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Abstract. In this paper we prove the existence and uniqueness of a strong solution to 3-D model of phase separation in elastic solids. The model has the form of an initial-boundary-value problem for nonlinear coupled system of hyperbolic-parabolic type. The key idea of the proof is based on the analysis of the system once- and twice-differentiated with respect to time variable. The paper develops results of the previous work [PawZaj06b].

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1. Introduction

In this paper we are concerned with the strong solvability of a three-dimensional (3-D) initial-boundary-value problem for hyperbolic-parabolic system arising as a model of phase separation in deformable binary alloys. The system under consideration combines the linear momentum balance, represented by the nonstationary elasticity system, with the mass balance, described by the Cahn-Hilliard problem. We prove that such nonlinear system has a unique strong solution in the sense that all derivatives that appear in the equations are at least in L_2.

In the previous paper [PawZaj06b] we have proved the existence and some time regulartiy of weak solutions to such problem. The proof of the regularity result was based on an analysis of a time-differentiated system. The analysis was performed by means of the Faedo-Galerkin approximation and energy methods. In the present paper we prove an additional time regularity of weak solutions by considering system twice differentiated with respect to time variable. As in [PawZaj06b] we apply the Faedo-Galerkin approximation and energy methods.

The idea of the proof of the strong solvability is based on an apparent observation that having the weak solutions with sufficiently regular time derivatives one can look at the hyperbolic-parabolic problem under consideration as on an elliptic system with the right-hand side including, in addition to the nonlinear terms, all time derivatives. Then the application of the standard elliptic regularity theory allows to deduce the existence of solutions with further space regularity and consequently the classical solvability of the problem.

We remark that the main difficulties in the analysis of the problem come from the 3-D setting and the hyperbolic nature of the elasticity system. In three space dimensions the coupled system shows features that cannot be found in its one-dimensional setting (see comments following equation (1.17) below). The classical solvability of the problem in 1-D was proved by the authors in [PawZaj06a] by means of a method specific for the single space dimension.

The place of our study in the present theory of the Cahn-Hilliard systems in elastic solids is discussed in the previous paper [PawZaj06b]. As mentioned there, in view of the fact that the mechanical equilibrium is usually attained on a much faster time scale than diffusion, in most of the literature on the subject a quasi-stationary approximation of elasticity system is assumed. At the initial stages of phase separation process, however, the formation of the microstructure is on a very fast time scale and thus the nonstationary elastic effects may become of importance.

The model problem under consideration has the following form:

$$
\begin{aligned}
&\boldsymbol{u}_{tt} - \nabla\cdot W_{,\varepsilon}(\varepsilon(\boldsymbol{u}),\chi) = \boldsymbol{b} && \text{in } \Omega^T = \Omega\times(0,T),\\
&\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega,\\
&\boldsymbol{u} = 0 && \text{on } S^T = S\times(0,T),
\end{aligned} \tag{1.1}
$$

$$
(1.2) \qquad \begin{aligned} &\chi_t - \nabla\cdot M\nabla\mu = 0 && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n}\cdot M\nabla\mu = 0 && \text{on } S^T, \end{aligned}
$$

$$
(1.3) \qquad \begin{aligned} &\mu = -\nabla\cdot\boldsymbol{\Gamma}\nabla\chi + \psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi) && \text{in } \Omega^T, \\ &\boldsymbol{n}\cdot\boldsymbol{\Gamma}\nabla\chi = 0 && \text{on } S^T. \end{aligned}
$$

Here $\Omega\subset\mathbb{R}^3$ is a bounded domain with a smooth boundary S, occupied by a body in a reference configuration with constant mass density $\varrho = 1$; $\boldsymbol{n}$ is the unit outward normal to S, and $T > 0$ is an arbitrary fixed time. The body is a binary $a-b$ alloy.

The unknows are the fields $\boldsymbol{u},\chi$ and μ, where $\boldsymbol{u}:\Omega^T\to\mathbb{R}^3$ is the *displacement vector*, $\chi:\Omega^T\to\mathbb{R}$ is the *order parameter* (phase ratio) and $\mu:\Omega^T\to\mathbb{R}$ is the chemical potential difference between the components, shortly referred to as the *chemical potential*. The second order tensor

$$
\varepsilon = \varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla u + (\nabla u)^T)
$$

denotes the linearized strain tensor.

In case of a binary $a-b$ alloy the order parameter is related to the volumetric fraction of one of the two phases, characterized by different crystalline structures of the components, e.g. $\chi=-1$ is identified with the phase a and $\chi = 1$ with the phase b.

The function $W(\varepsilon(\boldsymbol{u}),\chi)$ denotes the elastic energy defined by

$$
(1.4) \qquad W(\varepsilon(\boldsymbol{u}),\chi) = \frac{1}{2}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi))\cdot\boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi)).
$$

The corresponding derivatives

$$
W_{,\varepsilon}(\varepsilon(\boldsymbol{u}),\chi) = \boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi))
$$

and

$$
W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi) = -\bar{\varepsilon}'(\chi)\cdot\boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi))
$$

represent respectively the stress tensor and the elastic part of the chemical potential. The fourth order tensor $\boldsymbol{A} = (A_{ijkl})$ denotes a constant elasticity tensor given by

$$
(1.5) \qquad \varepsilon(\boldsymbol{u}) \mapsto \boldsymbol{A}(\varepsilon(\boldsymbol{u}) = \bar{\lambda}tr\varepsilon(\boldsymbol{u})\boldsymbol{I} + 2\bar{\mu}\varepsilon(\boldsymbol{u}),
$$

where $\boldsymbol{I} = (\delta_{ij})$ is the identity tensor, and $\bar{\lambda},\bar{\mu}$ are the Lamé constants with values within elasticity range (see Section 2). The form (1.5) refers to the isotropic, homogeneous medium with the same elastic properties of the phases.

The second order tensor $\bar{\varepsilon}(\chi)$ denotes the eigenstrain, i.e. the stress free strain corresponding to the phase ratio χ, defined by

$$\bar{\varepsilon}(\chi) = (1 - z(\chi))\bar{\varepsilon}_a + z(\chi)\bar{\varepsilon}_b, \tag{1.6}$$

with $\bar{\varepsilon}_a, \bar{\varepsilon}_b$ denoting constant eigenstrains of the phases a, b, and $z : \mathbb{R} \to [0, 1]$ being a sufficiently smooth interpolation function (called shape function) satisfying

$$z(\chi) = 0 \quad \text{for} \quad \chi \leq -1 \quad \text{and} \quad z(\chi) = 1 \quad \text{for} \quad \chi \geq 1. \tag{1.7}$$

Furthermore, the function $\psi : \mathbb{R} \to \mathbb{R}$ denotes the chemical energy of the system at zero stress. This function depends on temperature and is convex above a critical temperature and a nonconvex for temperatures less that the critical one. Here we assume it in the simplest double-well form

$$\psi(\chi) = \frac{1}{4}(1 - \chi^2)^2 \tag{1.8}$$

with two minima at $\chi = -1$ and $\chi = 1$.

The second order tensors $\boldsymbol{M} = (M_{ij})$ and $\boldsymbol{\Gamma} = (\Gamma_{ij})$ represent respectively the mobility matrix and the interfacial energy matrix. For simplicity, we shall confine ourselves to the isotropic, homogeneous case assuming that

$$\boldsymbol{M} = M\boldsymbol{I}, \quad \boldsymbol{\Gamma} = \Gamma\boldsymbol{I}, \quad M = \Gamma = 1 \tag{1.9}$$

with positive constants M, Γ normalized to unity.

System (1.1)–(1.3) represents respectively the linear momentum balance, the mass balance and a generalized equation for the chemical potential. In a thermodynamical theory due to Gurtin [Gur96] equation (1.3) is identified with a microforce balance. The free energy density underlying system (1.1)–(1.3) has the Landau-Ginzburg-Cahn-Hilliard form

$$f(\varepsilon(\boldsymbol{u}), \chi, \nabla\chi) = W(\varepsilon(\boldsymbol{u}), \chi) + \psi(\chi) + \frac{1}{2}\nabla\chi \cdot \boldsymbol{\Gamma}\nabla\chi \tag{1.10}$$

with the three terms on the right-hand side representing respectively the elastic, chemical and interfacial energy.

The remaining quantities in (1.1)–(1.3) have the following meaning: $\boldsymbol{b} : \Omega^T \to \mathbb{R}^3$ represents the external body force, and $\boldsymbol{u}_0, \boldsymbol{u}_1 : \Omega \to: \mathbb{R}^3$, $\chi_0 : \Omega \to \mathbb{R}$ are the initial conditions respectively for the displacement, the velocity and the order parameter.

The homogenous boundary conditions in (1.1)–(1.3) are chosen for the sake of simplicity. The condition $(1.1)_3$ means that the body is fixed at the boundary S, $(1.2)_3$ reflects the mass isolation at S, and $(1.3)_3$ is the natural boundary condition for (1.10).

Similarly as in [PawZaj06b], we introduce a simplified formulation of problem (1.1)-(1.3) which result on account of the constitutive equations (1.4)–(1.6) and (1.9). Let $\boldsymbol{Q}$ be the linear elasticity operator defined by

$$\boldsymbol{u} \mapsto \boldsymbol{Q}\boldsymbol{u} = \nabla \cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \bar{\mu}\Delta\boldsymbol{u} + (\bar{\lambda} + \bar{\mu})\nabla(\nabla \cdot \boldsymbol{u}). \tag{1.11}$$

Moreover, let us denote

$$\boldsymbol{B} = -\boldsymbol{A}(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad D = -\boldsymbol{B} \cdot (\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad E = -\boldsymbol{B} \cdot \bar{\boldsymbol{\varepsilon}}_a \tag{1.12}$$

where $\boldsymbol{B} = (B_{ij})$ is a symmetric, second order tensor and D, E are two scalars. With such notation we have

$$\nabla \cdot W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = \nabla \cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{A}(\bar{\boldsymbol{\varepsilon}}_a + z(\chi)(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a)) = \boldsymbol{Q}\boldsymbol{u} + z'(\chi)\boldsymbol{B}\nabla\chi, \tag{1.13}$$

and

$$W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E),$$

so that (1.1)–(1.3) simplifies to

$$\begin{aligned} &\boldsymbol{u}_{tt} - \boldsymbol{Q}\boldsymbol{u} = z'(\chi)\boldsymbol{B}\nabla\chi + \boldsymbol{b} && \text{in } \Omega^T, \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega, \\ &\boldsymbol{u} = 0 && \text{on } S^T, \end{aligned} \tag{1.14}$$

$$\begin{aligned} &\chi_t - \Delta\mu = 0 && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla\mu = 0 && \text{on } S^T, \end{aligned} \tag{1.15}$$

$$\begin{aligned} &\mu = -\Delta\chi + \psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) && \text{in } \Omega^T, \\ &\boldsymbol{n} \cdot \nabla\chi = 0 && \text{on } S^T \end{aligned} \tag{1.16}$$

with $W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)$ given by $(1.13)_2$.

Let us note that the combined systems (1.15) and (1.16) yield the following Cahn--Hilliard problem

$$\begin{aligned} &\chi_t + \Delta^2\chi = \Delta[\psi'(\chi) + z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E) && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla\chi = 0 && \text{on } S^T, \\ &\boldsymbol{n} \cdot \nabla\Delta\chi = z'(\chi)\boldsymbol{n} \cdot \nabla(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u})) && \text{on } S^T, \end{aligned} \tag{1.17}$$

coupled with the elasticity system (1.14). It is seen that the problems are coupled not only through the right-hand sides but also through the boundary conditions.
Moreover, by definition (1.7) of the shape function z, the problems uncouple for $\chi \leq -1$ and $\chi \geq 1$. As already mentioned, the boudnary coupling is characteristic for multidimensional problem and does not appear in the one-dimensional setting. In fact, in 1-D case assuming that $b = 0$ on S^T, it follows from $(1.14)_1$, $(1.14)_3$ and $(1.17)_3$ that $u_{xx} = 0$ on S^T, and consequently condition $(1.17)_4$ yields $\chi_{xxx} = 0$ on S^T. This fact was used in [PawZaj06a] in the analysis of the 1-D version of problem (1.1)–(1.3).

A time-differentiated system, analysed in [PawZaj06b], was considered with the following initial conditions corresponding respectively to $\boldsymbol{u}_{tt}(0)$, $\chi_t(0)$ and $\mu(0)$:

$$
\begin{aligned}
&\boldsymbol{u}_2 := \boldsymbol{Q}\boldsymbol{u}_0 + z'(\chi_0)\boldsymbol{B}\nabla\chi_0 + \boldsymbol{b}(0),\\
&\chi_1 := \Delta\mu_0,\\
&\mu_0 := -\Delta\chi_0 + \psi'(\chi_0) + z'(\chi_0)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + Dz(\chi_0) + E).
\end{aligned}
\tag{1.18}
$$

These conditions arise in compatibility with equations $(1.14)_1$, $(1.15)_1$ and $(1.6)_1$.

The analysis of the present paper involves twice time-differentiated system (1.14)–(1.16). To this purpose, in addition to (1.18), we define the initial conditions corresponding respectively to $\boldsymbol{u}_{ttt}(0)$, $\chi_{tt}(0)$ and $\mu_t(0)$:

$$
\begin{aligned}
&\boldsymbol{u}_3 := \boldsymbol{Q}\boldsymbol{u}_1 + z''(\chi_0)\chi_1\boldsymbol{B}\nabla\chi_0 + z'(\chi_0)\boldsymbol{B}\nabla\chi_1 + \boldsymbol{b}_t(0),\\
&\chi_2 := \Delta\mu_1,\\
&\mu_1 := -\Delta\chi_1 + \psi''(\chi_0)\chi_1 + z''(\chi_0)\chi_1(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + Dz(\chi_0) + E)\\
&\qquad + z'(\chi_0)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_1) + Dz'(\chi_0)\chi_1).
\end{aligned}
\tag{1.19}
$$

Expressions (1.19) arise as compatibility conditions for time-differentiated equations $(1.14)_1$, $(1.15)_1$ and $(1.16)_1$.

The paper is organized as follows: In Section 2 we present our main assumptions and results, stated in Theorems 2.1, 2.2 and 2.3. Theorem 2.1 asserts an improved time regularity of weak solutions to system (1.14)–(1.16), obtained by twice differentiation with respect to time variable. Theorem 2.2 states the existence of a strong solution, and Theorem 2.3 its uniqueness. In Section 3 we recall the existence and regularity results proved in [PawZaj06b] by the analysis of system (1.14)–(1.16) once differentiated with respect to time. Besides, we collect there some known results for linear elliptic problems as well as interpolation inequalities and imbeddings used in the paper. In Section 4 we introduce a Faedo-Galerkin approximation of the problem and study its twice time-differentated version. The subsequent sections 5, 6 and 7 provide the proofs respectively of Theorem 2.1, 2.2 and 2.3.

We remark that having in mind a future examination of a long time behaviour of solutions we shall record time-dependences of various constants. The obtained regularity estimates turn out to depend exponentially on time, thus in the present form are not useful for the long time analysis.

We use the following notation:

$$\begin{aligned}
&\boldsymbol{x} = (x_i)_{i=1,2,3} \quad \text{the material point,}\\
&f_{,i} = \frac{\partial f}{\partial x_i}, \quad f_t = \frac{df}{dt} \quad \text{the material space and time derivatives,}\\
&\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,2,3}, \quad W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi) = \left(\frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial \varepsilon_{ij}}\right)_{i,j=1,2,3},\\
&W_{,\chi}(\boldsymbol{\varepsilon},\chi) = \frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial \chi}, \quad \psi'(\chi) = \frac{d\psi(\chi)}{d\chi}.
\end{aligned}$$

For simplicity, whenever there is no danger of confusion, we omit the arguments $(\boldsymbol{\varepsilon},\chi)$. The specification of tensor indices is omitted as well.
Vector- and tensor-valued mappings are denoted by bold letters.

The summation convention over repeated indices is used, ad well as the notation: for vectors $\boldsymbol{a} = (a_i)$, $\tilde{\boldsymbol{a}} = (a_i)$ and tensors $\boldsymbol{B} = (B_{ij})$, $\tilde{\boldsymbol{B}} = (\tilde{B}_{ij})$, $\boldsymbol{A} = (A_{ijkl})$, we write

$$\begin{aligned}
&\boldsymbol{a}\cdot\tilde{\boldsymbol{a}} = a_i\tilde{a}_i, && \boldsymbol{B}\cdot\tilde{\boldsymbol{B}} = B_{ij}\tilde{B}_{ij},\\
&\boldsymbol{AB} = (A_{ijkl}B_{kl}), && \boldsymbol{BA} = (B_{ij}A_{ijkl}),\\
&|\boldsymbol{a}| = (a_ia_i)^{1/2}, && |\boldsymbol{B}| = (B_{ij}B_{ij})^{1/2}.
\end{aligned}$$

The symbols ∇ and $\nabla\cdot$ denote the gradient and the divergence operators with respect to the material point $\boldsymbol{x}$. For the divergence of a tensor field we use the convention of the contraction over the last index, e.g. $\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{x}) = (\varepsilon_{ij,j}(\boldsymbol{x}))$.

We use the standard Sobolev spaces notation $H^m(\Omega) = W_2^m(\Omega)$ for $m \in \mathbb{N}$. Besides,

$$\begin{aligned}
H_0^1(\Omega) &= \{v \in H^1(\Omega):\ v = 0 \text{ on } S\},\\
H_N^2(\Omega) &= \{v \in H^2(\Omega):\ \boldsymbol{n}\cdot\nabla v = 0 \text{ on } S\},
\end{aligned}$$

where $\boldsymbol{n}$ is the outward unit normal to $S = \partial\Omega$, denote the subspaces respectively of $H^1(\Omega)$ and $H^2(\Omega)$, with the standard norms of $H^1(\Omega)$ and $H^2(\Omega)$.

By bold letters we denote the spaces of vector- or tensor-valued functions, e.g.

$$\boldsymbol{L}_2(\Omega) = (L_2(\Omega))^n, \quad \boldsymbol{H}^1(\Omega) = (H^1(\Omega))^n, \quad n \in \mathbb{N};$$

if there is no confusion we do not specify dimension n. Moreover, we write

$$\|\boldsymbol{a}\|_{L_2(\Omega)} = \|\,|a|\,\|_{L_2(\Omega)}, \quad \|\boldsymbol{a}\|_{H^1(\Omega)} = \|\,|\boldsymbol{a}|\,\|_{L_2(\Omega)} + \|\,|\nabla a|\,\|_{L_2(\Omega)}$$

for the corresponding norms of a vector-valued function $\boldsymbol{a}(\boldsymbol{x}) = (a_i(\boldsymbol{x}))$; similarly for tensor-valued functions.

As common, the symbol $(\cdot,\cdot)$ denotes the scalar product in $L_2(\Omega)$. For simplicity, we use the same symbol to denote scalar products in $\boldsymbol{L}_2(\Omega) = (L_2(\Omega))^n$, e.g. we write

$$(a, \tilde{a}) = \int_\Omega a(\boldsymbol{x})\tilde{a}(\boldsymbol{x})dx, \quad (\boldsymbol{a}, \tilde{\boldsymbol{a}}) = \int_\Omega a_i(\boldsymbol{x})\tilde{a}_i(\boldsymbol{x})dx,$$
$$(\boldsymbol{B}, \tilde{\boldsymbol{B}}) = \int_\Omega B_{ij}(\boldsymbol{x})\tilde{B}_{ij}(\boldsymbol{x})dx.$$

The dual of the space V is denoted by V', and $\langle\cdot,\cdot\rangle_{V',V}$ stands for the duality pairing between V' and V.
By c and $c(T)$ we denote generic positive constants different in various instances, depending on the data of the problem and domain Ω; whenever it is of interest their dependence on parameters is specified. The argument T indicates the time horizon dependence. Moreover, δ denotes a generic, sufficiently small positive constant.

2. Assumptions and main results

System (1.1)–(1.3) (in simplified form (1.14)–(1.16)) is studied under the following assumptions:

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary S of class at least C^2; $T > 0$ is an arbitrary final time.

(A2) The coefficients of the elasticity operator $\boldsymbol{Q}$ defined by (1.11) satisfy

$$\bar{\mu} > 0, \quad 3\bar{\lambda} + 2\bar{\mu} > 0 \text{ (elasticity range)}. \tag{2.1}$$

These two conditions assure the following:

(i) Coercivity and boundedness of the operator $\boldsymbol{A}$

$$\underline{c}|\boldsymbol{\varepsilon}|^2 \le \boldsymbol{\varepsilon}\cdot\boldsymbol{A}\boldsymbol{\varepsilon} \le \bar{c}|\boldsymbol{\varepsilon}|^2 \quad \text{for all } \boldsymbol{\varepsilon} \in \boldsymbol{S}^2, \tag{2.2}$$

where $\boldsymbol{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^3$, and

$$\underline{c} = \min\{3\bar{\lambda} + 2\bar{\mu}, 2\bar{\mu}\}, \quad \bar{c} = \max\{3\bar{\lambda} + 2\bar{\mu}, 2\bar{\mu}\};$$

(ii) Strong ellipticity of the operator $\boldsymbol{Q}$ (property holding true under weaker assumption $\bar{\mu} > 0$, $\bar{\lambda}+2\bar{\mu} > 0$, see [PawZoch02], Section 7). Thanks to this property the following estimate holds true (see [Nec67], Lemma 3.2):

$$c\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \le \|\boldsymbol{Q}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for } \boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega) \tag{2.3}$$

with constant c depending on Ω.

Hence, since clearly $\|\boldsymbol{Q}\boldsymbol{u}\|_{L_2(\Omega)} \le \bar{\bar{c}}\|\boldsymbol{u}\|_{H^2(\Omega)}$, it follows that the norms $\|\boldsymbol{Q}\boldsymbol{u}\|_{L_2(\Omega)}$ and $\|\boldsymbol{u}\|_{H^2(\Omega)}$ are equivlaent on $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$.

The next two assumptions concern the ingredients of the free energy (see (1.10) with $\Gamma = \boldsymbol{I}$)

$$f(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi, \nabla\chi) = W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) + \psi(\chi) + \frac{1}{2}|\nabla\chi|^2. \tag{2.4}$$

(A3) The elastic energy $W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)$ is given by (1.4)-(1.6). The interpolation function $z : \mathbb{R} \to [0,1]$ in definition (1.6) of $\bar{\boldsymbol{\varepsilon}}(\chi)$ is at least of class C^1 with the property (1.7). Hence,

$$0 \le z(\chi) \le 1 \quad \text{and} \quad |z'(\chi)| \le c \quad \text{for all} \quad \chi \in \mathbb{R}. \tag{2.5}$$

(A4) The chemical energy $\psi(\chi)$ has the form of standard double-well potential (1.8), so

$$\psi'(\chi) = \chi^3 - \chi, \quad \psi''(\chi) = 3\chi^2 - 1, \quad \psi'''(\chi) = 6\chi. \tag{2.6}$$

Moreover, for simplicity it is assumed that

(A5) The mobility tensor $\boldsymbol{M}$ and the interfacial tensor Γ are the identities matrices $\boldsymbol{M} = \boldsymbol{I}$, $\Gamma = \boldsymbol{I}$. The second order symmetric tensor $\boldsymbol{B}$ and scalars D, E are defined in (1.12).

We note that assumptions (A3) and (A4) imply the following bounds for all $\boldsymbol{\varepsilon} \in \boldsymbol{S}^2$ and $\chi \in \mathbb{R}$:

$$\begin{aligned}
&|\bar{\boldsymbol{\varepsilon}}(\chi)| \le |\bar{\boldsymbol{\varepsilon}}_a| + |\bar{\boldsymbol{\varepsilon}}_b| \le c,\\
&|\bar{\boldsymbol{\varepsilon}}'(\chi)| = |z'(\chi)(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a)| \le c,\\
&|W(\boldsymbol{\varepsilon}, \chi)| \le \frac{1}{2}\bar{c}|\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}(\chi)|^2 \le c(|\boldsymbol{\varepsilon}|^2 + 1),\\
&|W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi)| + |W_{,\chi}(\boldsymbol{\varepsilon}, \chi)| \le c(|\boldsymbol{\varepsilon}| + 1),\\
&|\psi(\chi)| \le c(\chi^4 + 1), \quad |\psi'(\chi)| \le c(|\chi|^3 + 1)
\end{aligned} \tag{2.7}$$

with some positive constant c.

Moreover, by the Young inequality, we have

$$W(\boldsymbol{\varepsilon}, \chi) \ge \frac{1}{2}\underline{c}|\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}(\chi)|^2 \ge \frac{1}{4}\underline{c}|\boldsymbol{\varepsilon}|^2 - \frac{1}{2}\underline{c}|\bar{\boldsymbol{\varepsilon}}(\chi)|^2 \ge \frac{1}{4}\underline{c}|\boldsymbol{\varepsilon}|^2 - \underline{c}(|\bar{\boldsymbol{\varepsilon}}_a|^2 + |\bar{\boldsymbol{\varepsilon}}_b|^2), \tag{2.8}$$

and

$$\psi(\chi) \ge \frac{1}{8}\chi^4 - \frac{1}{4}.$$

This shows that free energy (2.4) satisfies the following structure condition

$$\begin{aligned}
f(\boldsymbol{\varepsilon}, \chi, \nabla\chi) &\ge \frac{1}{4}\underline{c}|\boldsymbol{\varepsilon}|^2 + \frac{1}{8}\chi^4 + \frac{1}{2}|\nabla\chi|^2 - \underline{c}(|\bar{\boldsymbol{\varepsilon}}_a|^2 + |\bar{\boldsymbol{\varepsilon}}_b|^2) - \frac{1}{4}\\
&\ge c_f(|\boldsymbol{\varepsilon}|^2 + \chi^4 + |\nabla\chi|^2) - c_f'
\end{aligned} \tag{2.9}$$

with constants $c_f > 0$ and c'_f given by

$$c_f = \min\left\{\frac{1}{4}\underline{c}, \frac{1}{8}\right\}, \quad c'_f = \underline{c}(|\bar{\varepsilon}_a|^2 + |\bar{\varepsilon}_b|^2) - \frac{1}{4}.$$

This bound plays the key role in the derivation of energy estimates for problem (1.1)–(1.3) (see Section 4).

For further purposes we recall here the following two additional properties of the operator $\boldsymbol{Q}$:

$$\begin{aligned} &\boldsymbol{Q} \text{ is selfadjoint on } \boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}^1_0(\Omega), \text{ i.e.} \\ &(\boldsymbol{Qu},\boldsymbol{v}) = -\bar{\mu}(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\bar{\lambda}+\bar{\mu})(\nabla\cdot\boldsymbol{u}, \nabla\cdot\boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{Qv}) \\ &\text{for } \ \boldsymbol{u},\boldsymbol{v}\in \boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}^1_0(\Omega), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &-\boldsymbol{Q} \text{ is positive on } \boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}^1_0(\Omega), \text{ i.e.} \\ &(-\boldsymbol{Qu},\boldsymbol{u}) = \bar{\mu}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)} + (\bar{\lambda}+\bar{\mu})\|\nabla\cdot\boldsymbol{u}\|^2_{L_2(\Omega)} \ge 0 \\ &\text{for } \ \boldsymbol{u}\in \boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}^1_0(\Omega). \end{aligned} \tag{2.11}$$

We state now the main results of the paper. The first theorem extends the regularity result of [PawZaj06b], Theorem 2.2.

Theorem 2.1. *Improved time regularity*
Let (A1)–(A5) hold, the boundary S of domain Ω be of class C^8, and

$$\begin{aligned} &z:\mathbb{R}\to[0,1] \ \text{ be of class } C^3 \ \text{ with} \\ &|z'(\chi)+|z''(\chi)|+|z'''(\chi)|\le c \quad \text{for all } \ \chi\in\mathbb{R}. \end{aligned} \tag{2.12}$$

Moreover, let the data satisfy

$$\begin{aligned} &\boldsymbol{b}\in H^1(0,T;\boldsymbol{H}^1(\Omega))\cap H^2(0,T;\boldsymbol{L}_2(\Omega)), \\ &\boldsymbol{u}_0\in \boldsymbol{H}^7(\Omega)\cap \boldsymbol{H}^1_0(\Omega), \quad \boldsymbol{u}_1\in \boldsymbol{H}^3(\Omega)\cap \boldsymbol{H}^1_0(\Omega), \quad \chi_0\in H^8\cap H^2_N(\Omega), \\ &\boldsymbol{u}_2\in \boldsymbol{H}^1_0(\Omega), \quad \chi_1\in H^4(\Omega)\cap H^2_N(\Omega), \quad \mu_0\in H^6(\Omega)\cap H^2_N(\Omega), \\ &\boldsymbol{u}_3\in \boldsymbol{L}_2(\Omega), \quad \chi_2\in L_2(\Omega), \quad \mu_1\in H^2_N(\Omega), \end{aligned} \tag{2.13}$$

where $\boldsymbol{u}_2, \chi_1, \mu_0$ and $\boldsymbol{u}_3, \chi_2, \mu_1$ are defined by (1.18), (1.19). Then there exist functions $(\boldsymbol{u}, \chi, \mu)$ such that

(2.14)

$$\begin{aligned} &\boldsymbol{u}\in Lip([0,T];\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega)), \quad \boldsymbol{u}_t\in L_\infty(0,T;\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega)), \\ &\boldsymbol{u}_{tt}\in L_\infty(0,T;\boldsymbol{H}^1_0(\Omega)), \quad \boldsymbol{u}_{ttt}\in L_\infty(0,T;\boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{tttt}\in L_2(0,T;(\boldsymbol{H}^1_0(\Omega))'), \\ &\chi\in C^1([0,T];H^2_N(\Omega)), \quad \chi_t\in C^{1/2}([0,T];H^2_N(\Omega)), \\ &\chi_{tt}\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;H^2_N(\Omega)), \quad \chi_{ttt}\in L_2(0,T;(H^2_N(\Omega))'), \\ &\mu\in Lip([0,T];H^2_N(\Omega)), \quad \mu_t\in L_\infty(0,T;H^2_N(\Omega)), \quad \mu_{tt}\in L_2(\Omega^T), \end{aligned}$$

$$
(2.15)\qquad \begin{aligned}
&\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad \boldsymbol{u}_t(0)=\boldsymbol{u}_1,\quad \boldsymbol{u}_{tt}(0)=\boldsymbol{u}_2,\quad \boldsymbol{u}_{ttt}(0)=\boldsymbol{u}_3,\\
&\chi(0)=\chi_0,\quad \chi_t(0)=\chi_1,\quad \chi_{tt}(0)=\chi_2,\\
&\mu(0)=\mu_0,\quad \mu_t(0)=\mu_1,
\end{aligned}
$$

which satisfy problem (1.14)-(1.16) in the sense of the identities

$$
(2.16)\qquad \begin{aligned}
&\int_0^T \langle \boldsymbol{u}_{tttt},\boldsymbol{\eta}\rangle_{(\boldsymbol{H}_0^1(\Omega))',\boldsymbol{H}_0^1(\Omega)}dt+\int_0^T(\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_{tt}),\boldsymbol{\varepsilon}(\boldsymbol{\eta}))dt\\
&=\int_0^T([z'(\chi)\boldsymbol{B}\nabla\chi]_{,tt}+\boldsymbol{b}_{tt},\boldsymbol{\eta})dt\quad \forall\boldsymbol{\eta}\in L_2(0,T;\boldsymbol{H}_0^1(\Omega)),\\
&\int_0^T\langle\chi_{ttt},\xi\rangle_{(H_N^2(\Omega))',H_N^2(\Omega)}dt=\int_0^T(\mu_{tt},\Delta\xi)dt\\
&\forall\xi\in L_2(0,T;H_N^2(\Omega)),\\
&\int_0^T(\mu_{tt},\zeta)dt=-\int_0^T(\Delta\chi_{tt},\zeta)dt+\int_0^T([\psi'(\chi)+W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)]_{,tt},\zeta)dt\\
&\forall\zeta\in L_2(0,T;L_2(\Omega)),
\end{aligned}
$$

where

$$
(2.17)\quad [z'(\chi)\boldsymbol{B}\nabla\chi]_{,tt}=z'''(\chi)\chi_t^2\boldsymbol{B}\nabla\chi+z''(\chi)\chi_{tt}\boldsymbol{B}\nabla\chi+2z''(\chi)\chi_t\boldsymbol{B}\nabla\chi_t+z'(\chi)\boldsymbol{B}\nabla\chi_{tt},
$$

$$
\begin{aligned}
&[\psi'(\chi)+W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)]_{,tt}=\psi'''(\chi)\chi_t^2+\psi''(\chi)\chi_{tt}\\
&\quad+z'''(\chi)\chi_t^2(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})+Dz(\chi)+E)+z''(\chi)\chi_{tt}(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})+Dz(\chi)+E)\\
&\quad+2z''(\chi)\chi_t(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t)+Dz'(\chi)\chi_t)+z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_{tt})+Dz''(\chi)\chi_t^2+Dz'(\chi)\chi_{tt}).
\end{aligned}
$$

Moreover, $(\boldsymbol{u},\chi,\mu)$ *satisfy the following estimates:*

$$
(2.18)\qquad \begin{aligned}
&\|\boldsymbol{u}_t\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))}+\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))}+\|\chi\|_{L_\infty(0,T;L_4(\Omega))}\\
&\quad+\|\nabla\chi\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))}+\|\nabla\mu\|_{\boldsymbol{L}_2(\Omega^T)}+\|\chi_t\|_{L_2(0,T;(H^1(\Omega))')}\le c_0,\\
&\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}_0^1(\Omega))}+\|\chi\|_{L_\infty(0,T;H^1(\Omega))}\le c_1,\\
&\|\chi\|_{L_2(0,T;H_N^2(\Omega))}+\|\mu\|_{L_2(0,T;H^1(\Omega))}\le c_2(T),\\
&\|\boldsymbol{u}_{tt}\|_{L_2(0,T;(\boldsymbol{H}_0^1(\Omega))')}\le c_3(T),
\end{aligned}
$$

$$
(2.19)\qquad \begin{aligned}
&\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}^2(\Omega))}+\|\boldsymbol{u}_t\|_{L_\infty(0,T;\boldsymbol{H}_0^1(\Omega))}+\|\boldsymbol{u}_{tt}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))}\le c_5(T)\\
&\|\chi\|_{C^{1/2}([0,T];H_N^2(\Omega))}+\|\chi_t\|_{L_\infty(0,T;L_2(\Omega))}+\|\chi_t\|_{L_2(0,T;H_N^2(\Omega))}\\
&\quad+\|\mu\|_{L_\infty(0,T;H_N^2(\Omega))}\le c_4(T),\\
&\|\boldsymbol{u}_{ttt}\|_{L_2(0,T;(\boldsymbol{H}_0^1(\Omega))')}+\|\chi_{tt}\|_{L_2(0,T;(H_N^2(\Omega))')}+\|\mu_t\|_{L_2(\Omega^T)}\le T^{1/2}c_5(T),
\end{aligned}
$$

and

$$
(2.20)\quad
\begin{aligned}
&\|\boldsymbol{u}\|_{Lip([0,T];\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}_t\|_{L_\infty(0,T;\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}_{tt}\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))}\\
&\quad + \|\boldsymbol{u}_{ttt}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} \le c_7(T),\\
&\|\chi\|_{C^1([0,T;H^2_N(\Omega))} + \|\chi_t\|_{C^{1/2}([0,T];H^2_N(\Omega))} + \|\chi_{tt}\|_{L_\infty(0,T;L_2(\Omega))}\\
&\quad + \|\chi_{tt}\|_{L_2(0,T;H^2_N(\Omega))} + \|\mu_t\|_{L_\infty(0,T;H^2_N(\Omega))} \le c_6(T),\\
&\|\mu_{tt}\|_{L_2(\Omega^T)} + \|\chi_{ttt}\|_{L_2(0,T;(H^2_N(\Omega))')} \le c_8(T),\\
&\|\boldsymbol{u}_{tttt}\|_{L_2(0,T;(\boldsymbol{H}^1_0(\Omega))')} \le c_9(T),
\end{aligned}
$$

with positive constants c_k, $k = 0, 1, \dots, 9$, given by

$$
(2.21)\quad
\begin{aligned}
&c_0 = c(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}, \ \|\boldsymbol{u}_1\|_{\boldsymbol{L}_2(\Omega)}, \ \|\chi_0\|_{H^1(\Omega)}, \ \|\boldsymbol{b}\|_{L_1(0,T;\boldsymbol{L}_2(\Omega))}, \ c_f, c_f'),\\
&c_1 = c(c_0, \Omega), \quad c_2(T) = c(c_1)T^{1/2}, \quad c_3(T) = c(c_0, \|\boldsymbol{b}\|_{\boldsymbol{L}_2(\Omega^T)})T^{1/2},\\
&c_4(T) = c(T^{1/2}E_1(T) + \|\chi_1\|_{L_2(\Omega)})[\exp a(T)]^{1/2}, \quad a(T) = c_0T^8\exp(cT),\\
&c_5(T) = T^{1/2}c_4(T),\\
&c_6(T) = c(T^{1/2}E_2(T) + \|\chi_2\|_{L_2(\Omega)} + T^{1/2}c_4^3(T))[\exp(c(c_1)T^2c_4^4(T))]^{1/2},\\
&c_7(T) = T^{1/2}c_6(T),\\
&c_8(T) = T^{1/2}c_4(T)c_6(T),\\
&c_9(T) = c(c_1)T^{1/2}c_6^2(T),
\end{aligned}
$$

where

$$
\begin{aligned}
E_1(T) &= T^{1/2}\|\boldsymbol{b}_t\|_{\boldsymbol{L}_2(\Omega^T)} + \|\boldsymbol{u}_2\|_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)\|_{\boldsymbol{L}_2(\Omega)},\\
E_2(T) &= T^{1/2}\|\boldsymbol{b}_{tt}\|_{\boldsymbol{L}_2(\Omega^T)} + \|\boldsymbol{u}_3\|_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_2)\|_{\boldsymbol{L}_2(\Omega)}.
\end{aligned}
$$

The next theorem asserts the existence of a strong solution to problem (1.14)–(1.16).

Theorem 2.2. *Strong solutions*
Let assumptions of Theorem 2.1 hold. then a solution $(\boldsymbol{u}, \chi, \mu)$ in Theorem 2.1 has in addition to (2.14) the following regularity

$$
(2.22)\quad
\begin{aligned}
&\boldsymbol{u} \in L_\infty(0,T;\boldsymbol{H}^3(\Omega)),\\
&\chi \in L_\infty(0,T;H^4(\Omega)),\\
&\mu \in L_\infty(0,T;H^4(\Omega)), \quad \mu_t \in L_2(0,T;H^4(\Omega)).
\end{aligned}
$$

Such solution satisfies equations $(1.14)_1$, $(1.15)_1$, $(1.16)_1$ almost everywhere, boundary conditions $(1.14)_3$, $(1.15)_3$, $(1.16)_3$ and initial conditions (2.15) in the sense of appropriate traces. Moreover, $(\boldsymbol{u}, \chi, \mu)$ satisfy estimates (2.18)–(2.20) and

$$
(2.23)\quad
\begin{aligned}
&\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}^3(\Omega))} \le c_7(T),\\
&\|\chi\|_{L_\infty(0,T;H^4(\Omega))} \le c_4^2(T)c_7(T),\\
&\|\mu\|_{L_\infty(0,T;H^4(\Omega))} + \|\mu_t\|_{L_2(0,T;H^4(\Omega))} \le c_6(T).
\end{aligned}
$$

The last theorem concerns the uniqueness of the solution.

Theorem 2.3. *Uniqueness.*
Let assumptions (A1)–(A5) hold,

$$z : \mathbb{R} \to [0,1] \text{ be of class } C^2 \text{ with} \quad |z'(\chi)| + |z''(\chi)| \le c \text{ for all } \chi \in \mathbb{R}, \tag{2.24}$$

and $(\boldsymbol{u}, \chi, \mu)$ be a solution of problem (1.14)–(1.16) such that

$$\boldsymbol{u} \in L_2(0,T; \boldsymbol{W}^1_\infty(\Omega)), \quad \chi \in L_4(0,T; L_\infty(\Omega)) \cap L_2(0,T; W^1_6(\Omega)) \tag{2.25}$$

with

$$\|\boldsymbol{u}\|_{L_2(0,T;\boldsymbol{W}^1_\infty(\Omega))} + \|\chi\|_{L_4(0,T;L_\infty(\Omega))} + \|\chi\|_{L_2(0,T;W^1_6(\Omega))} \le c(T).$$

Then the solution $(\boldsymbol{u}, \chi, \mu)$ is unique.

Corollary 2.1. *The strong solution in Theorem 2.2 is unique. The above uniqueness result does not apply to weak solutions in Theorem 2.1 since they do not satisfy the first regularity requirement in (2.25).*

3. Auxiliary results

In this section we recall first the existence and regularity results for system (1.14)–(1.16), proved in [PawZaj06b]. Besides, we collect some known results for linear elliptic problems as well as some interpolation inequalities and imbeddings which are used in the paper.

The first result concerns the existence of weak solutions bo (1.14)–(1.16).

Theorem 3.1. *[PawZaj06b] Weak solutions*
Let assumptions (A1)–(A5) hold true. Moreover, let the data satisfy

$$\begin{aligned} &\boldsymbol{b} \in \boldsymbol{L}_2(\Omega^T), \\ &\boldsymbol{u}_0 \in \boldsymbol{H}^1_0(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega), \quad \chi_0 \in H^1(\Omega). \end{aligned} \tag{3.1}$$

Then there exist functions $(\boldsymbol{u}, \chi, \mu)$ such that

$$\begin{aligned} &\boldsymbol{u} \in L_\infty(0,T; \boldsymbol{H}^1_0(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T; \boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{tt} \in L_2(0,T; (\boldsymbol{H}^1_0(\Omega))'), \\ &\chi \in L_\infty(0,T; H^1(\Omega)) \cap L_2(0,T; H^2_N(\Omega)), \quad \chi_t \in L_2(0,T; (H^1(\Omega))'), \\ &\mu \in L_2(0,T; H^1(\Omega)), \end{aligned} \tag{3.2}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \chi(0) = \chi_0, \tag{3.3}$$

which satisfy system (1.14)–(1.16) in the sense of the identities

$$
(3.4)\quad
\begin{aligned}
&\int_0^T \langle \boldsymbol{u}_{tt}, \boldsymbol{\eta}\rangle_{(\boldsymbol{H}^1_0(\Omega))',\boldsymbol{H}^1_0(\Omega)} dt + \int_0^T (\boldsymbol{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{\eta}))dt \\
&\quad = \int_0^T (z'(\chi)B\nabla\chi + \boldsymbol{b}, \boldsymbol{\eta})dt \quad \forall \boldsymbol{\eta} \in L_2(0,T; \boldsymbol{H}^1_0(\Omega)), \\
&\int_0^T \langle \chi_t, \xi\rangle_{(H^1(\Omega))',H^1(\Omega)} dt + \int_0^T (\nabla\mu, \nabla\xi)dt = 0 \\
&\forall \xi \in L_2(0,T; H^1(\Omega)), \\
&\int_0^T (\mu,\zeta)dt = -\int_0^T (\Delta\chi,\zeta)dt + \int_0^T (\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi),\zeta)dt \\
&\forall \zeta \in L_2(0,T; L_2(\Omega)).
\end{aligned}
$$

Moreover, $(\boldsymbol{u},\chi,\mu)$ satisfy a priori estimates (2.18) with constants $c_0, c_1, c_2(T), c_3(T)$ specified in (2.21).

The second theorem states time regularity whcich follows from time-differentiated system (1.14)–(1.16)

Theorem 3.2. *[PawZaj 06b] Time regularity*
Let (A1)–(A5) hold, the boundary S of domain Ω be of class C^4, and

$$
(3.5)\quad
\begin{aligned}
&z : \mathbb{R} \to [0,1] \text{ be of class } C^2 \text{ with} \\
&\quad |z'(\chi)| + |z''(\chi)| \le c \text{ for all } \chi \in \mathbb{R}.
\end{aligned}
$$

Moreover, let the data satisfy

$$
(3.6)\quad
\begin{aligned}
&\boldsymbol{b} \in H^1(0,T; \boldsymbol{L}_2(\Omega)), \\
&\boldsymbol{u}_0 \in \boldsymbol{H}^3(\Omega) \cap \boldsymbol{H}^1_0(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{H}^1(\Omega), \quad \chi_0 \in H^4(\Omega) \cap H^2_N(\Omega), \\
&\boldsymbol{u}_2 \in \boldsymbol{L}_2(\Omega), \quad \chi_1 \in L_2(\Omega), \quad \mu_0 \in H^2_N(\Omega).
\end{aligned}
$$

Then there exist functions $(\boldsymbol{u},\chi,\mu)$ such that

$$
(3.7)\quad
\begin{aligned}
&\boldsymbol{u} \in L_\infty(0,T; \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T; \boldsymbol{H}^1_0(\Omega)), \\
&\boldsymbol{u}_{tt} \in L_\infty(0,T; \boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{ttt} \in L_2(0,T; (\boldsymbol{H}^1_0(\Omega)'), \\
&\chi \in C^{1/2}([0,T]; H^2_N(\Omega)), \quad \chi_t \in L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; H^2_N(\Omega)), \\
&\chi_{tt} \in L_2(0,T; (H^2_N(\Omega))'), \\
&\mu \in L_\infty(0,T; H^2_N(\Omega)), \quad \mu_t \in L_2(\Omega^T),
\end{aligned}
$$

$$
\begin{aligned}
(3.8) \qquad & \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \boldsymbol{u}_{tt}(0) = \boldsymbol{u}_2, \\
& \chi(0) = \chi_0, \quad \chi_t(0) = \chi_1, \quad \mu(0) = \mu_0,
\end{aligned}
$$

which satisfy problem (1.14)–(1.16) in the sense of the identities

$$
\begin{aligned}
& \int_0^T \langle \boldsymbol{u}_{ttt}, \boldsymbol{\eta} \rangle_{(\boldsymbol{H}_0^1(\Omega))', \boldsymbol{H}_0^1(\Omega)} dt + \int_0^T (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))dt \\
& \quad = \int_0^T ([z'(\chi)\boldsymbol{B}\nabla\chi]_{,t} + \boldsymbol{b}_t, \boldsymbol{\eta})dt \quad \forall \boldsymbol{\eta} \in L_2(0,T; \boldsymbol{H}_0^1(\Omega)),
\end{aligned}
$$

$$
\begin{aligned}
(3.9) \qquad & \int_0^T \langle \chi_{tt}, \xi \rangle_{(H_N^2(\Omega))', H_N^2(\Omega)} dt = \int_0^T (\mu_t, \Delta\xi)dt \\
& \forall \xi \in L_2(0,T; H_N^2(\Omega)), \\
& \int_0^T (\mu_t, \zeta)dt = -\int_0^T (\Delta\chi_t, \zeta)dt + \int_0^T ([\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)]_{,t}, \zeta)dt \\
& \forall \zeta \in L_2(0,T; L_2(\Omega)),
\end{aligned}
$$

where

$$
\begin{aligned}
& [z'(\chi)\boldsymbol{B}\nabla\chi]_{,t} = z''(\chi)\chi_t\boldsymbol{B}\nabla\chi + z'(\chi)\boldsymbol{B}\nabla\chi_t, \\
(3.10) \qquad & [\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)]_{,t} = \psi''(\chi)\chi_t + z''(\chi)\chi_t(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E) \\
& \quad + z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + Dz'(\chi)\chi_t).
\end{aligned}
$$

Moreover, $(\boldsymbol{u}, \chi, \mu)$ *satisfy a priori estimates (2.18) and (2.19) with constanst* c_0, c_1, $c_2(T), c_3(T), c_4(T)$ *and* $c_5(T)$ *specified in (2.21).*

We recall now a standard elliptic regularity result for the problem

$$
\begin{aligned}
(3.11) \qquad & \Delta\chi = f \quad \text{in } \Omega, \\
& \boldsymbol{n}\cdot\nabla\chi = 0 \quad \text{on } S.
\end{aligned}
$$

Lemma 3.1. *(see e.g. [LM72a] I, Chap. 2) Let* $\Omega \subset \mathbb{R}^3$ *be a domain with boundary* S *of class* C^{2+l}, $l \geq 0$ *integer,* $f \in H^l(\Omega)$, $\int_\Omega f dx = 0$ *and* $\chi \in L_2(\Omega)$. *Then solutions of problem (3.11) satisfy the inequality*

$$
(3.12) \qquad \|\chi\|_{H^{2+l}(\Omega)} \leq c(\|f\|_{H^l(\Omega)} + \|\chi\|_{L_2(\Omega)}).
$$

The next result states the elliptic regularity for the system

$$
\begin{aligned}
(3.13) \qquad & \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \\
& \boldsymbol{u} = \boldsymbol{0} \quad \text{on } S,
\end{aligned}
$$

with the elliptic operator $\boldsymbol{Q}$ defined by (1.11). The lemma presented below is a particular case of general results due to Lions-Magenes [LionsMag72] I, Chap. 2 (see also Solonnikov [Sol66], §2, Thm. 2.2 for L_p approach).

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a domain with boundary S of class C^{2+l}, $l \geq 0$ integer, the coefficients of the operator $\boldsymbol{Q}$ satisfy $\bar{\mu} > 0$, $\bar{\lambda} + 2\bar{\mu} > 0$. Moreover, let $\boldsymbol{f} \in \boldsymbol{H}^l(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{L}_2(\Omega)$. Then solutions of problem (3.13) satisfy the inequality*

$$\|\boldsymbol{u}\|_{H^{2+l}(\Omega)} \leq c(\|\boldsymbol{f}\|_{H^l(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)}). \tag{3.14}$$

For the case $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$ we recall also the following result due to Nečas (see [Nec67], Lemma 3.2):

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^3$ be a domain with boundary S of class C^2, $\bar{\mu} > 0$, $\bar{\lambda}+2\bar{\mu} > 0$, $\boldsymbol{f} \in L_2(\Omega)$. then for solutions of problem (3.13) the following estimate holds*

$$\|\boldsymbol{u}\|_{H^2(\Omega)} \leq c\|\boldsymbol{f}\|_{L_2(\Omega)}. \tag{3.15}$$

Next, we recall the Gagliardo-Nirenberg inequality (see e.g. [BIN96], Chap. III, Sec. 15).

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with a smooth boundary. Then, for any $u \in W^l_{p_2}(\Omega) \cap L_{p_1}(\Omega)$, there exist two positive constants c_1, c_2 such that the following inequality holds:*

$$\sum_{|\alpha|=r} \|D^\alpha u\|_{L_p(\Omega)} \leq c_1 \|u\|^{1-\theta}_{L_{p_1}(\Omega)} \Big(\sum_{|\alpha|=l} \|D^\alpha u\|_{L_{p_2}(\Omega)}\Big)^\theta + c_2\|u\|_{L_q(\Omega)}, \tag{3.16}$$

provided the conditions

$$1 \leq p_1, p_2, p \leq \infty, \quad 0 \leq r < l, \quad q > 0,$$
$$\frac{n}{p} - r = (1-\theta)\frac{n}{p_1} + \theta\left(\frac{n}{p_2} - l\right), \quad \frac{r}{l} \leq \theta \leq 1,$$

with the following exception:
if $1 < p_2 < \infty$, $l - r - \frac{n}{p_2} = 0$, $p = \infty$, then (3.16) does not hold for $\theta = 1$.

The next is the following imbedding result (see e.g. [BIN96], Chap. III, Sec. 10):

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain satisfying the cone property. Let*

$$1 \leq p \leq q \leq \infty, \quad \varkappa = \left(\frac{n}{p} - \frac{n}{q}\right)\frac{1}{l} + \frac{\alpha}{l} < 1,$$

and $u \in W^l_p(\Omega)$. Then $D^\alpha u \in L_q(\Omega)$, and

$$\|D^\alpha u\|_{L_q(\Omega)} \leq \varepsilon^{1-\varkappa}\|D^l u\|_{L_p(\Omega)} + c\varepsilon^{-\varkappa}\|u\|_{L_p(\Omega)} \tag{3.17}$$

for any $\varepsilon \in (0, h_0)$, where h_0 is the height of the cone.

For later purposes we prepare also a lemma which states the imbedding $L_q(0,T; L_p(\Omega)) \subset L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; W^1_6(\Omega))$.

Lemma 3.6. *Let $u \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;W_6^1(\Omega))$ where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary. Then there exist positive constants c_1, c_2 such that*

$$\begin{aligned}\|u\|_{L_q(0,T;L_p(\Omega))} \le c_1 \|u\|^{1-2/q}_{L_\infty(0,T;L_2(\Omega))} \|\nabla u\|^{2/q}_{L_2(0,T;L_6(\Omega))} \\ + c_2 \|u\|_{q(0,T;L_2(\Omega))},\end{aligned} \tag{3.18}$$

where real numbers p, q are subject to the conditions

$$0 \le \frac{2}{q} \le 1, \quad \frac{3}{2p} + \frac{2}{q} = \frac{3}{4}. \tag{3.19}$$

Proof. According to the interpolation inequality (3.16), we have

$$\|u\|_{L_p(\Omega)} \le c_1 \|u\|^{1-\theta}_{L_2(\Omega)} \|\nabla u\|^{\theta}_{L_6(\Omega)} + c_2 \|u\|_{L_2(\Omega)}, \tag{3.20}$$

with θ satisfying

$$\theta = \frac{3}{4} - \frac{3}{2p} \quad \text{and} \quad 0 \le \theta \le 1. \tag{3.21}$$

From (3.20) it folows that

$$\begin{aligned}\|u\|_{L_q(0,T;L_p(\Omega))} &\le c_1 \left(\int_0^T \|u\|^{(1-\theta)q}_{L_2(\Omega)} \|\nabla u\|^{\theta q}_{L_6(\Omega)} dt \right)^{1/q} + c_2 \|u\|_{L_q(0,T;L_2(\Omega))} \\ &\le c_1 \|u\|^{1-\theta}_{L_\infty(0,T;L_2(\Omega))} \left(\int_0^T \|\nabla u\|^{\theta q}_{L_6(\Omega)} dt \right)^{1/q} + c_2 \|u\|_{L_q(0,T;L_2(\Omega))}.\end{aligned} \tag{3.22}$$

Hence, setting $\theta q = 2$, that is $\theta = 2/q$, we conclude from (3.21) and (3.22) the assertion of the lemma. □

4. The Faedo-Galerkin approximation

In this section we introduce first Faedo-Galerkin approximations of problem (1.14)–(1.16) and its time-differentated form. Next, we recall from [PawZaj06b] the main a priori estimates for these approximations. After such preparations we proceed to the main part of the present paper, namely the examination of an approximation corresponding to twice time-differentated problem.

4.1. Approximation

We consider the following two eigenvalue problems

$$
\begin{aligned}
-\boldsymbol{Q}\boldsymbol{v}_j &= \lambda_j \boldsymbol{v}_j \quad \text{in } \Omega,\\
\boldsymbol{v}_j &= \boldsymbol{0} \quad \text{on } S, \quad j\in\mathbb{N},
\end{aligned} \tag{4.1}
$$

where $\boldsymbol{Q}$ is the elliptic operator defined by (1.11), and

$$
\begin{aligned}
-\Delta w_j &= \lambda_j w_j \quad \text{in } \Omega,\\
\boldsymbol{n}\cdot\nabla w_j &= 0 \quad \text{on } S, \quad j\in\mathbb{N}.
\end{aligned} \tag{4.2}
$$

We recall that, by virtue of the elliptic regularity theory, if the domain Ω has the boundary of class C^l, $l\in\mathbb{N}$, then the solutions of (4.1) and (4.2) satisfy

$$
\boldsymbol{v}_j \in \boldsymbol{H}^l(\Omega), \quad w_j \in H^l(\Omega). \tag{4.3}
$$

As shown in [PawZaj06b], after normalization, the family $\{\boldsymbol{v}_j\}_{j\in\mathbb{N}}$ forms a basis of the space $\boldsymbol{H}^1_0(\Omega)$, orthonormal in $\boldsymbol{L}_2(\Omega)$ and orthogonal in $\boldsymbol{H}^1(\Omega)$ scalar products.

The family $\{w_j\}_{j\in\mathbb{N}}$ forms a basis of the space

$$
H^2_N(\Omega) = \{w\in H^2(\Omega) : \boldsymbol{n}\cdot\nabla w = 0 \text{ on } S\},
$$

and after normalization becomes orthonormal in $L_2(\Omega)$, and orthogonal in $H^1(\Omega)$ and $H^2(\Omega)$ scalar products. Furthermore, we assume without loss of generality that $w_1 = 1$.

For $m\in\mathbb{N}$ we denote by

$$
\boldsymbol{V}_{0m} = span\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\} \quad \text{and} \quad V_m = span\{w_1,\dots,w_m\}
$$

the finite dimensional subspaces, respectively of $\boldsymbol{H}^1_0(\Omega)$ and $H^2_N(\Omega)$, spanned by $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\}$ and $\{w_1,\dots,w_m\}$.

We intoduce the following Faedo-Galerkin approximation of (1.14)–(1.16): For any $m\in\mathbb{N}$ find a triple of functions $(\boldsymbol{u}^m,\chi^m,\mu^m)$ of the form

$$
\begin{aligned}
&\boldsymbol{u}^m(\boldsymbol{x},t) = \sum_{i=1}^m e_i(t)\boldsymbol{v}_i(\boldsymbol{x}), \quad \chi^m(\boldsymbol{x},t) = \sum_{i=1}^m c_i^m(t) w_i(\boldsymbol{x}),\\
&\mu^m(\boldsymbol{x},t) = \sum_{i=1}^m d_i^m(t) w_i(\boldsymbol{x}),
\end{aligned} \tag{4.4}
$$

with $e_i^m(t)$, $c_i^m(t)$, $d_i^m(t)$ being determined so that

$$
\begin{aligned}
&(\boldsymbol{u}^m_{tt},\boldsymbol{v}_j) + (\boldsymbol{A}\varepsilon(\boldsymbol{u}^m),\varepsilon(\boldsymbol{v}_j)) = (z'(\chi^m)B\nabla\chi^m + \boldsymbol{b},\boldsymbol{v}_j),\\
&(\chi^m_t, w_j) + (\nabla\mu^m,\nabla w_j) = 0,\\
&(\mu^m, w_j) = -(\Delta\chi^m, w_j) + (\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m),\chi^m), w_j), \quad j=1,\dots,m,\\
&\boldsymbol{u}^m(0) = \boldsymbol{u}^m_0, \quad \boldsymbol{u}^m_t(0) = \boldsymbol{u}^m_1, \quad \chi^m(0) = \chi^m_0,
\end{aligned} \tag{4.5}
$$

where $\boldsymbol{u}_0^m, \boldsymbol{u}_1^m \in \boldsymbol{V}_{0m}$ and $\chi_0^m \in V_m$ are the projections respectively of $\boldsymbol{u}_0, \boldsymbol{u}_1$ and χ_0 satisfying for $m \to \infty$

$$
\begin{aligned}
\boldsymbol{u}_0^m &\to \boldsymbol{u}_0 \quad \text{strongly in} \quad \boldsymbol{H}_0^1(\Omega), \\
\boldsymbol{u}_1^m &\to \boldsymbol{u}_1 \quad \text{strongly in} \quad \boldsymbol{L}_2(\Omega), \\
\chi_0^m &\to \chi_0 \quad \text{strongly in} \quad H^1(\Omega).
\end{aligned} \tag{4.6}
$$

A Faedo-Galerkin approximation corresponding to time-differentiated system (1.14)–(1.16), studied in [PawZaj06b], is

$$
\begin{aligned}
&(\boldsymbol{u}_{ttt}^m, \boldsymbol{v}_j) + (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m), \boldsymbol{\varepsilon}(\boldsymbol{v}_j)) = ([z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,t} + \boldsymbol{b}_t, \boldsymbol{v}_j), \\
&(\chi_{tt}^m, w_j) - (\mu_t^m, \Delta w_j) = 0, \\
&(\mu_t^m, w_j) = (-\Delta\chi_t^m, w_j) + ([\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,t}, w_j), \quad j = 1, \dots, m,
\end{aligned} \tag{4.7}
$$

where the explicit forms of $[z'(\chi)\boldsymbol{B}\nabla\chi]_{,t}$ and $[\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)]_{,t}$ are given in (3.10). System (4.7) is considered with the initial conditions

$$
\begin{aligned}
&\boldsymbol{u}^m(0) = \boldsymbol{u}_0^m, \quad \boldsymbol{u}_t^m(0) = \boldsymbol{u}_1^m, \quad \chi^m(0) = \chi_0, \\
&\boldsymbol{u}_{tt}^m(0) = \boldsymbol{u}_2^m, \quad \chi_t^m(0) = \chi_1^m, \quad \mu^m(0) = \mu_0^m,
\end{aligned} \tag{4.8}
$$

where $\boldsymbol{u}_2^m \in \boldsymbol{V}_{0m}$, $\chi_1^m, \mu_0^m \in V_m$ are the projections respectively of $\boldsymbol{u}_2, \chi_1$ and μ_0 defined in (1.18), and such that the following convergences in the strong sense hold:

$$
\begin{aligned}
&\boldsymbol{u}_0^m \to \boldsymbol{u}_0 \ \text{in} \ \boldsymbol{H}^3(\Omega) \cap \boldsymbol{H}_0^1(\Omega), && \boldsymbol{u}_1^m \to \boldsymbol{u}_1 \ \text{in} \ \boldsymbol{H}_0^1(\Omega), \\
&\chi_0^m \to \chi_0 \ \text{in} \ H^4(\Omega) \cap H_N^2(\Omega), && \boldsymbol{u}_2^m \to \boldsymbol{u}_2 \ \text{in} \ \boldsymbol{L}_2(\Omega), \\
&\chi_1^m \to \chi_1 \ \text{in} \ L_2(\Omega), && \mu_0^m \to \mu \ \text{in} \ H_N^2(\Omega).
\end{aligned} \tag{4.9}
$$

4.2. The energy and time-regularity estimates

It has been proved in [PawZaj06b] (see Lemmas 4.1–4.4) that under assumptions in Theorem 3.2 a solution $(\boldsymbol{u}^m, \chi^m, \mu^m)$ of system (4.7), (4.8) satisfies the following uniform (in m) energy estimates

$$
\begin{aligned}
&\|\boldsymbol{u}_t^m\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}^m)\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\chi^m\|_{L_\infty(0,T;L_4(\Omega))} \\
&\quad + \|\nabla\chi^m\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\nabla\mu^m\|_{\boldsymbol{L}_2(\Omega^T)} + \|\chi_t^m\|_{L_2(0,T;(H^1(\Omega))')} \le c_0, \\
&\|\boldsymbol{u}^m\|_{L_\infty(0,T;\boldsymbol{H}_0^1(\Omega))} + \|\chi^m\|_{L_\infty(0,T;H^1(\Omega))} \le c_1, \\
&\|\chi^m\|_{L_2(0,t;H_N^2(\Omega))} + \|\mu^m\|_{L_2(0,t;H^1(\Omega))} \le c_2(t), \\
&\|u_{t't'}^m\|_{L_2(0,t;(\boldsymbol{H}_0^1(\Omega))')} \le c_3(t) \quad \text{for} \ \ t \in (0, T],
\end{aligned} \tag{4.10}
$$

and time-regularity estimates

$$
(4.11)\quad
\begin{aligned}
&\|\boldsymbol{u}^m\|_{L_\infty(0,t;\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}^m_{t'}\|_{L_\infty(0,t;\boldsymbol{H}^1_0(\Omega))} + \|\boldsymbol{u}^m_{t't'}\|_{L_\infty(0,t;\boldsymbol{L}_2(\Omega))} \le c_5(t),\\
&\|\chi^m\|_{C^{1/2}([0,t];H^2_N(\Omega))} + \|\chi^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))} + \|\chi^m_{t'}\|_{L_2(0,t;H^2_N(\Omega))}\\
&\quad + \|\mu^m\|_{L_\infty(0,t;H^2_N(\Omega))} + \|\boldsymbol{u}^m_{t't't'}\|_{L_2(0,t;(\boldsymbol{H}^1_0(\Omega))')} \le c_4(t),\\
&\|\mu^m_{t'}\|_{L_2(\Omega^t)} + \|\chi^m_{t't'}\|_{L_2(0,t;(H^2_N(\Omega))')} \le t^{1/2}c_5(t)
\end{aligned}
$$

for $t \in (0,T]$, with positive constants independent of m, given by

$$
(4.12)\quad
\begin{aligned}
&c_0 = c_0(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}, \|\boldsymbol{u}_1\|_{\boldsymbol{L}_2(\Omega)}, \|\chi_0\|_{H^1(\Omega)}, \|\boldsymbol{b}\|_{L_1(0,t;\boldsymbol{L}_2(\Omega))}, c_f, c_f'),\\
&c_1 = c(c_0,\Omega), \quad c_2(t) = c(c_1)t^{1/2}, \quad c_3(t) = c(c_0, \|\boldsymbol{b}\|_{\boldsymbol{L}_2(\Omega^t)})t^{1/2},\\
&c_4(t) = c(t^{1/2}E_1(t) + \|\chi_1\|_{L_2(\Omega)})[\exp a(t)]^{1/2}, \quad a(t) = c_0t^8\exp(ct),\\
&c_5(t) = t^{1/2}c_4(t),
\end{aligned}
$$

where

$$
E_1(t) = t^{1/2}\|\boldsymbol{b}_{t'}\|_{\boldsymbol{L}_2(\Omega^t)} + \|\boldsymbol{u}_2\|_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)\|_{\boldsymbol{L}_2(\Omega)}.
$$

4.3. The improved time-regularity estimates

In this section we assume that the boundary S of domain Ω is at least of class C^8. We introduce a Faedo-Galerkin approximation corresponding to twice time-differentiated system:

$$
(4.13)\quad
\begin{aligned}
&(\boldsymbol{u}^m_{tttt}, \boldsymbol{v}_j) + (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}^m_{tt}), \boldsymbol{\varepsilon}(\boldsymbol{v}_j)) = ([z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,tt} + \boldsymbol{b}_{tt}, \boldsymbol{v}_j),\\
&(\chi^m_{ttt}, w_j) - (\mu^m_{tt}, \Delta w_j) = 0,\\
&(\mu^m_{tt}, w_j) = -(\Delta\chi^m_{tt}, w_j) + ([\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,tt}, w_j), \quad j = 1,\dots,m,
\end{aligned}
$$

where the explicit forms of $[z'(\chi)\boldsymbol{B}\nabla\chi]_{,tt}$ and $[\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)]_{,tt}$ are given in (2.17). System (4.13) is considered with the initial conditions

$$
(4.14)\quad
\begin{aligned}
&\boldsymbol{u}^m(0) = \boldsymbol{u}^m_0, \quad \boldsymbol{u}^m_t(0) = \boldsymbol{u}^m_1, \quad \chi^m(0) = \chi^m_0,\\
&\boldsymbol{u}^m_{tt}(0) = \boldsymbol{u}^m_2, \quad \chi^m_t(0) = \chi^m_1, \quad \mu^m(0) = \mu^m_0,\\
&\boldsymbol{u}^m_{ttt}(0) = \boldsymbol{u}^m_3, \quad \chi^m_{tt}(0) = \chi^m_2, \quad \mu^m_t(0) = \mu^m_1,
\end{aligned}
$$

where $\boldsymbol{u}^m_0, \boldsymbol{u}^m_1, \boldsymbol{u}^m_2, \boldsymbol{u}^m_3 \in \boldsymbol{V}_{0m}$, $\chi^m_0, \chi^m_1, \chi^m_2, \mu^m_0, \mu^m_1 \in V_m$ are the projections of the corresponding data, with $\boldsymbol{u}_2, \chi_1, \mu_0$ and $\boldsymbol{u}_3, \chi_2, \mu_1$ defined in (1.18), (1.19).
We assume that the following convergences in the strong sense are satisfied:

$$
(4.15)\quad
\begin{aligned}
&\boldsymbol{u}^m_0 \to \boldsymbol{u}_0 \text{ in } \boldsymbol{H}^7(\Omega)\cap\boldsymbol{H}^1_0(\Omega), && \boldsymbol{u}^m_1 \to \boldsymbol{u}_1 \text{ in } \boldsymbol{H}^3(\Omega)\cap\boldsymbol{H}^1_0(\Omega),\\
&\chi^m_0 \to \chi_0 \text{ in } H^8(\Omega)\cap H^2_N(\Omega), && \boldsymbol{u}^m_2 \to \boldsymbol{u}_2 \text{ in } \boldsymbol{H}^1_0(\Omega),\\
&\chi^m_1 \to \chi_1 \text{ in } H^4(\Omega)\cap H^2_N(\Omega), && \mu^m_0 \to \mu_0 \text{ in } H^6(\Omega)\cap H^2_N(\Omega),\\
&\boldsymbol{u}^m_3 \to \boldsymbol{u}_3 \text{ in } \boldsymbol{L}_2(\Omega), && \chi^m_2 \to \chi_2 \text{ in } L_2(\Omega),\\
&\mu^m_1 \to \mu_1 \text{ in } H^2_N(\Omega).
\end{aligned}
$$

System (4.13), (4.14) represents an initial value problem for a system of ordinary differential equations with the nonlinear terms on the right-hand side being continuous functions of their arguments. Hence, it has a solution on an interval $[0, T_m]$, $T_m > 0$. A priori estimates (uniform in m) proved in Lemmas 4.1–4.4 below show that $T_m = T$.

Lemma 4.1. *Let (A1)–(A5) hold, the boundary S of domain Ω be of class C^8, and*

$$\begin{aligned} &z : \mathbb{R} \to [0,1] \text{ be of class } C^3 \text{ with} \\ &|z'(\chi)| + |z''(\chi)| + |z'''(\chi)| \le c \quad \text{for all } \chi \in \mathbb{R}. \end{aligned} \tag{4.16}$$

Moreover, let the data satisfy

$$\begin{aligned} &\boldsymbol{b} \in \boldsymbol{H}^1(0,T;\boldsymbol{H}^1(\Omega)) \cap \boldsymbol{H}^2(0,T;\boldsymbol{L}_2(\Omega)), \\ &\boldsymbol{u}_0 \in \boldsymbol{H}^7(\Omega) \cap \boldsymbol{H}_0^1(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{H}^3(\Omega) \cap \boldsymbol{H}_0^1(\Omega), \quad \chi_1 \in H^8(\Omega) \cap H_N^2(\Omega), \\ &\boldsymbol{u}_2 \in \boldsymbol{H}_0^1(\Omega), \quad \chi_1 \in H^4(\Omega) \cap H_N^2(\Omega), \quad \mu_0 \in H^6(\Omega) \cap H_N^2(\Omega), \\ &\boldsymbol{u}_3 \in \boldsymbol{L}_2(\Omega), \quad \chi_2 \in L_2(\Omega), \quad \mu_1 \in H_N^2(\Omega). \end{aligned} \tag{4.17}$$

Then solutions $(\boldsymbol{u}^m, \chi^m, \mu^m)$ of system (4.13), (4.14) satisfy estimates (4.10), (4.11) with constants $c_0, c_1, c_2(t), c_3(t)$ and $c_4(t)$ given in (4.12), and

$$\begin{aligned} &\|\boldsymbol{u}^m_{t't't'}\|_{L_\infty(0,t;L_2(\Omega))} + \|\boldsymbol{u}^m_{t't'}\|_{L_\infty(0,t;H^1(\Omega))} \le c_7(t), \\ &\|\chi^m_{t't'}\|_{L_\infty(0,t;L_2(\Omega))} + \|\chi^m_{t't'}\|_{L_2(0,t;H^2_N(\Omega))} \le c_6(t), \quad t \in (0,T], \end{aligned} \tag{4.18}$$

with constants $c_6(t), c_7(t)$ independent of m, given by

$$\begin{aligned} &c_6(t) = c(t^{1/2}E_2(t) + \|\chi_2\|_{L_2(\Omega)} + t^{1/2}c_4^3(t))[\exp(c(c_1)t^2c_4^4(t))]^{1/2}, \\ &c_7(t) = t^{1/2}c_6(t), \end{aligned} \tag{4.19}$$

where

$$E_2(t) = t^{1/2}\|\boldsymbol{b}_{t't'}\|_{L_2(\Omega^t)} + \|\boldsymbol{u}_3\|_{L_2(\Omega)} + \|\varepsilon(\boldsymbol{u}_2)\|_{L_2(\Omega)}.$$

Proof. Estimates (4.10), (4.11) were proved in [PawZaj06b], Lemmas 3.1–3.4, 4.1–4.4. To show (4.18) we proceed similarly as in [PawZaj06b], Lemma 4.1.

In the first step we estimate $\boldsymbol{u}^m_{tt}$ in terms of the $L_2(0,t;H^2(\Omega))$-norm of χ^m_{tt}. Setting $\boldsymbol{u}^m_{ttt}(t)$ as test function in $(4.13)_1$ gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (|\boldsymbol{u}^m_{ttt}|^2 + \varepsilon(\boldsymbol{u}^m_{tt}) \cdot \boldsymbol{A}\varepsilon(\boldsymbol{u}^m_{tt}))dx = \int_\Omega ([z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,tt} + \boldsymbol{b}_{tt}) \cdot \boldsymbol{u}^m_{ttt}dx. \tag{4.20}$$

Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned} &\frac{d}{dt}\bigg(\int_\Omega (|\boldsymbol{u}^m_{ttt}|^2 + \varepsilon(\boldsymbol{u}^m_{tt}) \cdot \boldsymbol{A}\varepsilon(\boldsymbol{u}^m_{tt}))dx\bigg)^{1/2} \\ &\le \bigg(\int_\Omega (|[z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,tt}|^2 + |\boldsymbol{b}_{tt}|^2)dx\bigg)^{1/2}. \end{aligned} \tag{4.21}$$

Integrating (4.21) with respect to time leads to

$$
\begin{aligned}
&\int_\Omega (|u^m_{ttt}|^2 + |\varepsilon(u^m_{tt}|^2)dx \\
&\le c\Big[\int_0^t \Big(\int_\Omega |[z'(\chi^m)B\nabla\chi^m]_{,t't'}|^2 dx\Big)^{1/2} dt'\Big]^2 + c(E^m_2(t))^2 \\
&\le c\Big[\int_0^t \Big(\int_\Omega (\chi^m_{t'})^4|\nabla\chi^m|^2 dx\Big)^{1/2} dt'\Big]^2 + c\Big[\int_0^t \Big(\int_\Omega (\chi^m_{t't'})^2|\nabla\chi^m|^2 dx\Big)^{1/2} dt'\Big]^2 \\
&\quad + c\Big[\int_0^t \Big(\int_\Omega (\chi^m_{t'})^2|\nabla\chi^m_{t'}|^2 dx\Big)^{1/2} dt'\Big]^2 + c\Big[\int_0^t \Big(\int_\Omega |\nabla\chi^m_{t't'}|^2 dx\Big)^{1/2} dt'\Big]^2 \\
&\quad + c(E^m_2(t))^2 \equiv c\sum_{k=1}^4 I_k + c(E^m_2(t))^2,
\end{aligned}
\tag{4.22}
$$

where

$$E^m_2(t) := t^{1/2}b_{t't'}\|_{L_2(\Omega^t)} + \|u^m_3\|_{L_2(\Omega)} + \|\varepsilon(u^m_2)\|_{L_2(\Omega)}.$$

Clearly, due to convergences (4.15), $E^m_2(t) \le cE_2(t)$. We proceed to estimate the terms I_k, $k = 1,\dots,4$. An application of the Hölder inequality yields

$$
\begin{aligned}
I_1 &\le \Big[\int_0^t \|\chi^m_{t'}\|^2_{L_6(\Omega)}\|\nabla\chi^m\|_{L_6(\Omega)}dt'\Big]^2 \\
&\le \|\chi^m_{t'}\|^4_{L_4(0,t;L_6(\Omega))}\|\nabla\chi^m\|^2_{L_2(0,t;L_6(\Omega))} \equiv J_1.
\end{aligned}
$$

Now, let us note that by virtue of estimate $(4.10)_3$ and the Sobolev imbedding,

$$\|\nabla\chi^m\|_{L_2(0,t;L_6(\Omega))} \le c_2(t) = c(c_1)t^{1/2}. \tag{4.23}$$

Further, with the help of the interpolation result in Lemma 3.6, we obtain

$$\|\chi^m_{t'}\|_{L_4(0,t;L_6(\Omega))} \le c\|\chi^m_{t'}\|^{1/2}_{L_\infty(0,t;L_2(\Omega))}\|\nabla\chi^m_{t'}\|^{1/2}_{L_2(0,t;L_6(\Omega))} + c\|\chi^m_{t'}\|_{L_4(0,t;L_2(\Omega))},$$

which on account of estimate $(4.11)_2$ implies that

$$\|\chi^m_{t'}\|_{L_4(0,t;L_6(\Omega))} \le c_4(t). \tag{4.24}$$

Hence, by (4.23) and (4.24),

$$J_1 \le c_2(t)c_4(t).$$

Next, again with the help of the Hölder inequality,

$$I_2 \le \left[\int_0^t \|\chi^m_{t't'}\|_{L_3(\Omega)}\|\nabla\chi^m\|_{L_6(\Omega)}dt'\right]^2$$
$$\le \|\chi^m_{t't'}\|^2_{L_2(0,t;L_3(\Omega))}\|\nabla\chi^m\|^2_{L_2(0,t;L_6(\Omega))} \equiv J_2.$$

Using (4.23) and an interpolation inequality gives

$$J_2 \le c_2(t)\|\chi^m_{t't'}\|^2_{L_2(0,t;L_3(\Omega))}$$
$$\le c(c_1)t^{1/2}\int_0^t(\delta_1\|\nabla^2\chi^m_{t't'}\|^2_{L_2(\Omega)} + c(1/\delta_1)\|\chi^m_{t't'}\|^2_{L_2(\Omega)})dt',$$

where $\delta_1 > 0$. Turnig to the next term, we have

$$I_3 \le \left[\int_0^t \|\chi^m_{t'}\|_{L_3(\Omega)}\|\nabla\chi^m_{t'}\|_{L_6(\Omega)}dt'\right]^2$$
$$\le \|\chi^m_{t'}\|^2_{L_2(0,t;L_3(\Omega))}\|\nabla\chi^m_{t'}\|^2_{L_2(0,t;L_6(\Omega))} \equiv J_3.$$

Thanks to estimate $(4.11)_2$ and the Sobolev imbedding,

$$J_3 \le c_4^4(t).$$

The last term is estimated with the help of an interpolation inequality, similarly as J_2:

$$I_4 \le t\int_0^t \|\nabla\chi^m_{t't'}\|^2_{L_2(\Omega)}dt'$$
$$\le t\int_0^t(\delta_1\|\nabla^2\chi^m_{t't'}\|^2_{L_2(\Omega)} + c(1/\delta_1)\|\chi^m_{t't'}\|^2_{L_2(\Omega)})dt',$$

where $\delta_1 > 0$. Combining the above estimates in (4.22), we arrive at

$$\begin{aligned}&\|\boldsymbol{u}^m_{ttt}\|^2_{L_2(\Omega)} + \|\varepsilon(\boldsymbol{u}^m_{tt})\|^2_{L_2(\Omega)}\\ &\le c(c_1)(1+t)\int_0^t(\delta_1\|\nabla^2\chi^m_{t't'}\|^2_{L_2(\Omega)} + c(1/\delta_1)\|\chi^m_{t't'}\|^2_{L_2(\Omega)})dt'\\ &\quad + c_4^4(t) + c(E_2(t))^2.\end{aligned} \tag{4.25}$$

In the second step we consider system $(4.13)_2$, $(4.13)_3$ rewritten in the form of the following equation (due to (4.2)):

$$(\chi^m_{ttt}, w_j) = -(\Delta\chi^m_{tt}, \Delta w_j) + ([\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,tt}, \Delta w_j). \tag{4.26}$$

Testing (4.26) by $\chi^m_{tt}(t)$ yields

$$\frac{1}{2}\frac{d}{dt}\int\limits_\Omega (\chi^m_{tt})^2 dx + \int\limits_\Omega (\Delta\chi^m_{tt})^2 dx = \int\limits_\Omega [\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,tt}\Delta\chi^m_{tt} dx.$$

Hence, by the Young inequality it follows that

$$\frac{d}{dt}\int\limits_\Omega (\chi^m_{tt})^2 dx + \int\limits_\Omega (\Delta\chi^m_{tt})^2 dx \le \int\limits_\Omega [\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]^2_{,tt} dx. \tag{4.27}$$

Integrating (4.27) with respect to time, using expression $(2.17)_2$ for $[\psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}), \chi)]_{,tt}$ and the assumptions on ψ and z, we obtain

$$\begin{aligned}
&\int\limits_\Omega (\chi^m_{tt})^2 dx + \int\limits_0^t\int\limits_\Omega (\Delta\chi_{t't'})^2 dx dt' \\
&\le c\int\limits_0^t\int\limits_\Omega [(\chi^m)^2(\chi^m_{t'})^4 + (\chi^m)^4(\chi^m_{t't'})^2 + (\chi^m_{t't'})^2 + (\chi^m_{t'})^4|\varepsilon(\boldsymbol{u}^m)|^2 \\
&\quad + (\chi^m_{t'})^4 + (\chi^m_{t't'})^2|\varepsilon(\boldsymbol{u}^m)|^2 + (\chi^m_{t'})^2|\varepsilon(\boldsymbol{u}^m_{t'})|^2 + |\varepsilon(\boldsymbol{u}^m_{t't'})|^2] dx dt' \\
&\quad + \int\limits_\Omega (\chi^m_2)^2 dx \equiv c\sum_{i=1}^8 K_i + \|\chi^m_2\|^2_{L_2(\Omega)}.
\end{aligned} \tag{4.28}$$

Clearly, due to convergences (4.15), $\|\chi^m_2\|_{L_2(\Omega)} \le c\|\chi_2\|_{L_2(\Omega)}$.

We shall estimate now the subsequent terms K_i, $i = 1, \dots, 8$. Applying the Hölder inequality and then using estimates $(4.10)_2$, (4.24), yields

$$\begin{aligned}
K_1 &\le \int\limits_0^t \Big(\int\limits_\Omega (\chi^m)^6 dx\Big)^{1/3}\Big(\int\limits_\Omega \chi^m_{t'})^6 dx\Big)^{2/3} dt' \\
&\le \sup_{t'}\|\chi^m\|^2_{L_6(\Omega)}\int\limits_0^t \|\chi^m_{t'}\|^4_{L_6(\Omega)} dt' \le c_1^2 c_4^4(t).
\end{aligned}$$

Next,

$$\begin{aligned}
K_2 &\le \int\limits_0^t \Big(\int\limits_\Omega (\chi^m)^6 dx\Big)^{2/3}\Big(\int\limits_\Omega (\chi^m_{t't'})^6 dx\Big)^{1/3} dt' \\
&\le \sup_{t'}\|\chi^m\|^4_{L_6(\Omega)}\int\limits_0^t \|\chi^m_{t't'}\|^2_{L_6(\Omega)} dt' \equiv L_2.
\end{aligned}$$

Estimating the first factor of L_2 by $(4.10)_2$ and applying an interpolation inequality to the second one, gives

$$L_2 \le c_1^4 \int_0^t (\delta_2 \|\nabla^2 \chi^m_{t't'}\|^2_{L_2(\Omega)} + c(1/\delta_2)\|\chi^m_{t't'}\|^2_{L_2(\Omega)} dt', \tag{4.29}$$

where $\delta_2 > 0$. The next term

$$K_3 = \int_0^t \|\chi^m_{t't'}\|^2_{L_2(\Omega)} dt'$$

joins the second integral on the right-hand side of (4.29). Further, again by the Hölder inequality and on account of estimates $(4.11)_1$, (4.24), we obtain

$$K_4 \le \int_0^t \bigg(\int_\Omega (\chi^m_{t'})^6 dx\bigg)^{2/3} \bigg(\int_\Omega |\varepsilon(u^m)|^6 dx\bigg)^{1/3} dt'$$
$$\le \sup_{t'} \|\varepsilon(u^m\|^2_{L_6(\Omega)} \int_0^t \|\chi^m_{t'}\|^4_{L_6(\Omega)} dt' \le c_5^2(t) c_4^4(t) \le t c_4^6(t).$$

Similarly, due to (4.24),

$$K_5 = \int_0^t \|\chi^m_{t'}\|^4_{L_4(\Omega)} dt' \le c_4^4(t).$$

Continuing, we have

$$K_6 \le \int_0^t \bigg(\int_\Omega (\chi^m_{t't'})^3 dx\bigg)^{2/3} \bigg(\int_\Omega |\varepsilon(u^m)|^6 dx\bigg)^{1/3} dt'$$
$$\le \sup_{t'} \|\varepsilon(u^m)\|^2_{L_6(\Omega)} \int_0^t \|\chi^m_{t't'}\|^2_{L_3(\Omega)} dt' \equiv L_6.$$

Estimating the first factor of L_6 by $(4.11)_1$ and applying an interpolation inequality to the second one, it follows that

$$L_6 \le t c_4^2(t) \int_0^t (\delta_2 \|\nabla^2 \chi^m_{t't'}\|^2_{L_2(\Omega)} + c(1/\delta_2)\|\chi^m_{t't'}\|^2_{L_2(\Omega)}) dt',$$

where $\delta_2 > 0$. Further,

$$K_7 \leq \int_0^t \|\chi_{t'}^m\|^2_{L_\infty(\Omega)} \|\varepsilon(u_{t'}^m)\|^2_{L_2(\Omega)} dt'$$

$$\leq \sup_{t'} \|\varepsilon(u_{t'}^m)\|^2_{L_2(\Omega)} \int_0^t \|\chi_{t'}^m\|^2_{L_\infty(\Omega)} dt' \leq c_5^2(t) c_4^2(t) \leq t c_4^4(t),$$

where in the last inequality we applied estimates $(4.11)_{1,2}$ and the imbedding $L_\infty(\Omega) \subset H^2(\Omega)$. The last term

$$K_8 = \int_0^t \|\varepsilon(u_{t't'}^m)\|^2_{L_2(\Omega)} dt'$$

is left for further treatment by means of estimate (4.25). In conclusion, combining the obtained estimates in (4.28), yields

$$\begin{aligned} &\|\chi_{tt}^m\|^2_{L_2(\Omega)} + \int_0^t \|\Delta \chi_{t't'}\|^2_{L_2(\Omega)} dt' \\ &\leq t c_4^6(t) + c(c_1)(1 + t c_4^2(t)) \int_0^t (\delta_2 \|\nabla^2 \chi_{t't'}^m\|^2_{L_2(\Omega)} + c(1/\delta_2) \|\chi_{t't'}^m\|^2_{L_2(\Omega)}) dt' \\ &+ c \int_0^t \|\varepsilon(u_{t't'}^m)\|^2_{L_2(\Omega)} dt' + c\|\chi_2\|^2_{L_2(\Omega)}. \end{aligned} \tag{4.30}$$

Now we apply estimate (4.25) to the last but one term on the right-hand side of (4.30). Besides, we use the inequality

$$\|\chi_{tt}^m\|_{H^2(\Omega)} \leq c\|\Delta \chi_{tt}^m\|_{L_2(\Omega)},$$

which holds true in view of the fact that $\int_\Omega \chi_{tt}^m dx = 0$. The latter equality results from $(4.7)_2$ setting $w_j = 1$ (admissible by assumption). Consequently, we obtain

$$\begin{aligned} &\|\chi_{tt}^m\|^2_{L_2(\Omega)} + \int_0^t \|\chi_{t't'}^m\|^2_{H^2(\Omega)} dt' \\ &\leq c(c_1)(1 + t^2 + t c_4^2(t)) \int_0^t [(\delta_1 + \delta_2) \|\nabla^2 \chi_{t't'}^m\|^2_{L_2(\Omega)} \\ &+ (c(1/\delta_1) + c(1/\delta_2)) \|\chi_{t't'}^m\|^2_{L_2(\Omega)}] dt' \\ &+ t(c_4^4(t) + c_4^6(t)) + ct(E_2(t))^2 + c\|\chi_2\|^2_{L_2(\Omega)}. \end{aligned} \tag{4.31}$$

With an appropriate choice of constants δ_1, δ_2, e.g.

$$\delta_1 + \delta_2 = \frac{1}{2}(c(c_1)tc_4^2(t))^{-1},$$

the first term on the right-hand side of (4.31) can be absorbed by the left-hand side. This leads to

$$\|\chi_{tt}^m(t)\|_{L_2(\Omega)}^2 + \int_0^t \|\chi_{t't'}^m\|_{H^2(\Omega)}^2 dt' \le c(c_1)t^2c_4^4(t) \int_0^t \|\chi_{t't'}^m\|_{L_2(\Omega)}^2 dt'$$
$$+ tc_4^6(t) + ct(E_2(t))^2 + c\|\chi_2\|_{L_2(\Omega)}^2.$$

Hence, by the Gronwall lemma, it follows that

$$\begin{aligned} &\|\chi_{tt}^m\|_{L_2(\Omega)}^2 + \int_0^t \|\chi_{t't'}^m\|_{H^2(\Omega)}^2 dt' \\ (4.32)\qquad &\le c(t(E_2(t))^2 + \|\chi_2\|_{L_2(\Omega)}^2 + tc_4^6(t)) \exp(c(c_1)t^2c_4^4(t)) \\ &\le c_6^2(t) \quad \text{for} \quad t \in (0,T], \end{aligned}$$

where $c_6(t)$ is defined in (4.19). this proves estimate $(4.18)_2$. Applying (4.32) in (4.25) (with $\delta_1 = 1$) gives

$$\|u_{ttt}^m\|_{L_2(\Omega)}^2 + \|\varepsilon(u_{tt}^m)\|_{L_2(\Omega)}^2 \le c(c_1)(1+t)c_6^2(t) + c_4^4(t) + c(E_2(t))^2 \le c_7^2(t).$$

Hence, by virtue of Korn's inequality, we conclude estimate $(4.18)_1$. Thereby the proof is completed. □

4.4. Further estimates

Firstly, we note that in view of the inequality

$$|\chi_{txx}(t) - \chi_{txx}(t')| \le |t-t'|^{1/2}\left(\int_{t'}^t \chi_{t''txx}^2 dt''\right)^{1/2},$$

estimate $(4.18)_2$ implies that $\chi_t^m \in C^{1/2}([0,t]; H_N^2(\Omega))$, and

$$\|\chi_{t'}^m\|_{C^{1/2}([0,t];H_N^2(\Omega))} \le c_6(t). \qquad (4.33)$$

Next, we prove the following

Lemma 4.2. *Let assumptions of Lemma 4.1 hold. Then, for* $t \in (0,T]$,

$$\|u^m_{t'}\|_{L_\infty(0,t;H^2(\Omega))} \le c_7(t). \tag{4.34}$$

Proof. On account of (4.1), identity $(4.7)_1$ can be rewritten in the form

$$(u^m_{ttt}, Qv_j) - (Qu^m_t, Qv_j) = ([z'(\chi^m)B\nabla\chi^m]_{,t} + b_t, Qv_j).$$

Testing the above equation by $u^m_t(t)$ gives

$$\|Qu^m_t\|^2_{L_2(\Omega)} = (u^m_{ttt} - [z'(\chi^m)B\nabla\chi^m]_{,t} - b_t, Qu^m_t) \quad \text{for a.a. } t \in (0,T].$$

Hence, using the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} \|Qu^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))} &\le \|u^m_{t't't'}\|_{L_\infty(0,t;L_2(\Omega))} \\ &+ \|[z'(\chi^m)B\nabla\chi^m]_{,t'}\|_{L_\infty(0,t;L_2(\Omega))} + \|b_{t'}\|_{L_\infty(0,t;L_2(\Omega))}. \end{aligned} \tag{4.35}$$

The first term on the right-hand side of (4.35) is, due to (4.18), bounded by constant $c_7(t)$. Next, in view of $(2.17)_1$, using assumptions on z and then estimates (4.10), (4.33), we obtain

$$\begin{aligned} &\|[z'(\chi^m)B\nabla\chi^m]_{,t'}\|_{L_\infty(0,t;L_2(\Omega))} \\ &\le c(\|\chi^m_{t'}\nabla\chi^m\|_{L_\infty(0,t;L_2(\Omega))} + \|\nabla\chi^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))}) \\ &\le c(\|\chi^m_{t'}\|_{L_\infty(\Omega^t)}\|\nabla\chi^m\|_{L_\infty(0,t;L_2(\Omega))} + \|\nabla\chi^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))}) \\ &\le c(c_0c_6(t) + c_6(t)) \le c_6(t). \end{aligned}$$

Consequently, we arrive at

$$\|Qu^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))} \le c_7(t).$$

Hence, by the ellipticity property of the operator Q (see (2.3)), assertion (4.34) follows. □

The next result provides an additional regularity estimate for μ^m_t.

Lemma 4.3. *Let assumptions of Lemma 4.1 hold. Then, for* $t \in (0,T]$,

$$\begin{aligned} &\|\mu^m_{t'}\|_{L_\infty(0,t;H^2_N(\Omega))} \le c_6(t), \\ &\|\mu^m_{t't'}\|_{L_2(\Omega^t)} \le c_8(t) \end{aligned} \tag{4.36}$$

where

$$c_8(t) = t^{1/2}c_4(t)c_6(t).$$

Proof. Using (4.2) we rewrite identity $(4.7)_2$ in the form

$$(\chi^m_{tt}, \Delta w_j) = (\Delta \mu^m_t, \Delta w_j).$$

Testing the above equality by $\mu^m_t(t)$, applying the Cauchy-Schwartz inequality and then estimate $(4.18)_2$, leads to

$$\|\Delta \mu^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))} \le \|\chi^m_{t't'}\|_{L_\infty(0,t;L_2(\Omega))} \le c_6(t). \tag{4.37}$$

Let us estimate now the mean value of μ^m_t. Setting $w_j = 1$ in $(4.7)_3$ (admissible by assumption), using $(2.17)_2$ and assumptions on z, yields

$$\begin{aligned}
\Big|\int\limits_\Omega \mu^m_t dx\Big| &\le \Big|\int\limits_\Omega [\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,t} dx\Big| \\
&\le c\Big|\int\limits_\Omega [(\chi^m)^2\chi^m_t + \chi^m_t + \chi^m_t \varepsilon(\boldsymbol{u}^m) + \varepsilon(\boldsymbol{u}^m_t)] dx\Big| \\
&\le c(\|\chi^m\|_{L_\infty(\Omega^t)} + 1 + \|\varepsilon(\boldsymbol{u}^m)\|_{L_\infty(0,T;L_2(\Omega))})\|\chi^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))} \\
&\quad + c\|\varepsilon(\boldsymbol{u}^m_{t'})\|_{L_\infty(0,t;L_1(\Omega))} \\
&\le c(c_4(t) + 1 + c_0)c_4(t) + c_5(t) \quad \text{for a.a. } t \in (0,T],
\end{aligned} \tag{4.38}$$

where in the last inequality we applied estimates $(4.10)_1$, $(4.11)_1$ and $(4.11)_2$. Consequently, we see that estimate $(4.36)_1$ results from (4.37) and (4.38) on account of the ellipticity property of the Laplace operator.

To show $(4.36)_2$ we test $(4.13)_3$ by $\mu^m_{tt}(t)$. Then, by the Cauchy-Schwartz inequality, it follows that

$$\|\mu^m_{t't'}\|_{L_2(\Omega^t)} \le \|\Delta \chi^m_{t't'}\|_{L_2(\Omega^t)} + \|[\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,t't'}\|_{L_2(\Omega^t)}. \tag{4.39}$$

The first term on the right-hand side of (4.39) is, by $(4.18)_2$, bounded by $c_6(t)$. The bound on the second term can be obtained directly by recalling estimates on the integrals K_i, $i = 1, \dots, 8$, in (4.28) (see (4.30) with $\delta_2 = 1$) and then using $(4.18)_{1,2}$. Then it follows that

$$\begin{aligned}
&\|[\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,t't'}\|^2_{L_2(\Omega^t)} \\
&\le 2(\|[\psi'(\chi^m)]_{,t't'}\|^2_{L_2(\Omega^t)} + \|[W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,t't'}\|^2_{L_2(\Omega^t)}) \\
&\le tc_4^6(t) + c(c_1)(1 + tc_4^2(t))c_6^2(t) + ct^2c_6^2(t) \le tc_4^2(t)c_6^2(t) \le c_8^2(t)
\end{aligned} \tag{4.40}$$

with constant $c_8(t)$ defined in (4.36). Consequently, estimate $(4.36)_2$ follows. □

Finally, we estimate the time derivatives u^m_{tttt} and χ^m_{ttt}.

Lemma 4.4. *Let assumptions of Lemma 4.1 hold. Then for* $t \in (0,T]$,

$$\begin{aligned} &\|\boldsymbol{u}^m_{t't't't'}\|_{L_2(0,t;(H^1_0(\Omega))')} \le c_9(t), \\ &\|\chi^m_{t't't'}\|_{L_2(0,t;(H^2_N(\Omega))')} \le c_8(t) \end{aligned} \tag{4.41}$$

where

$$c_9(t) = c(c_1)t^{1/2}c_6^2(t).$$

Proof. We use standard duality arguments, similarly as in [PawZaj06b], Lemmas 3.4, 4.4. For $\boldsymbol{\eta} \in L_2(0,T;\boldsymbol{H}^1_0(\Omega))$, we test $(4.13)_1$ by $\boldsymbol{\eta}^m = \boldsymbol{P}^m\boldsymbol{\eta}$, where $\boldsymbol{P}^m$ denotes the projection defined by

$$\boldsymbol{P}^m\boldsymbol{\eta} = \sum_{i=1}^m (\boldsymbol{\eta}, \boldsymbol{v}_i)\boldsymbol{v}_i.$$

Then intergrating with respect to time and using the Cauchy-Schwartz inequality yields

$$\begin{aligned} &\Big|\int_0^t (\boldsymbol{u}^m_{t't't't'}, \boldsymbol{\eta})dt'\Big| = \Big|\int_0^t (\boldsymbol{u}^m_{t't't't'}, \boldsymbol{P}^m\boldsymbol{\eta})dt'\Big| \\ &= \Big|\int_0^t \{-(\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}^m_{t't'}), \boldsymbol{\varepsilon}(\boldsymbol{P}^m\boldsymbol{\eta})) + ([z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,t't'} + \boldsymbol{b}_{t't'}, \boldsymbol{P}^m\boldsymbol{\eta})\}dt'\Big| \\ &\le c\|\boldsymbol{\varepsilon}(\boldsymbol{u}^m_{t't'})\|_{L_2(\Omega^t)}\|\nabla\boldsymbol{P}^m\boldsymbol{\eta}\|_{L_2(\Omega^t)} \\ &\quad + (\|[z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,t't'}\|_{L_2(\Omega^t)} + \|\boldsymbol{b}_{t't'}\|_{L_2(\Omega^t)})\|\boldsymbol{P}^m\boldsymbol{\eta}\|_{L_2(\Omega^t)}. \end{aligned}$$

Making use of estimates $(4.10)_{1,3}$, $(4.11)_2$, $(4.18)_2$ and (4.33), we obtain

$$\begin{aligned} &\|[z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,t't'}\|_{L_2(\Omega^t)} \\ &\le c(\|(\chi^m_{t'})^2\nabla\chi^m\|_{L_2(\Omega^t)} + \|\chi^m_{t't'}\nabla\chi^m\|_{L_2(\Omega^t)} + \|\chi^m_{t'}\nabla\chi^m_{t'}\|_{L_2(\Omega^t)} + \|\nabla\chi^m_{t't'}\|_{L_2(\Omega^t)}) \\ &\le c(\|\chi^m_{t'}\|^2_{L_\infty(\Omega^t)}\|\nabla\chi^m\|_{L_2(\Omega^t)} + \|\chi^m_{t't'}\|_{L_2(0,t;L_\infty(\Omega))}\|\nabla\chi^m\|_{L_\infty(0,t;L_2(\Omega))} \\ &\quad + \|\chi^m_{t'}\|_{L_\infty(\Omega^t)}\|\nabla\chi^m_{t'}\|_{L_2(\Omega^t)} + \|\nabla\chi^m_{t't'}\|_{L_2(\Omega^t)}) \\ &\le c_2(t)c_6^2(t) + c_0c_6(t) + c_4(t)c_6(t) + c_6(t) \le c(c_1)t^{1/2}c_6^2(t). \end{aligned} \tag{4.42}$$

Further, by $(4.18)_1$,

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}^m_{t't'})\|_{L_2(\Omega^t)} \le t^{1/2}c_7(t) = tc_6(t).$$

Hence, we conclude that

$$\begin{aligned} \Big|\int_0^t (\boldsymbol{u}^m_{t't't't'}, \boldsymbol{\eta})dt'\Big| &\le (t^{1/2}c_7(t) + c(c_1)t^{1/2}c_6^2(t))\|\boldsymbol{P}^m\boldsymbol{\eta}\|_{L_2(0,t;\boldsymbol{H}^1_0(\Omega))} \\ &\le c_9(t)\|\boldsymbol{\eta}\|_{L_2(0,t;(\boldsymbol{H}^1_0(\Omega))')} \end{aligned}$$

for any $\boldsymbol{\eta} \in L_2(0,T;\boldsymbol{H}_0^1(\Omega))$, with constant $c_9(t)$ defined in (4.41). This proves the first estimate in (4.41).

Similarly, for any $\xi \in L_2(0,T;H_N^2(\Omega))$, testing $(4.13)_2$ by

$$\xi^m = P^m\xi = \sum_{i=1}^m (\xi, w_i)w_i,$$

and making use of Lemma 4.3, we obtain

$$\left|\int_0^t (\chi^m_{t't't'}, \xi)dt'\right| = \left|\int_0^t (\chi^m_{t't't'}, P^m\xi)dt'\right| = \left|\int_0^t (\mu^m_{t't'}, \Delta P^m\xi)dt'\right|$$
$$\leq \|\mu^m_{t't'}\|_{L_2(\Omega^t)}\|\Delta P^m\xi\|_{L_2(\Omega^t)} \leq c_8(t)\|\xi\|_{L_2(0,t;H_N^2(\Omega))}.$$

This proves the second estimate in (4.41). □

5. Proof of Theorem 2.1

By uniform in m estimates (4.10), (4.11) and those in Lemmas 4.1–4.4 it follows that there exists a triple $(\boldsymbol{u},\chi,\mu)$ with

$$\begin{aligned}
&\boldsymbol{u} \in L_\infty(0,T;\boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}_0^1(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T;\boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}_0^1(\Omega)),\\
&\boldsymbol{u}_{tt} \in L_\infty(0,T;\boldsymbol{H}_0^1(\Omega)), \quad \boldsymbol{u}_{ttt} \in L_\infty(0,T;L_2(\Omega)),\\
&\boldsymbol{u}_{tttt} \in L_2(0,T;(\boldsymbol{H}_0^1(\Omega))'),\\
&\chi \in C^{1/2}[0,T];H_N^2(\Omega)), \quad \chi_t \in C^{1/2}([0,T];H_N^2(\Omega)),\\
&\chi_{tt} \in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;H_N^2(\Omega)), \quad \chi_{ttt} \in L_2(0,T;(H_N^2(\Omega))'),\\
&\mu \in Lip([0,T];H_N^2(\Omega)), \quad \mu_t \in L_\infty(0,T;H_N^2(\Omega)), \quad \mu_{tt} \in L_2(\Omega^T),
\end{aligned} \tag{5.1}$$

and a subsequence of solutions $(\boldsymbol{u}^m,\chi^m,\mu^m)$ of approximate system (4.13) (which we still denote by the same indices) such that as $m \to \infty$:

$$\begin{aligned}
&\boldsymbol{u}^m \to \boldsymbol{u}, \quad \boldsymbol{u}^m_t \to \boldsymbol{u}_t && \text{weakly} -^* \text{ in } L_\infty(0,T;\boldsymbol{H}^2(\Omega)),\\
&\boldsymbol{u}^m_{tt} \to \boldsymbol{u}_{tt} && \text{weakly} -^* \text{ in } L_\infty(0,T;\boldsymbol{H}_0^1(\Omega)),\\
&\boldsymbol{u}^m_{ttt} \to \boldsymbol{u}_{ttt} && \text{weakly} -^* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
&\boldsymbol{u}^m_{tttt} \to \boldsymbol{u}_{tttt} && \text{weakly in } L_2(0,T;(\boldsymbol{H}_0^1(\Omega))'),\\
&\chi^m \to \chi, \quad \chi^m_t \to \chi_t && \text{weakly} -^* \text{ in } L_\infty(0,T;H_N^2(\Omega)),\\
&\chi^m_{tt} \to \chi_{tt} && \text{weakly} -^* \text{ in } L_\infty(0,T;L_2(\Omega)) \text{ and}\\
& && \text{weakly in } L_2(0,T;H_N^2(\Omega)),\\
&\chi^m_{ttt} \to \chi_{ttt} && \text{weakly in } L_2(0,T;(H_N^2(\Omega))'),\\
&\mu^m \to \mu, \quad \mu^m_t \to \mu_t && \text{weakly} -^* \text{ in } L_\infty(0,T;H_N^2(\Omega)),\\
&\mu^m_{tt} \to \mu_{tt} && \text{weakly in } L_2(\Omega^T).
\end{aligned} \tag{5.2}$$

By virtue of the compactness results (see e.g. Lions [Lions69], Simon [Sim87], Sec. 8) it follows that for a subsequence (still denoted by the same indices)

$$
(5.3)\quad
\begin{aligned}
&\boldsymbol{u}^m \to \boldsymbol{u},\quad \boldsymbol{u}_t^m \to \boldsymbol{u}_t && \text{strongly in } L_2(0,T;\boldsymbol{H}_0^1(\Omega)) \cap C([0,T];\boldsymbol{H}_0^1(\Omega)) \\
& && \text{and a.e. in } \Omega^T, \\
&\boldsymbol{u}_{tt}^m \to \boldsymbol{u}_{tt} && \text{strongly in } \boldsymbol{L}_2(\Omega^T) \cap C([0,T];\boldsymbol{L}_2(\Omega)) \\
& && \text{and a.e. in } \Omega^T, \\
&\boldsymbol{u}_{ttt}^m \to \boldsymbol{u}_{ttt} && \text{strongly in } C([0,T];(\boldsymbol{H}_0^1(\Omega))'), \\
&\chi^m \to \chi,\quad \chi_t^m \to \chi_t && \text{strongly in } L_2(0,T;H^1(\Omega)) \cap C([0,T];H^1(\Omega)) \\
& && \text{and a.e. in } \Omega^T, \\
&\chi_{tt}^m \to \chi_{tt} && \text{strongly in } L_2(0,T;H^1(\Omega)) \cap C([0,T];(H_N^2(\Omega))') \\
& && \text{and a.e. in } \Omega^T, \\
&\mu^m \to \mu && \text{strongly in } L_2(0,T;H^1(\Omega)) \cap C([0,T];H^1(\Omega)) \\
& && \text{and a.e. in } \Omega^T, \\
&\mu_t^m \to \mu_t && \text{strongly in } L_2(0,T;H^1(\Omega)) \cap C([0,T];H^1(\Omega)) \\
& && \text{and a.e. in } \Omega^T.
\end{aligned}
$$

From convergences (5.3) it follows in particular that

$$
(5.4)\quad
\begin{aligned}
&\boldsymbol{u}^m(0) = \boldsymbol{u}_0^m \to \boldsymbol{u}(0) && \text{strongly in } \boldsymbol{H}_0^1(\Omega), \\
&\boldsymbol{u}_t^m(0) = \boldsymbol{u}_1^m \to \boldsymbol{u}_t(0) && \text{strongly in } \boldsymbol{H}_0^1(\Omega), \\
&\boldsymbol{u}_{tt}^m(0) = \boldsymbol{u}_2^m \to \boldsymbol{u}_{tt}(0) && \text{strongly in } \boldsymbol{L}_2(\Omega), \\
&\boldsymbol{u}_{ttt}^m(0) = \boldsymbol{u}_3^m \to \boldsymbol{u}_{ttt}(0) && \text{strongly in } (\boldsymbol{H}_0^1(\Omega))', \\
&\chi^m(0) = \chi_0^m \to \chi(0) && \text{strongly in } H^1(\Omega), \\
&\chi_t^m(0) = \chi_1^m \to \chi_t(0) && \text{strongly in } H^1(\Omega), \\
&\chi_{tt}^m(0) = \chi_2^m \to \chi_{tt}(0) && \text{strongly in } (H_N^2(\Omega))' \\
&\mu^m(0) = \mu_0^m \to \mu(0) && \text{strongly in } H^1(\Omega), \\
&\mu_t^m(0) = \mu_1^m \to \mu_t(0) && \text{strongly in } H^1(\Omega),
\end{aligned}
$$

what together with convergences (4.15) implies that

$$
(5.5)\quad
\begin{aligned}
&\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \boldsymbol{u}_{tt}(0) = \boldsymbol{u}_2, \quad \boldsymbol{u}_{ttt}(0) = \boldsymbol{u}_3, \\
&\chi(0) = \chi_0, \quad \chi_t = \chi_1, \quad \chi_{tt}(0) = \chi_2, \\
&\mu(0) = \mu_0, \quad \mu_t(0) = \mu_1.
\end{aligned}
$$

The relations (5.4) and (5.5) imply assertions (2.14), (2.15) of the theorem.

We introduce now the following weak formulation of (4.13):

$$
(5.6)\qquad
\begin{aligned}
&\int_0^T \langle \boldsymbol{u}^m_{tttt}, \boldsymbol{\eta}\rangle_{(H^1_0(\Omega))', H^1_0(\Omega)} dt + \int_0^T (A\boldsymbol{\varepsilon}(\boldsymbol{u}^m_{tt}), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))dt \\
&= \int_0^T ([z'(\chi^m)B\nabla\chi^m]_{,tt} + \boldsymbol{b}_{tt}, \boldsymbol{\eta})dt \quad \forall \boldsymbol{\eta} \in L_2(0,T; \boldsymbol{V}_{0m}), \\
&\int_0^T \langle \chi^m_{ttt}, \xi\rangle_{(H^2_N(\Omega))', H^2_N(\Omega)} dt = \int_0^T (\mu^m_{tt}, \Delta\xi)dt \quad \forall \xi \in L_2(0,T,V_m), \\
&\int_0^T (\mu^m_{tt}, \zeta)dt = -\int_0^T (\Delta\chi^m_{tt}, \zeta)dt + \int_0^T ([\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,tt}, \zeta)dt \\
&\forall \zeta \in L_2(0,T; V_m).
\end{aligned}
$$

Using standard procedure (see e.g. Lions-Magenes [LionsMag72] we pass to the limit $m \to \infty$ in (5.6). Clearly, due to the weak convergences (5.2), all linear terms in (5.6) converge to the corresponding limits. Thus, it remains to examine the convergence of the nonlinear terms $[z'(\chi^m)B\nabla\chi^m]_{,tt}$ and $[\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,tt}$ whose explicit expressions are given in (2.17).
The convergence of these terms can be deduced by virtue of the standard nonlinear convergence lemma (see [Lions69], Chap. 1, Lemma 1.3). In fact, recalling (4.40) and (4.42) we have the following uniform bounds

$$
(5.7)\qquad
\begin{aligned}
&\|[z'(\chi^m)B\nabla\chi^m]_{,tt}\|_{L_2(\Omega^T)} \le c(c_1)T^{1/2}c_6^2(T), \\
&\|[\psi'(\chi^m)]_{,tt}\|_{L_2(\Omega^T)} + \|[W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,tt}\|_{L_2(\Omega^T)} \le c_8(T).
\end{aligned}
$$

Thanks to such uniform bounds and the pointwise convergences in (5.3) the nonlinear convergence lemma implies that

$$
(5.8)\qquad
\begin{aligned}
&[z'(\chi^m)B\nabla\chi^m]_{,tt} = z'''(\chi^m)(\chi^m_t)^2 B\nabla\chi^m + z''(\chi^m)\chi^m_{tt}B\nabla\chi^m \\
&\quad + 2z''(\chi^m)\chi^m_t B\nabla\chi^m_t + z'(\chi^m)B\nabla\chi^m_{tt} \\
&\quad \to z'''(\chi)\chi_t^2 B\nabla\chi + z''(\chi)\chi_{tt}B\nabla\chi + 2z''(\chi)\chi_t B\nabla\chi_t + z'(\chi)B\nabla\chi_{tt} \\
&= [z'(\chi)B\nabla\chi]_{,tt} \quad \text{weakly in} \quad L_2(\Omega^T),
\end{aligned}
$$

$$
(5.9)\qquad
\begin{aligned}
&[\psi'(\chi^m)]_{,tt} = 6\chi^m(\chi^m_t)^2 + 3(\chi^m)^2\chi^m_{tt} - \chi^m_{tt} \\
&\quad \to 6\chi\chi_t^2 + 3\chi^2\chi_{tt} - \chi_{tt} = [\psi'(\chi)]_{,tt} \quad \text{weakly in} \quad L_2(\Omega^T),
\end{aligned}
$$

$$
\begin{aligned}
[W_{,\chi}(\varepsilon(\boldsymbol{u}^m),\chi^m)]_{,tt} &= z'''(\chi^m)(\chi_t^m)^2(B\cdot\varepsilon(\boldsymbol{u}^m)+Dz(\chi^m)+E)\\
&\quad + z''(\chi^m)\chi_{tt}^m(B\cdot\varepsilon(\boldsymbol{u}^m)+Dz(\chi^m)+E)\\
&\quad + 2z''(\chi^m)\chi_t^m(B\cdot\varepsilon(\boldsymbol{u}_t^m)+Dz'(\chi^m)\chi_t^m)\\
&\quad + z'(\chi^m)(B\cdot\varepsilon(\boldsymbol{u}_{tt}^m)+Dz''(\chi^m)(\chi_t^m)^2+Dz'(\chi^m)\chi_{tt}^m)\\
&\to z'''(\chi)\chi_t^2(B\cdot\varepsilon(\boldsymbol{u})+Dz(\chi)+E)\\
&\quad + z''(\chi)\chi_{tt}(B\cdot\varepsilon(\boldsymbol{u})+Dz(\chi)+E)\\
&\quad + 2z''(\chi)\chi_t(B\cdot\varepsilon(\boldsymbol{u}_t)+Dz'(\chi)\chi_t)\\
&\quad + z'(\chi)(B\cdot\varepsilon(\boldsymbol{u}_{tt})+Dz''(\chi)\chi_t^2+Dz'(\chi)\chi_{tt})\\
&= [W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi]_{,tt} \quad \text{weakly in} \quad L_2(\Omega^T).
\end{aligned} \tag{5.10}
$$

In view of (5.8)–(5.10), passing to the limit $m\to\infty$ in (5.6) we conclude by standard arguments identities (2.16). A priori estimates (2.18)–(2.20) result from estimates (4.10), (4.11), estimates in Lemmas 4.1–4.4 and the weak convergences (5.2). This completes the proof of the theorem. □

6. Proof of Theorem 2.2

Let us consider a solution $(\boldsymbol{u},\chi,\mu)$ of problem (1.14)–(1.16) constructed in Theorem 2.1. Our goal is to show that this solution has regularity (2.22) and satisfies estimates (2.23). This will result from the following three lemmas.

Lemma 6.1. *Let assumptions of Theorem 2.1 hold. Then solutions $\boldsymbol{u}$ of (1.14), treated as an elliptic system*

$$
\begin{aligned}
Q\boldsymbol{u} &= \boldsymbol{u}_{tt} - z'(\chi)B\nabla\chi - \boldsymbol{b} && \text{in } \Omega^T,\\
\boldsymbol{u} &= 0 && \text{on } S^T,
\end{aligned} \tag{6.1}
$$

satisfy $\boldsymbol{u}\in L_\infty(0,T;\boldsymbol{H}^3(\Omega))$, and

$$
\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}^3(\Omega))} \le c_7(T). \tag{6.2}
$$

Proof. By virtue of the elliptic regularity result in Lemma 3.2 if follows that

$$
\begin{aligned}
\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}^3(\Omega))} &\le c(\|\boldsymbol{u}_{tt} - z'(\chi)B\nabla\chi - \boldsymbol{b}\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))}\\
&\quad + \|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))}) \equiv R_1.
\end{aligned} \tag{6.3}
$$

Thanks to estimates $(2.18)_2$, $(2.19)_2$ and $(2.20)_1$,

$$
\begin{aligned}
R_1 &\le c(\|\boldsymbol{u}_{tt}\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))} + \|\nabla\chi\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))}\\
&\quad + \|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + 1)\\
&\le c(c_7(T) + c_4(T) + c_1 + 1) \le c_7(T).
\end{aligned}
$$

Consequently, (6.1) follows. □

From (6.2), by the Sobolev imbedding, it follows that

$$\|\varepsilon(\boldsymbol{u})\|_{L_\infty(0,T;H^2(\Omega))} + \|\varepsilon(\boldsymbol{u})\|_{L_\infty(\Omega^T)} \le c_7(T). \tag{6.4}$$

With estimate (6.4) we are ready to prove a regularity of χ.

Lemma 6.2. *Let assumptions of Theorem 2.1 hold. Then solutions χ of (1.16), treated as an elliptic system*

$$\begin{aligned} &\Delta\chi = -\mu + \psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi) \quad \text{in } \Omega^T,\\ &\boldsymbol{n}\cdot\nabla\chi = 0 \quad \text{on } S^T, \end{aligned} \tag{6.5}$$

satisfy $\chi \in L_\infty(0,T;H^4(\Omega))$, and

$$\|\chi\|_{L_\infty(0,T;H^4(\Omega))} \le c_4^2(T)c_7(T). \tag{6.6}$$

Proof. Due to the elliptic regularity (see Lemma 3.1) we have

$$\begin{aligned} &\|\chi\|_{L_\infty(0,T;H^4(\Omega))}\\ &\le c(\|-\mu + \psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi)\|_{L_\infty(0,T;H^2(\Omega))} + \|\chi\|_{L_\infty(0,T;H^2(\Omega))}) \equiv R_2. \end{aligned} \tag{6.7}$$

Takin into account that

$$\begin{aligned} [\psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi)]_{xx} &= \psi'''(\chi)\chi_x^2 + \psi''(\chi)\chi_{xx}\\ &+ z'''(\chi)\chi_x^2(B\cdot\varepsilon(\boldsymbol{u}) + Dz(\chi) + E)\\ &+ z''(\chi)[\chi_{xx}(B\cdot\varepsilon(\boldsymbol{u}) + Dz(\chi) + E) + 2\chi_x((B\cdot\varepsilon(\boldsymbol{u}))_x + Dz'(\chi)\chi_x)]\\ &+ z'(\chi)((B\cdot\varepsilon(\boldsymbol{u}))_{xx} + Dz''(\chi)\chi_x^2 + Dz'(\chi)\chi_{xx}), \end{aligned}$$

recalling assumptions on ψ and z (see (2.6), (2.12)), and using estimates $(2.19)_2$ and (6.4), we obtain

$$\begin{aligned} &\|\nabla^2[\psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi)]\|_{L_\infty(0,T;L_2(\Omega))}\\ &\le c\||\chi||\nabla\chi|^2 + (\chi^2+1)|\nabla^2\chi| + |\nabla\chi|^2|\varepsilon(\boldsymbol{u})| + |\nabla\chi|^2 + |\nabla^2\chi||\varepsilon(\boldsymbol{u})|\\ &\quad + |\nabla\chi||\nabla\varepsilon(\boldsymbol{u})| + |\nabla^2\varepsilon(\boldsymbol{u})|\|_{L_\infty(0,T;L_2(\Omega))}\\ &\le c[\|\chi\|_{L_\infty(\Omega^T)}\|\nabla\chi\|^2_{L_\infty(0,T;L_4(\Omega))} + (\|\chi\|^2_{L_\infty(\Omega)} + 1)\|\nabla^2\chi\|_{L_\infty(0,T;L_2(\Omega))}\\ &\quad + \|\varepsilon(\boldsymbol{u})\|_{L_\infty(\Omega^T)}\|\nabla\chi\|^2_{L_\infty(0,T;L_4(\Omega))} + \|\nabla\chi\|^2_{L_\infty(0,T;L_4(\Omega))}\\ &\quad + \|\varepsilon(\boldsymbol{u})\|_{L_\infty(\Omega^T)}\|\nabla^2\chi\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\chi\|_{L_\infty(0,T;L_4(\Omega))}\|\nabla\varepsilon(\boldsymbol{u})\|_{L_\infty(0,T;L_4(\Omega))}\\ &\quad + \|\nabla^2\varepsilon(\boldsymbol{u})\|_{L_\infty(0,T;L_2(\Omega))}]\\ &\le c[c_4^3(T) + (c_4(T)+1)c_4(T) + c_7(T)c_4^2(T) + c_4^2(T)\\ &\quad + c_7(T)c_4(T) + c_4(T)c_7(T) + c_7(T)]\\ &\le c_4^2(T)c_7(T). \end{aligned} \tag{6.8}$$

Besides, estimates $(2.19)_2$ imply that

$$
\begin{aligned}
&\|\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)\|_{L_\infty(0,T;L_2(\Omega))} \le c_4(T),\\
&\|\mu\|_{L_\infty(0,T;H^2(\Omega))} \le c_4(T),\\
&\|\chi\|_{L_\infty(0,T;H^2(\Omega))} \le c_4(T).
\end{aligned}
\tag{6.9}
$$

Thus, it follows from (6.8) and (6.9) that

$$R_2 \le c_4^2(T)c_7(T) + c_4(T) \le c_4^2(T)c_7(T)$$

which shows estimate (6.6). This completes the proof of the lemma. □

The next lemma proves regularity of μ.

Lemma 6.3. *Let assumptions of Theorem 2.1 hold. Then solutions μ of (1.15), treated as an elliptic system*

$$
\begin{aligned}
\Delta\mu &= \chi_t \quad \text{in } \Omega^T,\\
\boldsymbol{n}\cdot\nabla\mu &= 0 \quad \text{on } S^T,
\end{aligned}
\tag{6.10}
$$

satisfy $\mu \in L_\infty(0,T;H^4(\Omega))$, $\mu_t \in L_2(0,T;H^4(\Omega))$, and

$$\|\mu\|_{L_\infty(0,T;H^4(\Omega))} + \|\mu_t\|_{L_2(0,T;H^4(\Omega))} \le c_6(T). \tag{6.11}$$

Proof. On account of the elliptic regularity (see Lemma 3.1) it follows that solutions μ of (6.10) satisfy

$$
\begin{aligned}
&\|\mu\|_{L_\infty(0,T;H^4(\Omega))} \le c(\|\chi_t\|_{L_\infty(0,T;H^2(\Omega))} + \|\mu\|_{L_\infty(0,T;L_2(\Omega))}) \equiv R_3,\\
&\|\mu_t\|_{L_2(0,T;H^4(\Omega))} \le c(\|\chi_{tt}\|_{L_2(0,T;H^2(\Omega))} + \|\mu_t\|_{L_2(0,T;L_2(\Omega))}) \equiv R_4.
\end{aligned}
\tag{6.12}
$$

By virtue of estimates $(2.19)_{2,3}$, $(2.20)_2$, we have

$$
\begin{aligned}
R_2 &\le c(c_6(T) + c_4(T)) \le c_6(T),\\
R_3 &\le c(c_6(T) + T^{1/2}c_5(T)) \le c_6(T),
\end{aligned}
$$

which shows (6.11). □

Lemmas 6.1–6.3 imply regularity statements (2.22), (2.23). Thereby the proof of Theorem 2.2 is completed. □

7. Proof of Theorem 2.3

Let $(\boldsymbol{u}_1, \chi_1, \mu_1)$ and $(\boldsymbol{u}_2, \chi_2, \mu_2)$ be two solutions of problem (1.14)–(1.16) corresponding to the same data. Subtracting the corresponding equations and denoting

$$\boldsymbol{U} = \boldsymbol{u}_1 - \boldsymbol{u}_2, \quad H = \chi_1 - \chi_2, \quad M = \mu_1 - \mu_2,$$

we obtain the following system for $(\boldsymbol{U}, H, M)$:

$$\begin{aligned} &\boldsymbol{U}_{tt} - \boldsymbol{Q}\boldsymbol{U} = (z'(\chi_1) - z'(\chi_2))\boldsymbol{B}\nabla\chi_1 + z'(\chi_2)\boldsymbol{B}\nabla H && \text{in } \Omega^T, \\ &\boldsymbol{U}|_{t=0} = \boldsymbol{0} && \text{in } \Omega, \\ &\boldsymbol{U} = \boldsymbol{0} && \text{on } S^T, \end{aligned} \tag{7.1}$$

$$\begin{aligned} &H_t - \Delta M = 0 && \text{in } \Omega^T, \\ &H|_{t=0} = 0 && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla M = 0 && \text{on } S^T, \end{aligned} \tag{7.2}$$

$$\begin{aligned} &M = -\Delta H + \psi'(\chi_1) - \psi'(\chi_2) + (z'(\chi_1) - z'(\chi_2))(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_1) \\ &\qquad + Dz(\chi_1) + E) + z'(\chi_2)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{U}) + D(z(\chi_1) - z(\chi_2)) && \text{in } \Omega^T, \\ &\boldsymbol{n}\cdot\nabla H = 0 && \text{on } S^T. \end{aligned} \tag{7.3}$$

Multiplying $(7.1)_1$ by $\boldsymbol{U}_t$, integrating over Ω and by parts, using boundary condition $(7.1)_3$, yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega |\boldsymbol{U}_t|^2 dx + \int_\Omega \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{U})\cdot\boldsymbol{\varepsilon}(\boldsymbol{U})dx\Big] \\ &= \int_\Omega (z''(\chi_*)H\boldsymbol{B}\nabla\chi_1 + z'(\chi_2)\boldsymbol{B}\nabla H)\cdot\boldsymbol{U}_t dx \equiv R_1, \end{aligned} \tag{7.4}$$

where $\chi_* \in (\chi_1, \chi_2)$. By assumption (2.24) on z, the right-hand side of (7.4) is estimated with the help of the Hölder inequality by

$$R_1 \le c(\|H\|_{L_3(\Omega)}\|\nabla\chi_1\|_{L_6(\Omega)}\|\boldsymbol{U}_t\|_{L_2(\Omega)} + \|\nabla H\|_{L_2(\Omega)}\|\boldsymbol{U}_t\|_{L_2(\Omega)}).$$

Hence,

$$\begin{aligned} &\frac{d}{dt}[(\|\boldsymbol{U}_t\|^2_{L_2(\Omega)} + \int_\Omega \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{U})\cdot\boldsymbol{\varepsilon}(\boldsymbol{U})dx] \\ &\le c(\|H\|_{L_3(\Omega)}\|\nabla\chi_1\|_{L_6(\Omega)} + \|\nabla H\|_{L_2(\Omega)})\|\boldsymbol{U}_t\|_{L_2(\Omega)} \equiv R_2. \end{aligned} \tag{7.5}$$

Now, multiplying $(7.2)_1$ by H, integrating over Ω, and twice by parts, taking into account boundary conditions $(7.2)_3$, $(7.3)_3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega H^2 dx - \int_\Omega M\Delta H dx = 0. \tag{7.6}$$

Finally, multiplying $(7.3)_1$ by ΔH and integrating over Ω, in view of assumptions on ψ and z, gives

$$\begin{aligned}\int_\Omega M\Delta H dx = &-\int_\Omega (\Delta H)^2 dx + \int_\Omega H(\chi_1^2 + \chi_1\chi_2 + \chi_2^2 - 1)\Delta H dx \\ &+ \int_\Omega z''(\chi_*)H(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_1) + Dz(\chi_1) + E)\Delta H dx \\ &+ \int_\Omega z'(\chi_2)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{U}) + Dz'(\chi_*)H)\Delta H dx,\end{aligned} \tag{7.7}$$

where $\chi_* \in (\chi_1, \chi_2)$. Consequently, combining (7.6) and (7.7), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega H^2 dx + \int_\Omega (\Delta H)^2 dx = &\int_\Omega H(\chi_1^2 + \chi_1\chi_2 + \chi_2^2 - 1)\Delta H dx \\ + &\int_\Omega z''(\chi_*)H(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_1) + Dz(\chi_1) + E)\Delta H dx \\ + &\int_\Omega z'(\chi_2)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{U}) + Dz'(\chi_*)H)\Delta H dx \equiv R_3.\end{aligned} \tag{7.8}$$

Estimating the right-hand side of (7.8) with the help of the Young inequality (using assumptions on z) by

$$\begin{aligned}R_3 \le \delta_1 \int_\Omega (\Delta H)^2 dx + c(1/\delta_1)\Big[&\int_\Omega H^2(\chi_1^4 + \chi_2^4 + 1)dx + \int_\Omega H^2(|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)|^2 + 1)dx \\ &+ \int_\Omega (|\boldsymbol{\varepsilon}(\boldsymbol{U})|^2 + H^2)dx\Big]\end{aligned}$$

with $\delta_1 > 0$, and then choosing δ_1 sufficiently small, we obtain

$$\begin{aligned}&\frac{d}{dt}\|H\|^2_{L_2(\Omega)} + \|\Delta H\|^2_{L_2(\Omega)} \\ &\le c[(\|\chi_1\|^4_{L_\infty(\Omega)} + \|\chi_2\|^4_{L_\infty(\Omega)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)\|^2_{L_\infty(\Omega)} + 1)\|H\|^2_{L_2(\Omega)} \\ &\quad + \|\boldsymbol{\varepsilon}(\boldsymbol{U})\|^2_{L_2(\Omega)})].\end{aligned} \tag{7.9}$$

Hence, taking into account that

$$\int_\Omega H dx = 0 \quad \text{for} \quad t \in (0,T],$$

it follows by virtue of the ellipticity property of the Laplace operator that

$$\begin{aligned}(7.10)\quad &\frac{d}{dt}\|H\|^2_{L_2(\Omega)} + \|H\|^2_{H^2(\Omega)} \\ &\le c(\|\chi_1\|^4_{L_\infty(\Omega)} + \|\chi_2\|^4_{L_\infty(\Omega)} + \|\varepsilon(u_1)\|^2_{L_\infty(\Omega)} + 1)\|H\|^2_{L_2(\Omega)} + \|\varepsilon(U)\|^2_{L_2(\Omega)}).\end{aligned}$$

Now, we apply the Young inequality to the right-hand side of (7.5)

$$R_2 \le \delta_2(\|H\|^2_{L_3(\Omega)} + \|\nabla H\|^2_{L_2(\Omega)}) + c(1/\delta_2)(\|\nabla\chi_1\|^2_{L_6(\Omega)} + 1)\|U_t\|^2_{L_2(\Omega)}$$

where $\delta_2 > 0$, and then sum up (7.5) and (7.10). As a result, choosing δ_2 sufficiently small so that the term $\delta_2(\|H\|^2_{L_3(\Omega)} + \|\nabla H\|^2_{L_2(\Omega)})$ is absorbed by $\|H\|^2_{H^2(\Omega)}$, we arrive at the inequality

$$\begin{aligned}(7.11)\quad &\frac{d}{dt}(\|U_t\|^2_{L_2(\Omega)} + \|\varepsilon(U)\|^2_{L_2(\Omega)} + \|H\|^2_{L_2(\Omega)}) + \|H\|^2_{H^2(\Omega)} \\ &\le c[(\|\chi_1\|^4_{L_\infty(\Omega)} + \|\chi_2\|^4_{L_\infty(\Omega)} + \|\varepsilon(u_1)\|^2_{L_\infty(\Omega)} + 1)\|H\|^2_{L_2(\Omega)} \\ &\quad + \|\varepsilon(U)\|^2_{L_2(\Omega)} + (\|\nabla\chi_1\|^2_{L_6(\Omega)} + 1)\|U_t\|^2_{L_2(\Omega)}].\end{aligned}$$

Denoting

$$D(t) = \|U_t\|^2_{L_2(\Omega)} + \int_\Omega A\varepsilon(U)\cdot\varepsilon(U)dx + \|H\|^2_{L_2(\Omega)},$$

and

$$p(t) = c[\|\chi_1\|^4_{L_\infty(\Omega)} + \|\chi_2\|^4_{L_\infty(\Omega)} + \|\varepsilon(u_1)\|^2_{L_\infty(\Omega)} + \|\nabla\chi_1\|^2_{L_6(\Omega)} + 1],$$

it follows from (7.11) that

$$\frac{d}{dt}D(t) + \|H\|^2_{H^2(\Omega)} \le p(t)D(t).$$

Hence, by the Gronwall lemma,

$$D(t) \le D(0)\exp\int_0^t p(t')dt'.$$

Since $D(0) = 0$ and, by assumption (2.25), $\int_0^t p(t')dt' \le c(T) < \infty$, we conclude that

$$\|U_t\|^2_{L_2(\Omega)} + \underline{c}\|\varepsilon(U)\|^2_{L_2(\Omega)} + \|H\|^2_{L_2(\Omega)} \le D(t) = 0 \quad \text{for} \quad t\in[0,T],$$

that is $U = \mathbf{0}$ and $H = 0$ in Ω^T. Besides, from $(7.3)_1$ it follows immediately that $M = 0$ in Ω^T. Thereby the proof is completed. □

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