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# Bounds on trade-offs and stability in vector combinatorial problems

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**Abstract** We consider the multiple criteria linear problem of 0-1 programming. We study the stability of efficient solutions satisfying upper bounds on trade-off coefficients. The approach to bounding trade-offs is based on linear transformation of the criterion space. We obtain a formula of the stability radius to perturbations of the criterion function coefficients, thus establishing dependence between partial information about preferences and stability aspects.

**Keywords:** multiple objective optimization, MCDM, combinatorial optimization, bounds on trade-offs, stability radius

## 1 Introduction

The stability analysis is an important part of many fields of applied research. J. Hadamar [5] included the stability condition into the definition of a well-posed mathematical problem (along with the solution existence and uniqueness). The issue of stability in a multiple objective optimization problem arises when the feasible solution set and/or criteria functions depend on depend on uncertain parameters. Such uncertainty may be caused by inaccuracy of initial data, inadequacy of the model specification, rounding off errors and other factors. So it seams important to study conditions under which small changes of input data lead to small changes of the result. The problems satisfying such conditions are called stable. Clearly, any practical problem of decision making can not be correctly formulated and solved without addressing the issue of stability.

The stability analysis of a discrete optimization problem is usually aimed at calculating the stability radius, which is defined as the limit level of perturbations of problem parameters preserving a given property of the solution set. Stability radius is investigated for scalar problems of 0-1 programming, on systems of subsets and graphs, scheduling problems (see for example the surveys by Sotskov, Leontiev and Gordeev [10], by Sotskov, Tanaev and Werner [11] and the annotated bibliography by Greenberg [1]). In the case of multiple objectives, analogous results are obtained for very few types of problems (we refer to a short survey described in Emelichev et al. [2]). The objects of stability analysis were different types of efficient solutions. The stability of solutions obtained using some information about decision maker preferences has never been studied before.

In this paper we study the stability of efficient solutions satisfying upper bounds on trade-off coefficients. Bounding trade-offs is an approach to handling partial information about DM's preferences (see for example the book by Kaliszewski [6]). In particular, this approach is used in interactive methods of MCDM with relative preference expressing (see for example Kaliszewski and Zionts [7]). We use the technique of finding trade-off solutions based on linear transformation of the criterion space (see Podkopaev [8], [9]). We obtain a formula of stability radius of the transformed problem to perturbations of the criterion function coefficients. This formula allows establishing a dependence between the information about preferences (upper bounds on trade-offs) and the problem stability.

## 2 The problem statement

Consider a linear multiple objective combinatorial problem formulated as the 0-1 programming problem with the fixed feasible solution set:

$$Cx \to \max_{x \in X}, \tag{1}$$

Where

$X \subseteq \{0,1\}^n$, $n > 1$, is the set of feasible solutions, $| X | > 1$;

$C = [c_{ij}]_{k\times n} \in \mathbf{R}^{k\times n}$ is the matrix of vector criterion coefficients;

$k$ is the number of criteria.

We use the approach to bounding trade-off coefficients proposed by Podkopaev [8]. Let $\alpha_{ij}$ be the upper bound on the global trade-off coefficient of criterion $i$ with respect to criterion $j$ (see the definition in Kaliszewski [6]). Define the matrix $\mathrm{B} = [\beta_{ij}]_{k\times k}$ by $\beta_{ij} = \frac{1}{\alpha_{ij}}$, $\beta_{ij} = 0$ if $\alpha_{ij}$ is undefined (when the upper bound on the trade-off is infinite) and $\beta_{ii} = 1$. It was proved by Podkopaev [8] than under some conditions, the efficient (Pareto optimal) solutions of the transformed problem

$$\mathrm{B}Cx \to \max_{x \in X}, \tag{2}$$

are efficient solutions of (1) satisfying the upper bounds on trade-off coefficients.

The interpretation of solutions of problem (2) in terms of bounds of trade-offs and its comparison to the "traditional" concept of global trade-off coefficients is given in [9].

We call the efficient solution of problem (2) the B-efficient solution of problem (1). The set of B-efficient solutions is defined by

$$P_{\mathrm{B}}(C) = \{x \in X : \ \pi_{\mathrm{B}}(x, C) = \emptyset\},$$

where

$$\pi_{\mathrm{B}}(x, C) = \{x' \in X : \ \mathrm{B}Cx' \geq \mathrm{B}Cx, \ \mathrm{B}Cx' \neq \mathrm{B}Cx\}.$$

The problem of finding set $P_{\mathrm{B}}(C)$ is denoted by $Z_{\mathrm{B}}(C)$.

Both matrices $C$ and B contain parameters of problem $Z_{\mathrm{B}}(C)$. We place B in the subscript distinguishing the role of parameters $\beta_{ij}$. Matrix B presents the partial information about decision maker preferences, while $C$ contains the parameters of the mathematical model coming from the real data. We investigate the problem stability only to perturbations of the elements of matrix $C$.

We will use the following representation of the $i$-th criterion function, $i \in N_k := \{1, 2, \ldots, k\}$, of problem (2):

$$\mathrm{B}_i Cx, \ x \in X.$$

Here and henceforth a subscript at a matrix indicates the corresponding row of the matrix.

## 3 The stability radius

Let us introduce formal definitions of stability which corresponds to the notation presented in [2].

The perturbation of problem parameters is understood as an arbitrary independent change of coefficients of the criterion functions. A perturbation is modeled by adding perturbing matrix $D \in \mathbf{R}^{k \times n}$ to $C$. Thus the perturbed problem is defined by $Z_{\mathrm{B}}(C + D)$ and the perturbed set of B-efficient solutions by $P_{\mathrm{B}}(C + D)$.

For any natural $q$, we define the norms

$$\| z \|_{\infty} = \max\{| z_i |: \ i \in N_q\},$$

$$\| z \|_1 = \sum_{i \in N_p} | z_i |$$

in the space $\mathbf{R}^q$. Under the norm of a matrix we understand the norm of the vector composed from all its elements.

For any number $\varepsilon > 0$ presenting a certain level of the uncertainty, we define the set of perturbing matrices

$$\mathcal{D}(\varepsilon) = \{D \in \mathbf{R}^{k \times n} : \| D \|_{\infty} < \varepsilon\}.$$

Let (1) be a model used for solving a practical problem. Suppose that in fact the problem parameters are uncertain and are different from the parameters of (1). It makes sense to find the supremum level of uncertainty such that for any perturbation below this level, the set of B-efficient solutions of the perturbed problem includes in the set of B-efficient solutions of the initial problem. This supremum level is called the stability radius of problem (1). If the level of uncertainty in problem parameters is not greater than the stability radius, then solving problem (2) we are guaranteed that no B-efficient solution of the real-life problem will be missed.

**Definition 1** *Problem $Z_{\mathrm{B}}(C)$ is called stable (to perturbations of matrix $C$), if there exists a number $\varepsilon > 0$ such that*

$$\forall D \in \mathcal{D}(\varepsilon)\ (P_{\mathrm{B}}(C+D) \subseteq P_{\mathrm{B}}(C)).$$

**Definition 2** *The number*

$$\rho_{\mathrm{B}}(C) = \begin{cases} \sup \Omega_{\mathrm{B}}(\mathrm{C}), & \text{if } \Omega_{\mathrm{B}}(C) \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

*where $\Omega_{\mathrm{B}}(C) = \{\varepsilon > 0 :\ \forall D \in \mathcal{D}(\varepsilon)\ (P_{\mathrm{B}}(C+D) \subseteq P_{\mathrm{B}}(C))\}$, is called the stability radius of problem $Z_{\mathrm{B}}(C)$.*

In other words, the stability radius is the supremum level of perturbations which do not cause appearance of new B-efficient solutions. It is evident that the problem is stable if and only if its stability radius is positive.

Obviously, the stability radius is infinite if $P_{\mathrm{B}}(C) = X$.

Denote $\bar{P}_{\mathrm{B}}(C) = X \setminus P_{\mathrm{B}}(C)$.

By definition, put

$$\varphi_{\mathrm{B}}(C) = \min_{x \in \bar{P}_{\mathrm{B}}(C)} \ \max_{x' \in \pi_{\mathrm{B}}(x,C)} \ \min_{i \in N_k} \frac{\mathrm{B}_i C(x'-x)}{\|\ \mathrm{B}_i\ \|_1 \|\ x'-x\ \|_1}.$$

Observe that $\varphi_{\mathrm{B}}(C) \geq 0$ since $\mathrm{B}_i C(x'-x) \geq 0$ for any $x \in X$ and $x' \in \pi_{\mathrm{B}}(x,C)$.

**Theorem 1** . *Let $P_{\mathrm{B}}(C) \neq X$. Then $\rho_{\mathrm{B}}(C) = \varphi_{\mathrm{B}}(C)$.*

**Proof.** The inequality $\rho_{\mathrm{B}}(C) \geq \varphi_{\mathrm{B}}(C)$ is trivial if $\varphi_{\mathrm{B}}(C) = 0$. Let us prove this inequality in the case $\varphi_{\mathrm{B}}(C) > 0$. Suppose $D \in \mathcal{D}(\varphi_{\mathrm{B}}(C))$. Then for any $x \in \bar{P}_{\mathrm{B}}(C)$ there exists $x' \in \pi(x, C)$ such that for any $i \in N_k$ the following inequality holds:

$$\| D_i \|_\infty < \frac{\mathrm{B}_i C(x' - x)}{\| \mathrm{B}_i \|_1 \| x' - x \|_1}.$$

Then $\| D_i \|_\infty \| \mathrm{B}_i \|_1 \| x' - x \|_1 < \mathrm{B}_i C(x' - x)$. Using this inequality we obtain:
$\mathrm{B}_i(C+D)(x'-x) = \mathrm{B}_i C x' + \sum_{i \in N_k} \beta_{ij} D_j (x'-x) \geq \mathrm{B}_i C x' - \sum_{i \in N_k} \beta_{ij} \| D \|_\infty \| x'-x \|_1 > 0$.

It follows that if $\| D \|_\infty < \varphi_{\mathrm{B}}(C)$, then $\pi_{\mathrm{B}}(x, C + D) \neq \emptyset$ for any $x \in \bar{P}_{\mathrm{B}}(C)$, i. e. $P_{\mathrm{B}}(C + D) \subseteq P_{\mathrm{B}}(C)$. This yields $\rho_{\mathrm{B}}(C) \geq \varphi_{\mathrm{B}}(C)$.

It remains to prove the inequality $\rho_{\mathrm{B}}(C) \leq \varphi_{\mathrm{B}}(C)$. This will be done, if we prove that for any $\varepsilon > \varphi_{\mathrm{B}}(C)$ there exists $D \in \mathcal{D}(\varepsilon)$ such that $P_{\mathrm{B}}(C + D) \not\subseteq P_{\mathrm{B}}(C)$.

Let $\varepsilon > \delta > \varphi_{\mathrm{B}}(C)$. Consider a perturbing matrix $D = [d_{ij}]_{k \times n} \in \mathcal{D}(\varepsilon)$ defined as follows:

$$d_{ij} = \begin{cases} \delta, & \text{if } x_j = 1, \ i \in N_k, \\ -\delta, & \text{if } x_j = 0, \ i \in N_k. \end{cases}$$

Observe that since $\delta > \varphi_{\mathrm{B}}(C)$, there exists $x \in \bar{P}_{\mathrm{B}}(C)$ such that for any $x' \in \pi_{\mathrm{B}}(x, C)$ at least one index $i \in N_k$ satisfies the inequality

$$\delta > \frac{\mathrm{B}_i C(x' - x)}{\| \mathrm{B}_i \|_1 \| x' - x \|_1}.$$

From here we obtain

$\mathrm{B}_i(C + D)(x' - x) = \mathrm{B}_i C(x' - x) - \delta \| x' - x \|_1 \| \mathrm{B}_i \|_1 < 0,$

i. e. no one $x' \in \pi_{\mathrm{B}}(x, C)$ belongs to $\pi_{\mathrm{B}}(x, C+D)$. It is easy to see, that no one $x' \in X \setminus \pi_{\mathrm{B}}(x, C)$ belongs to $\pi_{\mathrm{B}}(x, C + D)$ too, since the perturbation can only increase the difference between the values of criteria in favor of $x$. Thus $\pi_{\mathrm{B}}(x, C + D) = \emptyset$ which means that $x \in P(C + D)$ and $P_{\mathrm{B}}(C + D) \not\subseteq P_{\mathrm{B}}(C + D)$.□

## 4 Conclusion

Theorem 1 provides a formula of stability radius for the linear multiple objective 0-1 programming problem of finding trade-off solutions. This formula allows one to analyze the stability radius depending on the information about decision maker preferences expressed as bounds on trade-off coefficients. Another application of this formula is regularizing problem (1). The regularization means transforming a possible unstable problem in such a way that the solution set does not change while the problem becomes stable. This approach is based on the following two evident propositions.

Denote by $E$ the identity matrix of size $k$ and by $\Theta$ the set of matrices $\mathrm{B} = [\beta_{ij}] \in \mathbf{R}^{k\times k}$ such that $\mathrm{B} \geq 0$ and $b_{ii} = 1$ for any $i \in N_k$.

**Proposition 1** *Let all the elements of* B *be positive. If* $P_{\mathrm{B}}(C) = P_E(C)$, *then* $\rho_{\mathrm{B}}(C) > 0$.

**Proposition 2** *For any problem (1) there exists a number* $\varepsilon > 0$ *such that* $P_{\mathrm{B}}(C) = P_E(C)$ *for any* $\mathrm{B} \in \Theta$, *whenever* $\beta_{ij} < \varepsilon$, $i, j \in N_k$, $i \neq j$.

It follows that having an unstable problem (1), one can find a matrix B such that the set of B-efficient solutions coincides with the set of efficient solutions and problem (2) is stable. Thus (2) for the given B serves as a regularized problem (1).

The described approach to regularization generalizes the technique based on vector criterion transformation developed by Emelichev and Yanushkevich [4] (see also [3]).

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