphi)$ satisfy the following system:

$$\mathbf{z}_{tt} - \nu\mathbf{Q}\mathbf{z}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{z} = \nabla \cdot (F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon}) \quad \text{in } Q_T, \tag{4.7}$$

$$\begin{aligned} c_v\varphi_t - k\Delta\varphi &= \mathbf{H}_1 : \epsilon(\mathbf{z}) + \mathbf{H}_2 : \epsilon(\mathbf{z}_t) + H_3\varphi + G^{1,2} \\ &=: R + G^{1,2} \quad \text{in } Q_T, \end{aligned} \tag{4.8}$$

with initial and boundary conditions

$$\mathbf{z}(0,\mathbf{x}) = 0, \quad \mathbf{z}_t(0,\mathbf{x}) = 0, \quad \varphi(0,\mathbf{x}) = 0 \quad \text{in } \Omega \tag{4.9}$$

$$\mathbf{z} = \mathbf{Q}\mathbf{z} = 0, \quad \nabla\varphi \cdot \mathbf{n} = 0 \quad \text{on } S_T, \tag{4.10}$$

where

$$\begin{aligned} F^{1,2}_{/\epsilon} &= F^2_{/\epsilon} - F^1_{/\epsilon} - F^1_{/\epsilon\epsilon}(\epsilon^2 - \epsilon^1) - F^1_{/\epsilon\theta}(\theta^2 - \theta^1), \\ G^{1,2} &= G^2 - G^1 - G^1_{/\epsilon} : (\epsilon^2 - \epsilon^1) - G^1_{/\epsilon_t} : (\epsilon^2_t - \epsilon^1_t) - G^1_{/\theta}(\theta^2 - \theta^1). \end{aligned}$$

In view of regularity of solutions $(\mathbf{u}^i, \theta^i)$, there exists the unique solution $(\mathbf{z}, \varphi) \in V(p)$ to the problem (4.7)–(4.10) for any $p > p_n$.
We shall show that

$$\|(\mathbf{z}, \varphi)\|_{V(p)} \leq \Lambda\tau^2.$$

By assumptions on $F(\epsilon, \theta)$ and the regularity of $(\mathbf{u}^i, \theta^i) \in V(p)$, $p_n < p < \infty$, the following bounds are valid:

$$|F^{1,2}_{/\epsilon}| \leq \Lambda(|\epsilon^2 - \epsilon^1|^2 + |\theta^2 - \theta^1|^2), \tag{4.11}$$

$$|G^{1,2}| \leq \Lambda(|\epsilon^2 - \epsilon^1|^2 + |\epsilon^2_t - \epsilon^1_t|^2 + |\theta^2 - \theta^1|^2). \tag{4.12}$$

From now on we will follow closely the proof of Theorem 3.1. We start from energy estimates for $\mathbf{z}$. Multiplying equation (4.7) by $\mathbf{z}_t$ and integrating over Q_t yields

$$\begin{aligned} &\int_{Q_t} \left(\frac{1}{2}\frac{d}{dt}|\mathbf{z}_t|^2 + \frac{\kappa}{8}\frac{d}{dt}|\mathbf{Q}\mathbf{z}|^2\right) dx dt' + \nu \int_{Q_t} (\mathbf{A}\epsilon(\mathbf{z}_t)) : \epsilon(\mathbf{z}_t)\, dx dt' \\ &= -\int_{Q_t} (F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi) : \epsilon(\mathbf{z}_t)\, dx dt' - \int_{Q_t} F^{1,2}_{/\epsilon} : \epsilon(\mathbf{z}_t)\, dx dt'. \end{aligned} \tag{4.13}$$

Adding to (4.13) identity (3.11) with $\mathbf{v}$ replaced by $\mathbf{z}$, by Young's inequality, we get

$$\int_{\Omega_t}\left(\frac{1}{2}|\mathbf{z}_t|^2+|\epsilon(\mathbf{z})|^2+\frac{\kappa}{8}|\mathbf{Q}\mathbf{z}|^2\right)dx+a_*\int_{Q_t}|\epsilon(\mathbf{z}_t)|^2\,dxdt'$$
$$\leq\delta\int_{Q_t}|\epsilon(\mathbf{z}_t)|^2\,dxdt'$$
$$+\Lambda c(\delta)\int_{Q_t}\left(|\epsilon(\mathbf{z})|^2+|F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z})+F^1_{/\epsilon\theta}\varphi|^2+|F^{1,2}_{/\epsilon}|^2\right)dxdt'.$$

Choosing $\delta=a_*/2$, in view of (4.11), by Gronwall's inequality we arrive at

$$\|\mathbf{z}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}+\|\epsilon(\mathbf{z})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}+\|\mathbf{Q}\mathbf{z}\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}+\|\epsilon(\mathbf{z}_t)\|_{\mathbf{L}_2(Q_T)}$$
$$\leq\Lambda\left(\|\varphi\|_{L_2(Q_T)}+\|\epsilon^2-\epsilon^1\|^2_{\mathbf{L}_4(Q_T)}+\|\theta^2-\theta^1\|^2_{L_4(Q_T)}\right)$$
$$\leq\Lambda\left(\|\varphi\|_{L_2(Q_T)}+\tau^2\right),\tag{4.14}$$

where in the last inequality we have applied stability estimate (4.2). Hence, by the ellipticity property of Q,

$$\|\mathbf{z}\|_{L_\infty(0,T;\mathbf{W}^2_2(\Omega))}\leq\Lambda\left(\|\varphi\|_{L_2(Q_T)}+\tau^2\right).$$

In order to obtain energy estimates for φ we multiply equation (4.8) by φ and integrate over Q_t to get

$$\frac{c_v}{2}\int_{\Omega_t}\varphi^2\,dx+k\int_{Q_t}|\nabla\varphi|^2\,dxdt'=\int_{Q_t}R\varphi\,dxdt'+\int_{Q_t}G^{1,2}\varphi\,dxdt'.\tag{4.15}$$

Using uniform bounds on $\mathbf{H}_1,\mathbf{H}_2,H_3$ and estimate (4.14), we obtain

$$\int_{Q_t}R\varphi\,dxdt'\leq\Lambda\int_{Q_t}(\varphi^2+|\epsilon(\mathbf{z})|^2+|\epsilon(\mathbf{z}_t)|^2)\,dxdt'\leq\Lambda\int_{Q_t}(\varphi^2+\tau^4)\,dxdt'.$$

Further, using (4.12), we have

$$\int_{Q_t}G^{1,2}\varphi\,dxdt'\leq\Lambda\int_{Q_t}(\varphi^2+|\epsilon^2-\epsilon^1|^4+|\epsilon^2_t-\epsilon^1_t|^4+|\theta^2-\theta^1|^4)\,dxdt'$$
$$\leq\Lambda\int_{Q_t}(\varphi^2+\tau^4)\,dxdt',$$

where in the last inequality we have applied the stability estimate (4.2).
Consequently, it follows from (4.15) that

$$\int_{\Omega_t}\varphi^2\,dxdt'+\int_{Q_t}|\nabla\varphi|^2\,dxdt'\leq\Lambda\int_{Q_t}(\varphi^2+\tau^4)\,dxdt'.$$

Hence, applying Gronwall's inequality,

$$\|\varphi\|_{L_\infty(0,T;L_2(\Omega))}+\|\nabla\varphi\|_{\mathbf{L}_2(Q_T)}\leq\Lambda\tau^2.\tag{4.16}$$

Substituting (4.16) into (4.14) yields

$$\|\mathbf{z}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}+\|\epsilon(\mathbf{z})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}+\|\mathbf{Q}\mathbf{z}\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))}+\|\epsilon(\mathbf{z}_t)\|_{\mathbf{L}_2(Q_T)}\leq\Lambda\tau^2.$$

Hence, the imbedding theorems imply the following bounds:

$$\|\epsilon(\mathbf{z})\|_{L_\infty(0,T;\mathbf{L}_{q_n}(\Omega))} \le \Lambda\tau^2, \tag{4.17}$$

$$\|\varphi\|_{L_{2p_n/n}(Q_T)} \le \Lambda\tau^2. \tag{4.18}$$

We employ now the parabolic decomposition of the system (4.7) into BVP's:

$$\begin{aligned} \mathbf{w}_t - \beta \mathbf{Q}\mathbf{w} &= \nabla\cdot(F^1_{/\epsilon\epsilon} : \epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon}) \quad \text{in} \quad Q_T, \\ \mathbf{w}(0,\mathbf{x}) &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{w} &= 0 \quad \text{on} \quad S_T, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} \mathbf{z}_t - \alpha\mathbf{Q}\mathbf{z} &= \mathbf{w} \quad \text{in} \quad Q_T, \\ \mathbf{z}(0,\mathbf{x}) &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{z} &= 0 \quad \text{on} \quad S_T. \end{aligned} \tag{4.20}$$

In view of (4.11) we have

$$\begin{aligned} &\|F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon}\|_{\mathbf{L}_p(Q_T)} \\ &\quad \le \Lambda(\|\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} + \|\varphi\|_{L_p(Q_T)} + \|\epsilon^2 - \epsilon^1\|^2_{\mathbf{L}_{2p}(Q_T)} \\ &\quad + \|\theta^2 - \theta^1\|^2_{L_{2p}(Q_T)}) \quad \text{for} \quad \frac{p_n}{2} < p < \infty. \end{aligned}$$

Hence, with the help of estimates (4.17),(4.18) and (4.2), since $2(2p_n/n) > p_n$, it follows that

$$\|F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon}\|_{\mathbf{L}_{2p_n/n}(Q_T)} \le \Lambda\tau^2.$$

Consequently, by regularity properties of parabolic systems (see [16], Lemmas 7.2,7.3), we conclude that

$$\|\nabla\mathbf{w}\|_{\mathbf{W}^{1,1/2}_{2p_n/n}(Q_T)} \le \Lambda\tau^2,$$

and

$$\|\epsilon(\mathbf{z})\|_{\mathbf{W}^{2,1}_{2p_n/n}(Q_T)} \le \Lambda\|\nabla\mathbf{z}\|_{\mathbf{W}^{2,1}_{2p_n/n}(Q_T)} \le \Lambda\tau^2. \tag{4.21}$$

Hence, by imbedding,

$$\|\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} \le \Lambda\tau^2 \qquad \text{for} \quad 1 < p < \infty, \tag{4.22}$$

and

$$\|\nabla\mathbf{z}\|_{\mathbf{L}_p(Q_T)} \le \Lambda\tau^2 \qquad \text{for} \quad 1 < p \le \frac{q_n p_n}{n}. \tag{4.23}$$

In the next step we improve the bounds for the function φ by applying the parabolic theory. In view of (4.12) the right-hand side of equation (4.8) is bounded by

$$\begin{aligned} \|R + G^{1,2}\|_{L_p(Q_T)} &\le \Lambda(\|\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} + \|\epsilon(\mathbf{z}_t)\|_{\mathbf{L}_p(Q_T)} + \|\varphi\|_{L_p(Q_T)} \\ &\quad + \|\epsilon^2 - \epsilon^1\|^2_{\mathbf{L}_{2p}(Q_T)} + \|\epsilon^2_t - \epsilon^1_t\|^2_{\mathbf{L}_{2p}(Q_T)} + \|\theta^2 - \theta^1\|^2_{L_{2p}(Q_T)}) \\ &\quad \text{for} \quad \frac{p_n}{2} < p < \infty. \end{aligned} \tag{4.24}$$

Hence, in view of estimates (4.18), (4.21), (4.22) and (4.2), it follows that

$$\|R+G^{1,2}\|_{L_{2p_n/n}(Q_T)} \le \Lambda\tau^2.$$

Consequently,

$$\|\varphi\|_{W^{2,1}_{2p_n/n}(Q_T)} \le \Lambda\tau^2, \tag{4.25}$$

so, by imbedding,

$$\|\varphi\|_{L_p(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad 1 < p < \infty, \tag{4.26}$$

and

$$\|\nabla\varphi\|_{\mathbf{L}_p(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad 1 < p \le \frac{q_n p_n}{n}. \tag{4.27}$$

We return now to the decomposed system (4.19), (4.20). In view of the bounds

$$\begin{aligned} &|\nabla\cdot(F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi)| \le \Lambda(|\epsilon(\mathbf{z})| + |\nabla\epsilon(\mathbf{z})| + |\varphi| + |\nabla\varphi|), \\ &\qquad |\nabla\cdot F^{1,2}_{/\epsilon}| \le \Lambda(|\epsilon^2-\epsilon^1|^2 + |\nabla(\epsilon^2-\epsilon^1)|^2 + |\theta^2-\theta^2|^2 + |\nabla(\theta^2-\theta^1)|^2), \end{aligned} \tag{4.28}$$

which follow by the regularity of $(\mathbf{u}^i, \theta^i)$,

$$\begin{aligned} &\|\nabla\cdot(F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon})\|_{\mathbf{L}_p(Q_T)} \\ &\le \Lambda\Big(\|\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} + \|\nabla\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} + \|\varphi\|_{L_p(Q_T)} + \|\nabla\varphi\|_{\mathbf{L}_p(Q_T)} \\ &+ \|\epsilon^2-\epsilon^1\|^2_{\mathbf{L}_{2p}(Q_T)} + \|\nabla(\epsilon^2-\epsilon^1)\|^2_{\mathbf{L}_{2p}(Q_T)} + \|\theta^2-\theta^1\|^2_{L_{2p}(Q_T)} + \|\nabla(\theta^2-\theta^1)\|^2_{\mathbf{L}_{2p}(Q_T)}\Big) \\ &\le \Lambda\tau^2 \quad \text{for} \quad \frac{p_n}{2} < p \le \frac{q_n p_n}{n}, \end{aligned} \tag{4.29}$$

where in the last inequality we have applied (4.22), (4.23), (4.26), (4.27) and (4.2). Thanks to above estimate, the theory of parabolic systems implies

$$\|\mathbf{z}\|_{\mathbf{W}^{4,2}_{q_n p_n/n}(Q_T)} \le \Lambda\|\mathbf{w}\|_{\mathbf{W}^{2,1}_{q_n p_n/n}(Q_T)} \le \Lambda\tau^2.$$

Hence, by imbedding,

$$\|\nabla\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} + \|\epsilon(\mathbf{z}_t)\|_{\mathbf{L}_p(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad 1 < p < \infty. \tag{4.30}$$

Now, in view of (4.22), (4.26), (4.30) and (4.2), repeating estimation (4.24) we obtain

$$\|R+G^{1,2}\|_{L_p(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad p_n < p < \infty.$$

This implies

$$\|\varphi\|_{W^{2,1}_p(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad p_n < p < \infty,$$

so, by imbedding,

$$\|\nabla\varphi\|_{\mathbf{L}_p(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad 1 < p < \infty. \tag{4.31}$$

Finally, repeating estimation (4.29), with the help of (4.22), (4.26), (4.30), (4.31) and (4.2), it follows that

$$\|\nabla\cdot(F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon})\|_{\mathbf{L}_p(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad p_n < p < \infty.$$

As a result,

$$\|\mathbf{z}\|_{\mathbf{W}^{4,2}_p(Q_T)} \le \Lambda\|\mathbf{w}\|_{\mathbf{W}^{2,1}_p(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad p_n < p < \infty.$$

This completes the proof. □

5 Optimal control problem

Let us denote the control space by

$$\mathcal{U} = \mathrm{L}_p(Q_T) \times L_p(Q_T) \qquad \text{for} \quad p > p_n,$$

and assume, that g is subject to the additional pointwise constraint, i.e.,

$$\mathbf{f} = (\mathbf{b}, g) \in \mathcal{U}_{ad} = \{ (\mathbf{b}, g) \in \mathcal{U} \mid 0 \leq g \text{ a.e. in } Q_T \}.$$

Let $\mathcal{S}$ denotes the solution operator, i.e. the map from the admissible set $\mathcal{U}_{ad}$ into $V(p)$, defined by

$$\mathcal{S}(\mathbf{f}) = (\mathbf{u}, \theta),$$

where $(\mathbf{u}, \theta)$ is the solution of (P) corresponding to $\mathbf{f} = (\mathbf{b}, g)$. From Theorem 3.1 it follows that the map $\mathcal{S}$ is Lipschitz continuous.
We have also the following weak continuity property.

Lemma 5.1 *Under assumptions of Theorem 2.2 the map $\mathcal{S}$ is continuous from $\mathcal{U}_{ad}$ (weak) into $V(p)$ (weak).*

Proof. Consider a sequence $(\mathbf{b}^n, g^n) \in \mathcal{U}$ such, that

$$(\mathbf{b}^n, g^n) \rightarrow (\mathbf{b}, g) \quad \text{weakly in } \mathcal{U}.$$

Let $(\mathbf{u}^n, \theta^n)$ be a sequence of solutions of (P^n) corresponding to $(\mathbf{b}^n, g^n)$. Since $(\mathbf{b}^n, g^n)$ is uniformly bounded in $\mathcal{U}$, the a priori bounds of Theorem 2.1 imply that $(\mathbf{u}^n, \theta^n)$ is also uniformly bounded in the space $V(p)$.
Therefore, after selecting the subsequence,

$$(\mathbf{u}^n, \theta^n) \rightarrow (\mathbf{u}, \theta) \quad \text{weakly in } V(p).$$

Since $p > p_n$, by the compact imbeddings,

$$\mathbf{u}^n,\ \nabla\mathbf{u}^n,\ \nabla^2\mathbf{u}^n,\ \nabla^3\mathbf{u}^n,\ \mathbf{u}^n_t,\ \nabla\mathbf{u}^n_t,\ \theta^n,\ \nabla\theta^n$$

are convergent in spaces of Hölder continuous functions. Therefore we have pointwise convergence for all terms entering the right–hand sides of equations (2.1)–(2.2). Then we can pass to the weak limit in (P^n) to conclude that $(\mathbf{u}, \theta)$ satisfies (P). □

By virtue of the stability estimate (4.2), the result of Theorem 4.1 can be easily reformulated in terms of $\mathcal{S}$ in the following way:
Let

$$\mathbf{f}, \mathbf{f} + \delta\mathbf{f} \in \mathcal{U}_{ad}, \quad \mathbf{f} = (\mathbf{b}, g), \quad \delta\mathbf{f} = (\phi, \psi),$$

and $\mathcal{S}(\mathbf{f})$, $\mathcal{S}(\mathbf{f} + \delta\mathbf{f})$ be the corresponding solutions of (P). Then

$$\|\mathcal{S}(\mathbf{f} + \delta\mathbf{f}) - \mathcal{S}(\mathbf{f}) - \mathcal{S}'(\mathbf{f})\,\delta\mathbf{f}\|_{V(p)} \leq \Lambda \|\delta\mathbf{f}\|^2_{\mathcal{U}}, \tag{5.1}$$

where $\mathcal{S}'(\mathbf{f}) : \mathcal{U}_{ad} \rightarrow V(p)$ is a linear operator, and $(\mathbf{v}, \eta) = \mathcal{S}'(\mathbf{f})\,\delta\mathbf{f}$ is the solution of the problem

$$\mathbf{v}_{tt} - \nu \mathrm{Q}\mathbf{v}_t + \frac{\kappa}{4}\mathrm{Q}\mathrm{Q}\mathbf{v} = \nabla \cdot (F_{/\epsilon\epsilon}\epsilon(\mathbf{v}) + F_{/\epsilon\theta}\eta) + \phi, \tag{5.2}$$

$$c_v \eta_t - k\Delta\eta = \mathrm{H}_1 : \epsilon(\mathbf{v}) + \mathrm{H}_2 : \epsilon(\mathbf{v}_t) + H_3\eta + \psi \quad \text{in} \quad Q_T, \tag{5.3}$$

$$\mathbf{v}(0, \mathbf{x}) = 0, \quad \mathbf{v}_t(0, \mathbf{x}) = 0, \quad \eta(0, \mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{5.4}$$

$$\mathbf{v} = \mathrm{Q}\mathbf{v} = 0, \quad \nabla\eta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T, \tag{5.5}$$

where the coefficients $\mathrm{F}_{/\epsilon\epsilon}, \mathrm{F}_{/\epsilon\theta}, \mathrm{H}_1, \mathrm{H}_2, H_3$ are evaluated at $(\mathbf{u}, \theta) = \mathcal{S}(\mathbf{f})$.

Remark 5.1 *The constant Λ on the right–hand side of estimate (5.1) does not depend on the norms of $\mathbf{v}, \eta$. Therefore the operator $\mathcal{S}'(\mathbf{f}) : \mathcal{U}_{ad} \to V(p)$ is the Fréchet derivative of the operator $\mathcal{S}$.*

We consider the following cost functional

$$J[\mathbf{u},\theta;\mathbf{f}] = \frac{1}{2}\int_{Q_T} \Phi(|\mathbf{u}-\bar{\mathbf{u}}|^2, |\epsilon(\mathbf{u}-\bar{\mathbf{u}})|^2, |\nabla(\theta-\bar{\theta})|^2)\,dxdt' + \frac{\rho}{2}\int_{Q_T} (|\mathbf{b}|^{2s} + g^{2s})\,dxdt', \tag{5.6}$$

where the function $\Phi(s_1, s_2, s_3)$ is assumed to be of a class $C^1(\bar{\mathbb{R}}^3_+)$, Lipschitz continuous, and the weight coefficient ρ is positive. Moreover, $s \in \mathcal{N}$ and $2s > p_n$. The functions $\bar{\mathbf{u}}, \bar{\theta}$ are given reference solutions of problem (P).
The following holds

Theorem 5.1 *There exists an optimal control $\hat{\mathbf{f}} \in \mathcal{U}_{ad}$ minimizing the cost functional (5.6) of the problem (P), i.e.*

$$J[\hat{\mathbf{u}},\hat{\theta};\hat{\mathbf{f}}] = \inf_{\mathbf{f}\in\mathcal{U}_{ad}} J[\mathbf{u},\theta;\mathbf{f}],$$

where $(\hat{\mathbf{u}},\hat{\theta}) = \mathcal{S}(\hat{\mathbf{f}})$ and $(\mathbf{u},\theta) = \mathcal{S}(\mathbf{f})$.

Proof. The proof follows by standard arguments. Let $(\mathbf{u}^n, \theta^n; \mathbf{f}^n)$, $(\mathbf{u}^n,\theta^n) = \mathcal{S}(\mathbf{f}^n)$ be a minimizing sequence for the functional J. Since $J[\mathbf{u}^n,\theta^n;\mathbf{f}^n] \le \Lambda$, thanks to the positivity of ρ we have

$$\|\mathbf{f}^n\|_{\mathcal{U}} \le \Lambda.$$

Due to the Lemma 5.1 we can select a subsequence of $\{\mathbf{f}^n\}$ and $\{(\mathbf{u}^n,\theta^n)\}$, denoted by the same index n, such that $\mathbf{f}^n \to \mathbf{f}$ weakly in $\mathcal{U}$, and

$$(\mathbf{u}^n,\theta^n) = \mathcal{S}(\mathbf{f}^n) \to (\mathbf{u},\theta) = \mathcal{S}(\mathbf{f}) \quad \text{weakly in} \quad V(p).$$

By the weak l.s.c. of $J[\mathbf{u},\theta;\mathbf{f}]$,

$$\liminf_{n\to\infty} J[\mathbf{u}^n,\theta^n;\mathbf{f}^n] \ge J[\mathbf{u},\theta;\mathbf{f}].$$

Thus $\hat{\mathbf{f}} := \mathbf{f}$ is an optimal control for (P). □

6 Necessary optimality conditions

We turn now to the neccessary optimality conditions which have to be satisfied by any optimal control $\mathbf{f}$. The variation of the goal functional (5.6) is given by

$$\begin{aligned} \delta J &= \frac{d}{d\tau} J[\mathcal{S}(\mathbf{f}+\tau\,\delta\mathbf{f}; \mathbf{f}+\tau\,\delta\mathbf{f}]|_{\tau=0} \\ &= \int_{Q_T} \left[\Phi_{/s_1}(\mathbf{u}-\bar{\mathbf{u}})\cdot\mathbf{v} + \Phi_{/s_2}\epsilon(\mathbf{u}-\bar{\mathbf{u}}) : \epsilon(\mathbf{v}) + \Phi_{/s_3}\nabla(\theta-\bar{\theta})\cdot\nabla\eta\right] dxdt' \\ &+ \rho s\int_{Q_T} (\mathbf{b}^{2s-1}\cdot\phi + g^{2s-1}\psi)\,dxdt'. \end{aligned}$$

Performing integration by parts gives

$$\delta J = \int_{Q_T} (\Phi_1 \cdot \mathbf{v} + \Phi_2 \eta)\, dxdt' + \rho s \int_{Q_T} (\mathbf{b}^{2s-1} \cdot \phi + g^{2s-1}\psi)\, dxdt', \tag{6.1}$$

where

$$\Phi_1 = \Phi_{/s_1}(\mathbf{u} - \bar{\mathbf{u}}) - \nabla \cdot [\Phi_{/s_2}\epsilon(\mathbf{u} - \bar{\mathbf{u}})],$$
$$\Phi_2 = -\nabla \cdot [\Phi_{/s_3}\nabla(\theta - \bar{\theta})].$$

We note, that by the regularity properties of solution $(\mathbf{u}, \theta) \in V(p)$ the function Φ_1 is continuous in Q_T, and $\Phi_2 \in L_p(Q_T)$.

In order to derive the adjoint equations it is advantageous to rewrite (5.2)–(5.5) as a first order system, introducing an artificial variable $\mathbf{z}$:

$$\mathbf{v}_t = \mathbf{z}, \tag{6.2}$$
$$\mathbf{z}_t = \nu \mathbf{Q}\mathbf{z} - \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{v} + \nabla \cdot (F_{/\epsilon\epsilon}\epsilon(\mathbf{v}) + F_{/\epsilon\theta}\eta) + \phi, \tag{6.3}$$
$$c_v \eta_t = k\Delta\eta + \mathbf{H}_1 : \epsilon(\mathbf{v}) + \mathbf{H}_2 : \epsilon(\mathbf{z}) + H_3\eta + \psi \quad \text{in} \quad Q_T, \tag{6.4}$$

with initial and boundary conditions

$$\mathbf{v}(0, \mathbf{x}) = 0, \quad \mathbf{z}(0, \mathbf{x}) = 0, \quad \eta(0, \mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{6.5}$$
$$\mathbf{v} = \mathbf{z} = \mathbf{Q}\mathbf{v} = 0, \quad \nabla\eta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T. \tag{6.6}$$

Here the coefficients $\mathbf{F}_{/\epsilon\epsilon}, \mathbf{F}_{/\epsilon\theta}, \mathbf{H}_1, \mathbf{H}_2, H_3$ are evaluated at $(\mathbf{u}, \theta) = \mathcal{S}(\mathbf{f})$.

Denoting the adjoint variables by $\mathbf{p}, \mathbf{r}, q$ we may formally write down the adjoint system as

$$\mathbf{p}_t = \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{r} - \nabla \cdot (F_{/\epsilon\epsilon}\epsilon(\mathbf{r}) + \mathbf{H}_1 q) - \Phi_1, \tag{6.7}$$
$$\mathbf{r}_t = -\mathbf{p} - \nu\mathbf{Q}\mathbf{r} + \nabla \cdot (\mathbf{H}_2 q), \tag{6.8}$$
$$c_v q_t = F_{/\epsilon\theta} : \epsilon(\mathbf{r}) - k\Delta q - H_3 q - \Phi_2 \quad \text{in} \quad Q_T, \tag{6.9}$$

with terminal and boundary conditions

$$\mathbf{p}(T, \mathbf{x}) = 0, \quad \mathbf{r}(T, \mathbf{x}) = 0, \quad q(T, \mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{6.10}$$
$$\mathbf{r} = \mathbf{Q}\mathbf{r} = 0, \quad \nabla q \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T. \tag{6.11}$$

The first order adjoint system (6.7)–(6.11) is equivalent to

$$\mathbf{r}_{tt} + \nu\mathbf{Q}\mathbf{r}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{r} = \nabla \cdot (F_{/\epsilon\epsilon}\epsilon(\mathbf{r}) - \mathbf{H}_1 q + (\mathbf{H}_2 q)_t) + \Phi_1, \tag{6.12}$$
$$c_v q_t + k\Delta q = F_{/\epsilon\theta} : \epsilon(\mathbf{r}) - H_3 q - \Phi_2 \quad \text{in} \quad Q_T, \tag{6.13}$$

with terminal and boundary conditions

$$\mathbf{r}(T, \mathbf{x}) = 0, \quad \mathbf{r}_t(T, \mathbf{x}) = 0, \quad q(T, \mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{6.14}$$
$$\mathbf{r} = \mathbf{Q}\mathbf{r} = 0, \quad \nabla q \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T. \tag{6.15}$$

Multiplying equations (6.2)–(6.4) correspondingly by $\mathbf{p}, \mathbf{r}, q$ and integrating over Q_T gives, after several integration by parts and use of boundary conditions, the identity

$$\begin{aligned}
&\int_\Omega (\mathbf{p}\cdot\mathbf{v}_t + \mathbf{r}\cdot\mathbf{z}_t + c_v q\eta_t)\,dx = \\
&\qquad \int_\Omega [-\frac{\kappa}{4}\mathbf{v}\cdot(\mathbf{Q}\mathbf{Q}\mathbf{r}) + \mathbf{v}\cdot(\nabla\cdot(F_{/\epsilon\epsilon}\epsilon(\mathbf{r}))) - \mathbf{v}\cdot(\nabla\cdot(\mathbf{H}_1 q))]\,dx \\
&\qquad + \int_\Omega [\mathbf{z}\cdot\mathbf{p} + \nu\mathbf{z}\cdot(\mathbf{Q}\mathbf{r}) - \mathbf{z}\cdot(\nabla\cdot(\mathbf{H}_2 q))]\,dx \\
&\qquad + \int_\Omega [-\eta F_{/\epsilon\theta} : \epsilon(\mathbf{r}) + \eta H_3 q + k\eta\Delta q]\,dx \\
&\qquad + \int_\Omega (\phi\cdot\mathbf{r} + \psi q)\,dx \\
&\quad = -\int_\Omega (\mathbf{v}\cdot\mathbf{p}_t + \mathbf{z}\cdot\mathbf{r}_t + c_v\eta q_t)\,dx \\
&\qquad - \int_\Omega (\Phi_1\cdot\mathbf{v} + \Phi_2\eta)\,dx + \int_\Omega (\phi\cdot\mathbf{r} + \psi q)\,dx,
\end{aligned} \tag{6.16}$$

or, equivalently

$$\frac{d}{dt}\int_\Omega (\mathbf{p}\cdot\mathbf{v} + \mathbf{r}\cdot\mathbf{z} + c_v q\eta)\,dx = -\int_\Omega (\Phi_1\cdot\mathbf{v} + \Phi_2\eta)\,dx + \int_\Omega (\phi\cdot\mathbf{r} + \psi q)\,dx.$$

Hence, in view conditions (6.5), (6.10) it follows that

$$\int_{Q_T} (\Phi_1\cdot\mathbf{v} + \Phi_2\eta)\,dxdt = \int_{Q_T} (\phi\cdot\mathbf{r} + \psi q)\,dxdt. \tag{6.17}$$

This identity corresponds to the definition of the solution $(\mathbf{r}, q)$ of the adjoint system (6.7)–(6.11) in the transposition method sense of Lions, Magenes [12].

As common in the control theory, despite the lower regularity of the solution $(\mathbf{r}, q)$, identity (6.17) allows to formulate the first order optimality condition.

Actually, according to (6.1), the first variation of the cost functional has the representation

$$\delta J = \int_{Q_T} (\phi\cdot\mathbf{r} + \psi q)\,dxdt' + \rho s\int_{Q_T} (\phi\cdot\mathbf{b}^{2s-1} + \psi g^{2s-1})\,dxdt.$$

Concluding, we get the following

Theorem 6.1 *Let $\mathbf{f} = (\mathbf{b}, g) \in \mathcal{U}_{ad}$ be an optimal control for problem (P). If $(\mathbf{u}, \theta) = \mathcal{S}(\mathbf{f})$ is the corresponding solution of (P) and $(\mathbf{r}, q)$ the corresponding solution of the adjoint system (6.12)–(6.15), then they satisfy the first order optimality condition*

$$\int_{Q_T} [(\mathbf{r} + \rho s\mathbf{b}^{2s-1})\cdot(\tilde{\mathbf{b}} - \mathbf{b}) + (q + \rho s g^{2s-1})(\tilde{g} - g)]\,dxdt \geq 0$$

for all $(\tilde{\mathbf{b}}, \tilde{g}) \in \mathcal{U}_{ad}$.

7 Existence of adjoint state for the control problem

The system (6.12)–(6.15) is linear with respect to r, q, but has a nonstandard form due to the presence of q_t on the right–hand side of (6.12). Therefore the existence of its solution requires justification. Besides, under previous assumptions on the data of problem (P), some of the coefficients $F_{/\epsilon\epsilon}, \mathbf{H}_1, \mathbf{H}_2, F_{/\epsilon\theta}, H_3, \Phi_2$ may have very low regularity, especially the derivative $(\nabla \cdot \mathbf{H}_2)_t$.

However, given higher regularity of data, we may obtain stronger a priori bounds for the solutions of the problem (P). Therefore we shall assume in this section that the space and time derivatives of $F_{/\epsilon\epsilon}, \mathbf{H}_1, \mathbf{H}_2, F_{/\epsilon\theta}, H_3$ as well as $(\nabla \cdot \mathbf{H}_2)_t, \Phi_2$ are continuous in Q_T.

For example it holds if the following conditions are satisfied:

- $D_t^s D_x^r \mathbf{u}$ are continuous for $2s + r \leq 6$;
- $D_t^s D_x^r \theta$ are continuous for $2s + r \leq 4$.

Theorem 7.1 *Under the assumptions stated above there exists a solution*

$$\mathbf{r} \in \mathbf{W}_2^{4,2}(Q_T), \qquad q, q_t \in W_2^{2,1}(Q_T)$$

to the problem (6.12)–(6.15).

Proof. Due to our regularity assumptions we may differentiate (6.13) and obtain for q_t the linear parabolic equation with continuous coefficients. Moreover, from (6.13) follows, by substituting $t = T$, that

$$q_t(T, \mathbf{x}) = -\frac{1}{c_v}\Phi_2(T, \mathbf{x}), \tag{7.1}$$

what supplies the neccessary end condition.

For simplicity of reasoning let us now change the time direction by substitution $t := T - t$, so that the end conditions become initial ones. The system (6.12)–(6.13) transforms to

$$\mathbf{r}_{tt} - \nu \mathbf{Q}\mathbf{r}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{r} = \nabla \cdot \left(F_{/\epsilon\epsilon}\epsilon(\mathbf{r}) - \mathbf{H}_1 q - (\mathbf{H}_2)_t q - \mathbf{H}_2 q_t\right) + \Phi_1, \tag{7.2}$$

$$c_v q_t - k\Delta q - H_3 q = -F_{/\epsilon\theta} : \epsilon(\mathbf{r}) + \Phi_2 \quad \text{in} \quad Q_T, \tag{7.3}$$

with unchanged boundary conditions.

By standard parabolic theory we have from (7.3) the estimate

$$\|q\|_{W_2^{2,1}(Q_T)} \leq \Lambda + \Lambda\|\epsilon(\mathbf{r})\|_{\mathbf{L}_2(Q_T)}. \tag{7.4}$$

Similarly, after differentiating (7.3) with respect to time, and using our regularity assumptions as well as initial condition (7.1), we get similar estimates for q_t:

$$\|q_t\|_{W_2^{2,1}(Q_T)} \leq \Lambda + \Lambda\|\epsilon(\mathbf{r}_t)\|_{\mathbf{L}_2(Q_T)} + \Lambda\|\epsilon(\mathbf{r})\|_{\mathbf{L}_2(Q_T)}. \tag{7.5}$$

These bounds are crucial for the proof. We shall concentrate on obtaining an a priori estimate for the solutions in the spaces given by the formulation of the theorem. The rest of the steps required by the Leray–Schauder theorem are easy due to the linearity of the problem.

First we multiply (7.2) by $\mathbf{r}_t$ and integrate over Q_t, using initial and boundary conditions. As a result for the left–hand side we obtain the identity

$$L = \frac{1}{2}\int_{\Omega_t} |\mathbf{r}_t|^2\, dx + \nu \int_{Q_t} \epsilon(\mathbf{r}_t) : \mathbf{A}\epsilon(\mathbf{r}_t)\, dx\, dt' + \frac{\kappa}{8}\int_{\Omega_t} |\mathbf{Q}\mathbf{r}|^2\, dx.$$

Therefore there exists $\Lambda > 0$ such that

$$L \geq \Lambda \left(\int_{\Omega_t} |\mathbf{r}_t|^2\, dx + \int_{Q_t} |\epsilon(\mathbf{r}_t)|^2\, dx\, dt' + \int_{\Omega_t} |\mathbf{Q}\mathbf{r}|^2\, dx \right) := L_1 + L_2 + L_3.$$

The right–hand side consists of five terms,

$$R := R_1 + R_2 + R_3 + R_4 + R_5$$

which will be considered one by one. For the first we have

$$R_1 = \int_{Q_t} \mathbf{r}_t \cdot (\nabla \cdot (F_{/\epsilon\epsilon}\epsilon(\mathbf{r})))\, dx\, dt' = -\int_{Q_t} \epsilon(\mathbf{r}_t) : (F_{/\epsilon\epsilon}\epsilon(\mathbf{r}))\, dx\, dt'.$$

Hence

$$|R_1| \leq \Lambda \left(\delta_1 \|\epsilon(\mathbf{r}_t)\|^2_{\mathbf{L}_2(Q_t)} + \delta_1^{-1} \|\epsilon(\mathbf{r})\|^2_{\mathbf{L}_2(Q_t)} \right),$$

where the first part may be absorbed by L_2 after suitable choice of δ_1.
Similarly, by (7.4),

$$|R_2| = |\int_{Q_t} \mathbf{r}_t \cdot (\nabla \cdot (\mathbf{H}_1 q))\, dx\, dt'| \leq \Lambda \left(\delta_2 \|\epsilon(\mathbf{r}_t)\|^2_{\mathbf{L}_2(Q_t)} + \delta_2^{-1} \|q\|^2_{\mathbf{L}_2(Q_t)} \right)$$
$$\leq \Lambda \left(\delta_2 \|\epsilon(\mathbf{r}_t)\|^2_{\mathbf{L}_2(Q_t)} + \delta_2^{-1} \|\epsilon(\mathbf{r})\|^2_{\mathbf{L}_2(Q_t)} + 1 \right),$$

and again the first part may be absorbed by L_2. For the third term we have, after application of (7.4),

$$|R_3| = |-\int_{Q_t} \mathbf{r}_t \cdot (\nabla \cdot (\mathbf{H}_2)_t q)\, dx\, dt'| = |\int_{Q_t} \epsilon(\mathbf{r}_t) : (\mathbf{H}_2)_t q\, dx\, dt'|$$
$$\leq \Lambda \left(\delta_3 \|\epsilon(\mathbf{r}_t)\|^2_{\mathbf{L}_2(Q_t)} + \delta_3^{-1} \|\epsilon(\mathbf{r})\|^2_{\mathbf{L}_2(Q_t)} + 1 \right),$$

Similar procedure for next term gives

$$|R_4| = |-\int_{Q_t} \mathbf{r}_t \cdot (\nabla \cdot (\mathbf{H}_2 q_t))\, dx\, dt'| = |\int_{Q_t} \epsilon(\mathbf{r}_t) : (\mathbf{H}_2) q_t\, dx\, dt'|$$
$$\leq \Lambda \left(\delta_4 \|\epsilon(\mathbf{r}_t)\|^2_{\mathbf{L}_2(Q_t)} + \delta_4^{-1} \|\epsilon(\mathbf{r})\|^2_{\mathbf{L}_2(Q_t)} + 1 \right).$$

The last term is simple,

$$|R_5| = |\int_{Q_t} \mathbf{r}_t \cdot \mathbf{\Phi}_1 dx\, dt'| \leq \Lambda \left(\|\mathbf{r}_t\|_{\mathbf{L}_2(Q_t)} + 1 \right).$$

Because of the strong ellipticity of the operator $\mathbf{Q}$ we have also, taking into account homogeneous boundary conditions,

$$\int_{Q_t} |\epsilon(\mathbf{r})|^2\, dx\, dt' \leq \Lambda \int_{Q_t} |\mathbf{Q}\mathbf{r}|^2\, dx\, dt'.$$

Therefore, by suitable choice of $\delta_1, \delta_3, \delta_4$ we get the inequality

$$L_1 + L_2 + L_3 \leq \Lambda \left(\int_{Q_t} |\mathbf{r}_t|^2 \, dx dt' + \int_{Q_t} |\mathbf{Q}\mathbf{r}|^2 \, dx dt' + 1 \right).$$

This allows us to apply Gronwall's inequality and obtain the bound

$$\int_{\Omega_t} |\mathbf{r}_t|^2 \, dx + \int_{Q_t} |\epsilon(\mathbf{r}_t)|^2 \, dx \, dt' + \int_{\Omega_t} |\mathbf{Q}\mathbf{r}|^2 \, dx \leq \Lambda.$$

Hence, using (7.5), we may uniformly estimate q in $W_2^{2,1}(Q_T)$, q_t in $W_2^{2,1}(Q_T)$ as well as the whole right–hand side of (7.2) in $\mathbf{L}_2(Q_T)$, what implies the required bound for $\mathbf{r}$ in $\mathbf{W}_2^{4,2}(Q_T)$. □

Theorem 7.1 ensures that the neccessary conditions of optimality given by Theorem 6.1 are meaningfull. We have not striven here for the highest generality (i.e. weakest regularity assumptions), but wanted to demonstrate the existence of situtations when the optimality conditions are valid.

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