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Abstract. The paper is devoted to the solution of the energy minimization problem for a moving train. The train movement is governed by the system of the first order ordinary differential equations where the train speed and the distance along the track are the state variables. The provided locomotive power depends on the control function. The generated traction force is assumed to depend on the velocity of the train and on the control function. Each non-negative value of the control function determines a traction force control while negative values determine a braking force control. The cost functional is defined as the train energy. It is dependent on traction force, speed and control functions. The speed, distance and control functions are assumed bounded. Using the maximum principle and Lagrangian multipliers the system of equations constituting the necessary optimality conditions is formulated. Based on the analysis of the train movement the optimal trajectories in terms of train speed and associated optimal control functions are calculated. A new simplified method is used to calculate the set of the switching times implementing the optimal control function. Numerical examples are provided and discussed.

Keywords: ODEs train model, semi analytical solutions, energy minimization, maximum principle, optimal switching, numerical algorithm

1 Introduction

The minimization of the energy consumption by a train moving from one station to the other is a central issue of the railway transport both from the environmental and economic perspective. The train movement between two stations has to be completed in a given time and to satisfy the infrastructure and traffic conditions. The train speed profile has to satisfy these imposed constraints.

A comprehensive review of the modern theory of optimal train control can be found in [1–4, 10, 11, 13, 18] and references contained therein. In general for zero slope tracks, the optimal strategy consists in *power*, *speed-hold*, *coast*, *brake* phases [3, 4, 13, 20, 22]. For non-zero slope tracks this strategy has to be updated, i.e., the speed-hold mode must be interrupted by phases of power for steep uphill track sections and coast for steep downhill track sections. Moreover, the optimal strategy is reduced to the defining the optimal switching strategy [4].

In this paper we consider the energy minimization problem for train moving on the different tracks. The movement of the train is governed by an ordinary differential equations where the velocity and the distance traveled along the track are the state functions. The traction force generated by the locomotive engine is assumed to depend on the train speed and the control function. The cost functional is defined as the integral from the train power function on the train movement time interval. Using maximum principle the set of the necessary optimality conditions is formulated and optimal strategies are proposed. The optimal control function is calculated using the analytical solutions to state and adjoint equations. Moreover the simplified algorithm is used to calculate the optimal switching times and optimal velocities profiles. These features make the proposed approach to solve this train optimal control problem different from the approaches already developed in literature [4, 5, 10, 19, 20]. Numerical results are provided and discussed.

2 Problem formulation

Consider the movement of the train having the mass M along the track between two stations during the time interval $[0, T]$, $T > 0$ is a given real. Let us denote the velocity of train by $v(t) : [0, T] \to [0, v_{max}] \subset R_+$ and by $s(t) : [0, T] \to [0, s_{max}] \subset R_+$ the distance traveled by the train along the track, where v_{max} and s_{max} are given real positive constants. The movement of the train modeled as a point mass is governed by the system of the state equations [10, 11, 14]:

$$\frac{ds(t)}{dt} = v(t), \tag{1}$$

$$M\frac{dv(t)}{dt} = F - F_R, \tag{2}$$

where F and F_R denote, respectively, traction and resistance forces. Moreover the following initial conditions are imposed:

$$v(0) = 0,\ v(T) = 0,\ s(0) = 0,\ \ s(T) = s_T, \tag{3}$$

where $s_T > 0$ is a given real. Practically $s_T = s_{max}$. The traction force F is generated by the electric or diesel engine of the locomotive. It depends on the control function $u(t) : [0, T] \to [-u_{min}, u_{max}] \subset R$, u_{min} and u_{max} are positive real constants and is strictly increasing function on interval $[-u_{min}, u_{max}]$. For $u \in [-u_{min}, 0)$ the train breaks and the breaking force $F = F^- \leq 0$ is applied. The generated energy due to it may be positive, i.e., it may reduce the total train energy requirement to move it along the track. For $u \in [0, u_{max}]$ the train accelerates and the constant traction force $F = F^+ > 0$ is applied [12], i.e.,

$$F = \begin{cases} F^+ > 0 \text{ for } u \geq 0, \\ F^- \leq 0 \text{ for } u < 0. \end{cases} \tag{4}$$

The graphs of the traction and braking forces with respect to the train speed are calculated experimentally for each type of locomotive (see references in [10]). Usually the traction force is constant for the velocities in the range of the constant torque and is decreasing for the higher velocities in the constant power range where the observed acceleration is inversely proportional to the speed. The graph of the braking force is usually strongly nonlinear [10]. Recall [7] the control function may be interpreted either as the traction force or as the applied fraction of the traction force or the locomotive power to generate the desired train velocity.
The power $P(t)$ of the locomotive engine depends on the traction force F and the train velocity v, i.e.,

$$P(u(t)) = F(u(t))v(t). \tag{5}$$

The resistance force F_R is the sum of line, curve and vehicle resistance forces [3, 5–7, 10–14, 17, 19, 22]. The line resistance force F_L depends on the train mass M and the slope angle α of the train track, i.e.,

$$F_L = Mg(\sin\alpha - \mu_T \cos\alpha), \tag{6}$$

where $g = 9.81m/s^2$ and μ_T denote the gravity constant and the friction coefficient, respectively. The curve resistance force F_C depends also on the train mass M, the curve radius R, the slope angle α, the vertical slope angle β, the velocity v and the friction coefficient μ_R, i.e.,

$$F_C = Mg[\sin\alpha - \mu_T(\frac{v^2}{gR}\sin\beta + \cos\beta)\cos\alpha - \mu_R(\frac{v^2}{gR}\cos\beta - \sin\beta)]. \tag{7}$$

In literature [7] the curve resistance force is usually approximated by a simpler model known as the Roeckl formula which does not depend on velocity v. Finally the vehicle resistance force F_V usually combines the rolling resistance force as well as the air resistance force [6, 7, 17]. It strongly depends on the current train speed, the speed and the direction of wind, the area adjacent to the track for example tunnels. Usually this force is approximated by quadratic Davis formula [7] $F_V^D = \tilde{A} + \tilde{B}v + \tilde{C}v^2$ where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are positive constants calculated experimentally. The values of these constants significantly depend on the track and environment conditions. We assume that the vehicle resistance force F_V is equal to

$$F_V = p(v - w)^2, \tag{8}$$

where p and w denote the air resistance coefficient and the wind velocity, respectively. Therefore using (6), (7), (8) we obtain that the resistance force F_R is equal to

$$\begin{aligned} F_R = F_L + F_C + F_V = -p(v - w)^2 + Mg\sin\alpha - \\ \mu_T Mg(\frac{v^2}{gR}\sin\beta + \cos\beta)\cos\alpha - \mu_R Mg(\frac{v^2}{gR}\cos\beta - \sin\beta) \stackrel{def}{=} \\ M[-Av^2(t) + Bv(t) + C] \qquad \text{where} \end{aligned} \tag{9}$$

$$A = \frac{p}{M} + \frac{\mu_T \sin\beta\cos\alpha + \mu_R \cos\beta}{MR}, \quad B = \frac{2pw}{M},$$
$$C = \frac{F(t)}{M} + g\sin\alpha - \mu_T g\cos\beta\cos\alpha + \mu_R g\sin\beta - \frac{pw^2}{M}.$$

The equation (2) with the resistance force (9) is Riccati equation type. For step-wise tracking force its analytical solution can be obtained using formulas from [16]. For detailed analytical solution formula see reference [14].

2.1 Optimal control problem for a train

Let us formulate the optimal control problem for the moving train governed by system (1)-(3). Define the sets of admissible velocities and distances:

$$V_{ad} = \{v \in R : g_{1v}(v) \geq 0, \;\; g_{2v}(v) \geq 0, \;\; \text{for } \; t \in [0,T]\}, \tag{10}$$
$$S_{ad} = \{s \in R : g_{1s}(s) \geq 0, \;\; g_{2s}(s) \geq 0, \;\; \text{for } \; t \in [0,T]\}, \tag{11}$$
$$g_{1v}(v(t)) \stackrel{def}{=} v(t), \;\; g_{2v}(v(t)) \stackrel{def}{=} v_{max} - v(t), \tag{12}$$
$$g_{1s}(s(t) \stackrel{def}{=} s(t), \;\; g_{2s}(s(t)) \stackrel{def}{=} s_T - s(t). \tag{13}$$

The state functions $v(t) \in V_{ad}$ and $s(t) \in S_{ad}$ for $t \in [0,T]$ are nonnegative and bounded. Moreover for a given final velocity v_T for $t = T$ we have $v_T \in V_{ad}$. Similarly $s_T = s(T) \in S_{ad}$. Since $v(t)$ is nonnegative due to state equation (1) it follows that $s(t)$ is the increasing function. It implies that conditions (11) are always satisfied.
The energy $\tilde{E}(v,u) : V_{ad} \times U_{ad} \to R_+$ needed to move the train along the track is equal to [4]

$$\tilde{E}(v,u) = \int_0^T P(t)dt = \int_0^T F(u(t))v(t)dt. \tag{14}$$

Consider the following optimal control problem:

> Find the control function $u^\star \in U_{ad}$ and the velocity function $v^\star \in V_{ad}$ satisfying the state equations (1)-(3) and minimizing the cost functional (14) describing the energy necessary to move a train along the track, i.e.,

$$\tilde{E}(v^\star, u^\star) = \min_{u \in U_{ad}} \int_0^T F(u(t))v(t)dt, \tag{15}$$

where $v^\star$ denotes the velocity corresponding to $u^\star$.

3 Necessary optimality conditions

We shall use the maximum principle [4, 8, 9, 14] to formulate the first order necessary optimality condition for the optimal control problem (15). Let us denote by $\eta(t)$ and $\lambda(t)$ the adjoint functions associated with the state equations (1)-(2), respectively, and by μ_i, and ν_i, $i = 1, 2$ Lagrange multipliers associated with the control and velocity constraints (9) and (10), respectively. It implies that the distance constraints (11) are also satisfied. Define the Hamiltonian H and the generalized Hamiltonian L functions for the optimal control problem (15) where for the sake of simplicity $A = 0$ in (9):

$$H(s, v, u, \eta, \lambda, t) = -F(u(t))v(t) + \eta s(t) + \lambda[\frac{F(u(t))}{M} - Bv(t) - C], \quad (16)$$

$$L(s, v, u, \eta, \lambda, \mu_1, \mu_2, t) = H(s, v, u, \eta, \lambda, t) + \mu_1 g_{1u}(u) + \mu_2 g_{2u}(u) + \nu_1 g_{1v}(v) + \nu_2 g_{2v}(v). \quad (17)$$

Therefore the first order necessary optimality conditions for the optimal control problem (11) have the form:

$$-\frac{dF}{du}v(t) + \frac{\lambda(t)}{M}\frac{dF}{dt} + \mu_1 - \mu_2 = 0, \quad t \in (0, T), \quad (18)$$

$$\frac{d\eta(t)}{dt} = -\eta(t), \quad t \in (0, T), \quad (19)$$

$$\frac{d\lambda(t)}{dt} = F(u(t)) - \lambda(t)B - \nu_1 + \nu_2, \quad t \in (0, T) \quad (20)$$

$$\eta(T) = \eta_T, \quad \text{and} \quad \lambda(T) = \lambda_T. \quad (21)$$

Moreover

$$\mu_1 \geq 0 \;\; \mu_2 \geq 0 \quad \text{and} \quad \mu_1 g_{1u}(u) = 0, \;\; \mu_2 g_{2u}(u) = 0, \quad (22)$$

$$\nu_1 \geq 0, \;\; \nu_2 \geq 0, \;\; \text{and} \;\; \nu_1 g_{1v}(v) = 0, \;\; \nu_2 g_{2v}(v) = 0, \quad (23)$$

and

$$\frac{dH(t)}{dt} = \frac{dL(t)}{dt}, \quad (24)$$

for optimal trajectories $u^\star \in U_{ad}$. The following transversality conditions are imposed [9]:

$$\eta(T^-) = \eta_T = \psi, \;\; \lambda(T^-) = \lambda_T = \sum_{i=1}^{2} \delta_i \frac{\partial g_{iv}^{k-1}(t_k)}{\partial v} + \gamma = \delta_1 - \delta_2 + \gamma, \quad (25)$$

$$\delta_i g_{iv}(T) = 0, \;\; \delta_i, \geq 0, \;\; i = 1, 2. \quad (26)$$

Recall [9] $\psi = \eta_T$ and γ, δ_i, $i = 1, 2$ are real numbers.

4 Optimal train strategies

The numerical solution of the system (18)-(26) to find the optimal control $u^\star$ is time consuming and rather complex. Taking into account the linear dependence of the optimal control problem (15) on the control function u and recalling the results from the literature [4, 8–10] it is sufficient to confine to piece-wise constant optimal control functions. Consider additionally special features of the train movement, based on conditions (18)-(26), the optimal control strategies ensuring minimal energy consumption of the train may be formulated (for details see discussion in references [4, 10]). These strategies consist of few phases among the following:

1. *power phase*
   when the train accelerates from the current velocity to a given higher one and the optimal control is $u^\star = u_{max}$,
2. *speed holding phase*
   when the train moves with the constant cruising speed between the given points on the track and the optimal control equals to $u^\star \in (0, u_{max}]$,
3. *coasting phase*
   when the train is moving with non zero velocity and the traction force is equal to zero, i.e., $u^\star = 0$,
4. *braking phase*
   when the velocity is reduced from a given nonzero value to zero and optimal control is equal to $u^\star = -u_{min}$.

Applying these strategies and using the maximum principle as well as taking into account the linear dependence on control, the optimal control to problem (15) takes the following piece-wise constant form:

$$u^\star(t) = \begin{cases} u_{max} & \text{for} \lambda > Mv, \\ u_{d1} \in [0, u_{max}] & \text{for } \lambda = Mv, \\ 0 & \text{for } 0 < \lambda < Mv, \\ u_{d2} \in [-u_{min}, 0] & \text{for } \lambda = 0, \\ -u_{min} & \text{for } \lambda < 0. \end{cases} \tag{27}$$

4.1 Optimal switching times

The implementation of the optimal control (27) requires the calculation of the adjoint variable $\lambda(t)$ and the velocity $v(t)$ satisfying the optimality conditions (18)-(26) and depending on the fulfillment by the adjoint variable suitable condition in (27) the value of optimal control $u^\star$ is calculated. It leads to the establishing of the switching times [4, 8–10] of the optimal control $u^\star$ to implement optimal control strategy. Therefore to implement optimal control (27) the set of the switching times $\{\tau_i\}$ among the different values of the optimal control $u^\star$ should be determined:

$$u^\star(t) = \begin{cases} u_{max} & \text{for } 0 \le t \le \tau_1, \\ u_{d1} \in [0, u_{max}] & \text{for } \tau_1 < t \le \tau_2, \\ 0 & \text{for } \tau_2 < t \le \tau_3, \\ u_{d2} \in [-u_{min}, 0] & \text{for } \tau_3 < t \le \tau_4, \\ -u_{min} & \text{for } \tau_4 < t \le T. \end{cases} \tag{28}$$

The switching times $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ usually (see [10, 11]) are calculated numerically using state and adjoint equations as well as the set of the necessary optimality conditions. In the paper, taking into account the special features of this optimal control problem as well as the requirement to calculate them in real time the other method is used based on studying the four basic optimal velocity trajectories of the train for the piecewise constant control function. Considering different cases of the constant control functions as well as associated admissible velocity, the acceleration or braking phases suitable optimal trajectories consisting from the provided phases are provided and suitable switching times are calculated analytically. When for a given velocity function and the calculated switching times the distance traveled by the train is equal to a given distance between the two stations or characteristic points the calculated switching times are correct. Otherwise the new switching times are calculated, new type of velocity function is selected and the calculated switching times are verified. For details concerning this algorithm for determining the train optimal velocity see reference [14].

5 Numerical Examples

The computations have been carried out for the following data: $M = 202000\ kg$, traction force $F = 310000\ N$, friction coefficient $\mu_T = 0.0015$, air resistance coefficient $p = 0.1\ kg/s$. Moreover $\mu_R = 0$ and $\beta = 0$. The state equations (1)-(3) as well as the adjoint equations (19)-(21) have been solved using analytical formulas [16] where the exact analytical solutions to different types of the standard ordinary differential equations are described. Based on this general formulas from [16] the analytical solutions to the ordinary differential equations (1)-(3) and (19)-(21) where the right hand side depends linearly and nonlinearly on velocity $v(t)$ have been provided in [14]. The numerical solution of the optimal control problem (15) includes the calculation of the track slopes from a given data, train optimal velocity along this track satisfying given constraints as well as optimal control ensuring this optimal velocity. Below are presented some results of the computations of track slopes and train velocity. Other computations are carried out and the results will be delivered soon. For the given distance 1833 m between two stations characteristic points have been selected in the vector $XI = [35511\ \ 37344]\ m$. The angle $\alpha = -0.0085\ rad$ for this part of the track.
Fig. 1 displays estimation of one part of the real railway track. The slope of this part is constant. The train velocity on this track is displayed in Fig. 2. The train accelerates to the first switching point $t_f = 20\ s$, next it moves with the maximal velocity 120 km/h which is admissible on this type of track and start to brake at the second switching point $t_H = 135\ s$ to reach velocity $v = 0$ at the end of the track.

6 Conclusions

The proposed method of the optimal control of the train in the movement seems to be well suited for the real time applications. It allows to calculate the switching times of the optimal control much quicker than standard methods. The further

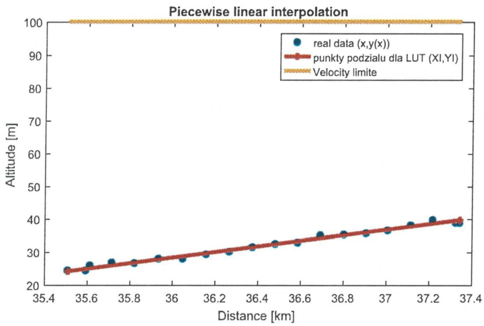

Fig. 1. Estimation of the train track.

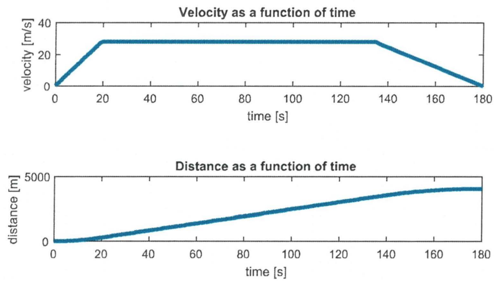

Fig. 2. a) (upper) Train velocity along the track as the function of time b) (lower) The distance traveled by train along the track as the function of time.

numerical tests are under investigation. Let us remark that similar approach may be used to other moving objects like unmanned cars or flying vehicles.

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