176/2007

Raport Badawczy

Research Report

RB/77/2007

Some properties of B-efficiency
preference relation in
multiobjective optimization
problems

D.Podkopaev

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

Kierownik Pracowni zgłaszający pracę:
Prof. dr hab. inż. Janusz Kacprzyk

Warszawa 2007

Some properties of B-efficiency preference relation

in multiobjective optimization problems

Dmitry Podkopaev

We study B-efficiency preference relation which is used for modeling trade-offs in multiobjective optimization problems. This relation is the weak Pareto dominance relation over transformed vector evaluations, where the transformation is linear and is defined by a matrix of special case. We describe the situation, where number of objective functions is reduced due to singularity of the transformation matrix and interpret this situation in decision making terms. We compare the domination cone of B-efficiency preference relation with domination cones of two well-known preference relations.

0. Introduction and problem setting

We consider the multiple objective optimization problem in the following general statement:

$$\max_{x \in X} f(x),\tag{1}$$

where

X is the set of feasible solutions;

$f(x) = (f_1(x), f_2(x), \ldots, f_k(x)), f_i: X \to \mathbf{R}, i \in N_k$, are objective functions;

$N_k = \{1, 2, \ldots, k\}$.

Solving this problem means deriving an element of X, which is the most preferred for the Decision Maker (DM). As a rule, it is impossible to obtain complete information about DM's preferences. Therefore methods of deriving and manipulating partial information about DM's preferences gain in importance.

One of approaches to handling partial information about DM's preferences is to set upper bounds on trade-offs between objective functions (see for example Kaliszewski 2006). This ap-

proach is widely used in interactive methods of MCDM with relative preference expressing (see for example Kaliszewski and Zionts 2004, Roy and Wallenius 1992). A technique of deriving trade-off solutions based on linear transformation of objective functions is proposed in Podkopaev, 2007a, 2007b. A transformation matrix of special kind is used to represent information of bounds on trade-offs. It was proved in Podkopaev, 2007a that weakly Pareto optimal solutions of the transformed problem are Pareto optimal solutions of the initial problem with predetermined bounds on trade-offs.

In this work we study properties of the transformation matrix in terms of DM's preferences. In particular, in Section 2 we interpret the case where the matrix is singular and obtain a condition of its singularity. In Section 3 we compare the transformation-based preference model with two other approaches of preference representation based on analysis of domination cones.

1. Transformation-based approach to finding trade-offs solutions

Let us observe that in problem (1) each feasible solution $x \in X$ is represented by its vector evaluation $y=f(x)$. Therefore instead of problem (1) we can consider the problem

$$\max_{y \in Y} y \qquad (2)$$

where

$$Y = \{f(x): x \in X\}$$

is the set of evaluations, $Y \subset R^k$.

Given Y, the set of Pareto optimal evaluations $P(Y)$ and the set of weakly Pareto optimal evaluations $W(Y)$ are defined as

$$P(Y) = \{ y \in Y: \not\exists \, y' \in Y \, (y' \geq y \, \& \, y' \neq y) \},$$

$$W(Y) = \{ y \in Y: \forall \, y' \in Y \backslash \{y\} \, \exists \, p \in N_k \, (y_p \geq y'_p) \}.$$

Feasible solution x is called *Pareto optimal* (*weakly Pareto optimal*) solution of problem (1), if $f(x)\in P(Y)$ ($f(x)\in W(Y)$).

For any $y^*\in Y$ and any $j \in N_k$, let

$$Z_j(y^*) = \{y\in \mathbf{R}^k: y_j<y_j^* \ \& \ \forall \ s \in N_k\backslash\{j\} \ (y_s \geq y_s^*)\}.$$

Definition 1 [3]. *Let $i, j \in N_k$, $i\neq j$. If $Z_j(y^*)\cap Y \neq \varnothing$, then the number*

$$T_{ij}(y^*,Y) = \sup_{y\in Z_j(y^*)\cap Y} \frac{y_i - y_i^*}{y_j^* - y_j}$$

is called trade-off between i-th and j-th objective function for evaluation y^*. If $Z_j(y^*)\cap Y = \varnothing$, then by definition we assume $T_{ij}(y^*,Y)=-\infty$.

Interpretation 1. *Trade-off $T_{ij}(y^*,Y)$ indicates how much at most evaluation y^* can be improved in i-th component relatively to its worsening in j-th component during passage from y^* to any other evaluation from Y, under the condition that the remaining components do not worsen.*

It was proposed in (Podkopaev 2007a) to impose constraints on upper trade-off bounds with the help of linear transformation of vector evaluations. The transformation matrix $B=(\beta_{ij})_{k\times k}\in \mathbf{R}^{k\times k}$ is to be positive, its main diagonal elements have to be equal to one, and the remaining elements have to satisfy following conditions:

$$\beta_{pj}\,\beta_{ji} \leq \beta_{pi}, \quad i,j,p \in N_k, i\neq j, j\neq p, \tag{3}$$

$$\beta_{ij} \leq 1/\beta_{ji}, \quad i,j, \in N_k, i\neq j. \tag{4}$$

The transformation of Y is defined by

$$Y_B = \{By: \ y \in Y\}.$$

Theorem 1 [5]. *Let $By^*\in W(Y_B)$. Then $y^* \in P(Y)$ and for any $i,j \in N_k$, $i\neq j$, we have*

$$T_{ij}(y^*,Y) \leq \frac{1}{\beta_{ji}}.$$

The approach to setting bounds on trade-offs based on Theorem 1 is described as follows.

Let for any $i, j \in N_k$ number $\alpha_{ij} > 0$ represents the needed upper bound on trade-off between i-th and j-th objective functions. Suppose that these numbers satisfy following conditions:

$$\alpha_{pj} \, \alpha_{ji} \geq \alpha_{pi}, \quad i, j, p \in N_k, i \neq j, j \neq p, \tag{5}$$

$$\alpha_{ij} \geq 1/\alpha_{ji}, \quad i, j, \in N_k, i \neq j. \tag{6}$$

Define the elements of matrix B as $\beta_{ij} = 1/\alpha_{ji}$, $i, j \in N_k$, $i \neq j$, and put $\beta_{ii} = 1$ for all $i \in N_k$. It is easy to see that those elements satisfy conditions (3) and (4). Then, by Theorem 1, any weakly Pareto optimal solution of the problem

$$\max_{x \in X} B f(x) \tag{7}$$

is a Pareto optimal solution of problem (1) satisfying following bounds on trade-offs:

$$T_{ij}(y^*, Y) \leq \alpha_{ij} \text{ for all } i, j \in N_k.$$

We call weakly Pareto optimal solutions of problem (7) *B-efficient solutions.*

A self-depended interpretation of elements β_{ij} in terms of decision making process and reasoning of conditions (5), (6) is presented in (Podkopaev 2007b).

Interpretation 2. *β_{ij} is the maximum loss in i-th objective which DM agrees to pay for unitary gain in j-th objective under the condition that all the other objectives do not worsen.*

The approach based on linear transformation provides some advantages in comparison to other approaches to handle trade-offs known before. In particular, it allows to set bounds on trade-offs with $k(k-1)$ degrees of freedom, i. e. independently for each pair of objective functions. Another advantage is enclosing the technique of finding trade-off solutions into the framework of linear algebra, which makes possible to apply this theory to analysis of preference relation.

2. Singularity of matrix B and reduction of the number of objectives

Here and henceforth we assume that transformation matrix $B=(\beta_{ij})_{k\times k}\in \mathbf{R}^{k\times k}$ is positive, its main diagonal elements equal to one, and the remaining elements satisfy conditions (3) and (4).

Theorem 2. *Let $i,j \in N_k$, $i\neq j$. Equality $\beta_{ij}\beta_{ji} = 1$ holds if and only if i-th and j-th rows of matrix B are proportional to each other.*

Proof. Suppose $\beta_{ij}\beta_{ji} = 1$ and let us prove that i-th and j-th rows of matrix B are proportional to each other. Let $s\in N_k$. Consider three possible cases.

Case 1. $s=i$. Then we have

$$\beta_{is} = 1 = \beta_{ij}\beta_{ji} = \beta_{ij}\beta_{js}.$$

Case 2. $s=j$. Then we have

$$\beta_{is} = \beta_{ij} = \beta_{ij}\beta_{js}.$$

Case 3. $s\neq i$, $s\neq j$. Then from (3) taking into account $\beta_{ij}\beta_{ji} = 1$ we have

$$\beta_{is} \geq \beta_{ij}\beta_{js}, \tag{8}$$

$$\beta_{js} \geq \beta_{ji}\beta_{is} = \beta_{is}/\beta_{ij}.$$

The last inequality implies $\beta_{is} \leq \beta_{ij}\beta_{js}$. Combining this inequality and (8) we obtain

$$\beta_{is} = \beta_{ij}\beta_{js}. \tag{9}$$

In each of three cases we get (9). This means that i-th and j-th rows of matrix B are proportional to each other with coefficient of proportionality β_{ij}.

Now suppose that there exists a constant $c>0$ such that $\beta_{is}=c\beta_{js}$ for any $s\in N_k$. Then we have $\beta_{ij} = c\beta_{jj} = c$ and $1 = \beta_{ii} = c\beta_{ji}$ which implies $\beta_{ij}\beta_{ji} = 1$. □

It follows from Theorem 2 that in the case $\beta_{ij}\beta_{ji} = 1$ matrix B is singular and problem (7) has i-th and j-th objective functions proportional to each other.

To analyze implications from this fact, let us recall that most of quantitative criteria in real-life optimization problems has ratio scale, i. e. meaning and properties of such a criterion are

invariant to multiplying the objective function to a constant. As an example, converting a quanti-tative criterion to another metric unit (meters to kilometers, kilograms to pounds etc.) does not change its properties. Using this argumentation we can formulate following assumption:

If two objective functions in a multiple objective optimization problem are proportional to each other, then one of these objective functions is redundant.

The situation where $\beta_{ij}\beta_{ji} = 1$ can be easily understood with the help of Interpretation 2. Decision Maker agrees to pay with β_{ij} units of i-th objective function for unitary gain of j-th ob-jective function and at the same time agrees to pay with $\beta_{ji} = 1/\beta_{ij}$ units of j-th objective function for unitary gain of i-th objective function. This implies

Interpretation 3. If $\beta_{ij}\beta_{ji} = 1$, then i-th and j-th objective function are perfectly substitut-able and so they can be reduced to one surrogate function $\beta_{ij}f_i + f_j$.

Let us introduce the binary *relation of substitutability* on the set of objective functions f_1, $f_2, ., f_k$. We say, that i-th and j-th objective functions are *substitutable*, if $\beta_{ij}\beta_{ji} = 1$. Since relation of proportionality is an equivalence relation (it is reflexive, symmetric and transitive), it follows from Theorem 2 that relation of substitutability is an equivalence relation too. Let us formulate this as a corollary.

Corollary 1. *Let* $i, j, s \in N_k$, $i \neq j$, $j \neq s$. *If* $\beta_{ij}\beta_{ji} = 1$ *and* $\beta_{is}\beta_{si} = 1$ *then* $\beta_{sj}\beta_{js} = 1$.

The properties of an equivalence relation imparts to Corollary 1 an important interpreta-tion.

Interpretation 4. *The relation of substitutability generates partitioning of the set of objec-tive functions into groups. In each of there groups the objective functions are perfectly substitut-able with each other and therefore each group can be reduced to one surrogate objective func-tion.*

On our hypothesis, equality $\beta_{ij}\beta_{ji} = 1$ for some $i,j \in N_k$, $i \neq j$, is not only sufficient (as Theorem 2 states), but also necessary condition of singularity of B. This would imply that rank of B is equal to the number of groups of substitutable criteria.

3. Domination cones of B-efficient solutions

Definition 2 [1]. *Let K be a convex cone in $\mathbf{R}^k$. Evaluation $y \in Y$ (feasible solution $x \in X$) is called efficient with respect to K, if*
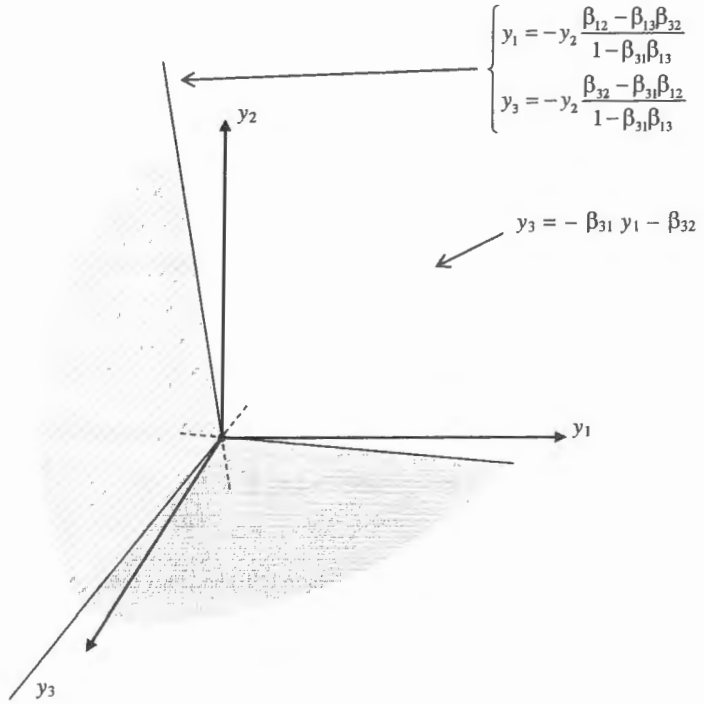
$$(\{y\}+K) \cap Y = \{\varnothing\} \ ((\{f(x)\}+K) \cap Y = \{\varnothing\}).$$

The cone associated with the notion of B-efficient solutions is defined as

$$C(B) = \{z \in \mathbf{R}^k : \forall \ i \in N_k \ \ z_i \geq -\sum_{\substack{j \in N_k \\ j \neq i}} \beta_{ij} z_j \ \}.$$

It is evident that solutions efficient with respect to $C(B)$ and only them are B-efficient solutions.

Let us depict the facets of $C(B)$ for the case of three objective functions.

$$\begin{cases} y_1 = -y_2 \dfrac{\beta_{12} - \beta_{13}\beta_{32}}{1 - \beta_{31}\beta_{13}} \\[3mm] y_3 = -y_2 \dfrac{\beta_{32} - \beta_{31}\beta_{12}}{1 - \beta_{31}\beta_{13}} \end{cases}$$

$$y_3 = -\beta_{31}\, y_1 - \beta_{32}$$

If condition (4) for $i=1$, $j=3$ turns to inequality $\beta_{13}\beta_{31}=1$, then planes $y_1 = \beta_{12}\,y_2 + \beta_{13}\,y_3$ and $y_3 = \beta_{31}\,y_1 + \beta_{32}\,y_2$ coincide and the domination cone has 2 facets. This situation is depicted on the following picture.
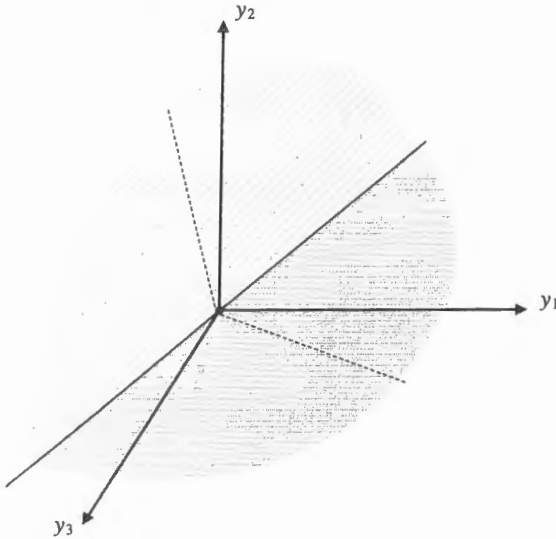
Let us compare $C(B)$ with two well-known domination cones associated with methods of finding trade-off solutions.

3.1. Proper efficiency with a prior bound ε

Wierzbicki (1986) introduced the concept of proper efficiency with a prior bound ε. This type of efficiency is based on the domination cone

$$D(\varepsilon) = \{\ z\in \mathbf{R}^k\colon \operatorname{dist}(z,C_+) \le \varepsilon\|z\|\ \},$$

where $0 < \varepsilon < 1$; $C_+ = \{\ z\in \mathbf{R}^k\colon \forall\ i\in N_k\ (z_i \ge 0)\ \}$; $\|\cdot\|$ is a norm in $\mathbf{R}^k$; $\operatorname{dist}(\cdot\,,\cdot)$ is Hausdorff distance between a point and a set defined by

$$\operatorname{dist}(z,Z) = \min\{\|z-z'\|^*\colon z'\in Z\},\ z\in\mathbf{R}^k,\ Z\subseteq\mathbf{R}^k.$$

We consider the case, where:

$$\|z\| = \|z\|_1 = \sum_{i=1}^{k} |z_i|, \ z \in \mathbf{R}^k,$$

$$\|z\|^* = \|z\|_\infty = \max\{|z_i| : \ i \in N_k\}.$$

The following lemma is evident.

Lemma 1. *If* $z \in C_+$, *then* $dist(z, C_+) = 0$. *Otherwise*

$$dist(z, C_+) = max\{ -z_i : i \in N_k \}.$$

Theorem 3. $C(B) \subseteq D(\varepsilon)$ *if and only if* $\beta_{ij} \leq \dfrac{\varepsilon}{1-\varepsilon}$ *for any* $i, j \in N_k$, $i \neq j$.

Proof. Sufficiency. Suppose that

$$\beta_{ij} \leq \frac{\varepsilon}{1-\varepsilon} \tag{10}$$

for any $i, j \in N_k$, $i \neq j$.

Let $y \in C(\mathrm{B})$. If $y \in C_+$, then it is clear that $y \in D(\varepsilon)$.

Assume that $y \notin C_+$. Let $i \in \mathrm{argmin}\{y_i : i \in N_k\}$. From the definition of $C(\mathrm{B})$ and from (10) we have $-y_i \leq \sum_{j \in N_k \setminus \{i\}} \beta_{ij} y_j \leq \sum_{j \in N_k \setminus \{i\}} \beta_{ij} |y_j| \leq \dfrac{\varepsilon}{1-\varepsilon} \sum_{j \in N_k \setminus \{i\}} |y_j|$. Multiplying this inequality to 1-ε we

obtain $-y_i \leq \varepsilon \sum_{j=1}^{k} |y_j| = \varepsilon \|y\|$. Taking into account Lemma 1 we get $y \in D(\varepsilon)$. Thus $C(B) \subseteq D(\varepsilon)$.

Necessity. Suppose that (10) does not hold. Let $-\beta_{ij} < y_i < -\dfrac{\varepsilon}{1-\varepsilon}$, $y_j = 1$ and $y_s = 0$ for any

$s \in N_k \setminus \{i, j\}$. Then we have $y_i \geq -\beta_{ij} y_j$ and $y_j \geq -\dfrac{1}{\beta_{ij}} y_i \geq \beta_{ji} y_i$ which implies $y \in C(\mathrm{B})$. But on the

other hand $\varepsilon \|y\| = \varepsilon \left(1 + \dfrac{\varepsilon}{1-\varepsilon} \right) = \dfrac{\varepsilon}{1-\varepsilon} < -y_i = \mathrm{dist}(y, C_+)$ which implies $y \notin C(\varepsilon)$. It follows that

$C(\mathrm{B}) \not\subseteq D(\varepsilon)$. $\qquad\square$

From Theorem 3 we obtain an approach to finding properly efficient solutions with prior bound ε.

Corollary 2. *Let $\varepsilon < 1$ and (10) holds for any $i,j \in N_k$, $i \neq j$. Then all B-efficient solutions are properly efficient solutions with the prior bound ε*

3.2. Finding trade-off solutions with a surrogate objective function

Consider the following cone:

$$G(\rho) = \{ \ y \in \mathbf{R}^k : \ \forall \ i \in N_k \ (y_i \geq \rho e^k y) \ \},$$

where $\rho > 0$ is a parameter, $e^k = (1,1,\ldots,1) \in \mathbf{R}^k$.

Let y^* be an element of $\mathbf{R}^k$ such that $y_i^* > y$ for any $y \in Y$

This cone corresponds to solutions of the following parametric optimization problem:

$$\min_{y \in Y} \max_{i \in N_k} \lambda_i \left(\left(y_i^* - y_i \right) + \sum_{j \in N_k} \rho_j \left(y_j^* - y_j \right) \right) \tag{11}$$

for all $\lambda_i > 0$, $i \in N_k$.

This problem is widely used in multiobjective optimization and particularly in interactive method of decision making for deriving solutions with predetermined bounds on trade-offs (see for example Kaliszewski 2006).

Theorem 4. *Let elements of B be defined by*

$$\beta_{ii} = 1, \ \beta_{ij} = \frac{\rho}{1+\rho}, \ i, j \in N_k, \ i \neq j. \tag{12}$$

Then $C(B) = G(\rho)$.

Indeed, if the matrix elements are defined as above, we have

$$z_i \geq -\sum_{\substack{j \in N_k \\ j \neq i}} \beta_{ij} z_j \ \Leftrightarrow \ z_i \geq -\frac{\rho}{1+\rho} \sum_{\substack{j \in N_k \\ j \neq i}} z_j \ \Leftrightarrow \ z_i \geq -\rho e^k z \ \text{ for any } z \in \mathbf{R}^k, i \in N_k.$$

Corollary 3. Let elements of B be defined by (12). Then B-efficient solutions and only them a solutions of problem (11) for all $\lambda_i > 0$, $i \in N_k$.

References

1. **Kaliszewski I.** (1994) Quantitative Pareto analysis by cone separation technique. *Boston : Kluwer. 160 p.*

2. **Kaliszewski I.** (2006) Soft Computing for Complex Multiple Criteria Decision Making. International Series in Operations Research & Management Science. Springer-Verlag, Berlin.

3. **Kaliszewski I., Michalowski W.** (1997) Efficient solutions and bounds on tradeoffs. *Journal of Optimization Theory and Applications, 94, 381–394.*

4. **Kaliszewski I., Zionts S.** (2004) A generalization of the Zionts-Wallenius multiple criteria decision making algorithm. *Control and Cybernetics, 33, 477–500.*

5. **Podkopaev D.** An approach to finding trade-off solutions by a linear transformation of objective functions (2007a). *Control and Cybernetics. V. 36. No 2. P. 347-356.*

6. **Podkopaev D.** Representing partial information on preferences with the help of linear transformation of objective space (2007b). *Proceedings of International Workshop on Multiple Criteria Decision Making '07.*

7. **Roy A., Wallenius J.** (1992) Nonlinear and unconstrained multiple-objective optimization: algorithm, computation and application. *Mathematical Programming, 55, 235–249.*

8. **Wierzbicki A. P.** (1986) On the completeness and constructiveness of parametric characterizations to vector optimization problems. *OR Spektrum, 8, 73–87.*