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Multimaterial topology optimization of variational inequalities

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Multimaterial Topology Optimization of Variational Inequalities

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Abstract. The paper is concerned with the analysis and numerical solution of multimaterial structural optimization problems for bodies in unilateral contact. Contact phenomenon is governed by the elliptic variational inequality. The structural optimization problem consists in finding such topology of the domain occupied by the body that the normal contact stress along the boundary of the body is minimized. The cost functional is regularized by the multiphase volume constrained Ginzburg-Landau energy functional. The first order necessary optimality condition is formulated. The optimal topology is obtained as the steady state of the phase transition governed by the generalized Allen-Cahn equation. The optimization problem is solved numerically using operator splitting approach combined with the projection gradient method. Numerical examples are provided and discussed.

Keywords: topology optimization, unilateral contact problems, multimaterials, phase field regularization

1 Introduction

Multimaterial topology optimization aims to find the optimal distribution of several elastic materials in a given design domain to minimize a criterion describing the mechanical or thermal properties of the structure or its cost under constraints imposed on the volume or the mass of the structure [1]. In recent years multiple phases topology optimization problems have become subject of the growing interest [1, 4, 7, 14, 18, 21]. The use of multiple number of phases during design of engineering structures opens a new opportunities in the design of smart and advanced structures in material science and/or industry. In contrast to single material design the use of multiple number of materials extends the design space and may lead to better design solutions.

Analytical and numerical aspects of the multimaterial structural optimization are subject of intensive research (see references in [1, 7, 14, 18, 21]). Many methods including the homogenization method [2], the Solid Isotropic Material Penalization (SIMP) method [20] or different methods based on the level set approach [9, 12, 13, 20], successful in single material optimization, have been extended to deal with the multimaterial optimization. The extension of these methods faces several challenges. A crucial issue in the solution of the multimaterial optimization problems is the lack of physically based parametrization of the phases mixture [1, 18]. Although different material interpolation schemes are proposed in the literature, in general, they may influence the optimization path in terms of the computational efficiency and the final design. The level set methods can eliminate the need of the material interpolation schemes provided that inter-phase interfaces are actually tracked explicitly [1]. Among others the level set method has been used in [1] to solve this problem. The elasticity tensor has been smeared out using the signed distance function and Hadamard derivative of the shape functional has been calculated. A generalized Cahn-Hilliard model of multiphase transition has been used in [21] to solve the multimaterial structural optimization problem. In [18] using phase field approach the optimality condition has been formulated as a generalized Allen–Cahn equation.

The paper is concerned with the structural topology optimization of systems governed by the variational inequalities. The class of such systems includes among others unilateral contact phenomenon [11] between the surfaces of the elastic bodies. This optimization problem consists in finding such topology of the domain occupied by the body that the normal contact stress along the boundary of the body is minimized. In literature [12] this problem usually is considered as two-phase material optimization problem with voids treated as one of the materials. In the paper the domain occupied by the body is assumed to consist from several elastic materials rather than two materials. Material fraction function is a variable subject to optimization. The regularization of the objective functional by the multiphase volume constrained Ginzburg-Landau energy functional is used. The derivative formula of the cost functional with respect to the material density function is calculated and is employed to formulate a necessary optimality condition for the topology optimization problem. This necessary optimality condition takes the form of the generalized Allen–Cahn equation. The derivative of the cost functional appears in the right hand side of this equation. Two step operator splitting approach [18] is used to solved this gradient flow equation. Finite difference and element methods are used as the approximation method. Numerical examples are reported and discussed.

2 Problem Formulation

Consider deformations of an elastic body occupying two–dimensional domain Ω with the smooth boundary Γ (see Fig. 1). Assume $\Omega \subset D$ where D is a bounded smooth hold–all subset of R^2. The body is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. We assume, that the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i, j = 0, 1, 2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

Assume that Ω is occupied by $s \geq 2$ distinct isotropic elastic materials. The void is considered as a separate phase. The materials distribution is described by

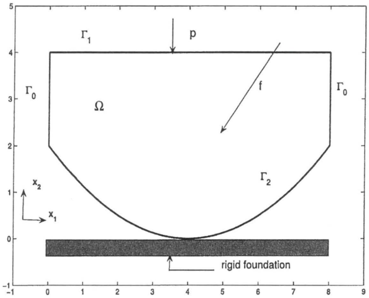

Fig. 1. Elastic body occupying domain Ω in unilateral contact with the foundation.

a phase field vector $\rho = \{\rho_i\}_{i=1}^s$ where the local fraction field $\rho_i = \rho_i(x) : \Omega \to R$, $i = 1, ..., s$, corresponds to the contributing phase. The phase field approach allows for a certain mixing between materials and between materials and void. This mixing is restricted only to a small interfacial region. In order to ensure that the phase field vector describes the fractions the following pointwise bound constraints are imposed on every ρ_i

$$\alpha_i \leq \rho_i \leq \beta_i, \text{ for } i = 1, ..., s, \text{ and } \sum_{i=1}^{s} \rho_i = 1, \tag{1}$$

where constants $0 < \alpha_i \leq \beta_i \leq 1$ are given and the summation operator is understood componentwise. The second condition in (1) ensures that no overlap and gap of fractions are allowed in the expected optimal domain. In material science field [4, 18] the intersection of the constraints (1) is called the Gibbs simplex. Moreover the total spatial amount of material fractions satisfies

$$\int_{\Omega} \rho_i(x)dx = m_i \mid \Omega \mid, \ \ 0 \leq m_i \leq 1, \text{ for } i = 1, ..., s, \text{ and } \sum_{i=1}^{s} m_i = 1, \tag{2}$$

where m_i are user defined parameters and $\mid \Omega \mid$ denotes the volume of domain Ω. The elastic tensor $\mathcal{A}$ of the material body is assumed to be a function depending on the fraction function ρ:

$$\mathcal{A}(\rho) = \sum_{i=1}^{s} g(\rho_i)\mathcal{A}_i, \ \ \mathcal{A}_i = \{a^i_{mjkl}\}^2_{m,j,k,l=1}, \tag{3}$$

with $g(\rho_i) = \rho_i^3$ as chosen function [3, 5, 17] and $\mathcal{A}_i$ is the constant stiffness tensor corresponding to the $i-th$ phase. For discussion of the interpolation of the elasticity tensor see [3, 19]. It is assumed, that elements $a^i_{mjkl}(x)$, $m, j, k, l = 1, 2$, $i = 1, ..., s$ of the elasticity tensor $\mathcal{A}_i$ satisfy [11] usual symmetry, boundedness and ellipticity conditions. Denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$,

$$\sigma_{ij}(u(x)) = \mathcal{A}(x)e_{kl}(u(x)), \quad e_{kl}(u(x)) \stackrel{def}{=} \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \tag{4}$$

where $u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}$. We use here and throughout the paper the summation convention over repeated indices [11]. The stress field σ satisfies the system of equations in the domain Ω [11]

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad \sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}, \quad x \in \Omega, \ i, j = 1, 2, \tag{5}$$

The following boundary conditions are imposed on the boundary $\partial\Omega$

$$u_i(x) = 0 \quad \text{on} \quad \Gamma_0, \quad \sigma_{ij}(x)n_j = p_i \quad \text{on} \quad \Gamma_1, \ \ i, j = 1, 2, \tag{6}$$
$$(u_N + v) \leq 0, \ \ \sigma_N \leq 0, \ \ (u_N + v)\sigma_N = 0 \quad \text{on } \Gamma_2, \tag{7}$$
$$| \sigma_T | \leq 1, \quad u_T\sigma_T + | u_T | = 0 \ \ \text{on } \Gamma_2, \tag{8}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij}n_i n_j$, $i, j = 1, 2$, represent [11] the normal components of displacement u and stress σ, respectively. The tangential components of displacement u and stress σ are given [11] by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij}n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. $| u_T |$ denotes the Euclidean norm in R^2 of the tangent vector u_T. Gap between the bodies is described by a given function v.

Let us formulate contact problem (5)-(8) in the variational form. Denote by V_{sp} and K the space and the set of kinematically admissible displacements and by Λ the set of tangential tractions on Γ_2:

$$V_{sp} = \{z \in H^1(\Omega; R^2) \ : \ z_i = 0 \text{ on } \Gamma_0, \ i = 1, 2\}, \tag{9}$$
$$K = \{z \in V_{sp} \ : \ z_N \leq 0 \ \text{ on } \ \Gamma_2\}, \quad \Lambda = \{\zeta \in L^2(\Gamma_2; R^2) \ : | \zeta | \leq 1\}. \tag{10}$$

Variational formulation of problem (5)-(8) has the form: for given $(f, p, \rho) \in L^2(\Omega; R^2) \times L^2(\Gamma_2; R^2) \times L^\infty(\Omega; R^s)$ *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying*

$$\int_\Omega \mathcal{A}(\rho)e_{ij}(u)e_{kl}(\varphi - u)dx - \int_\Omega f(\varphi - u)dx -$$
$$\int_{\Gamma_1} p(\varphi - u)ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T)ds \geq 0 \ \ \forall \varphi \in K, \tag{11}$$
$$\int_{\Gamma_2} (\zeta - \lambda)u_T ds \leq 0 \quad \forall \zeta \in \Lambda, \tag{12}$$

$i, j, k, l = 1, 2$. Function λ is interpreted as a Lagrange multiplier corresponding to term $| u_T |$ in equality constraint in (8) [11]. This function is equal to tangent stress along the boundary Γ_2, i.e., $\lambda = \sigma_{T|\Gamma_2}$. Function λ belongs to the space $H^{-1/2}(\Gamma_2; R^2)$. Here following [11] function λ is assumed to be more regular, i.e., $\lambda \in L^2(\Gamma_2; R^2)$. Recall from [11,17] by standard arguments there exists a unique solution $(u, \lambda) \in K \times \Lambda$ to the system (11)-(12).

3 Phase field based topology optimization problem

Before formulating a structural optimization problem for (11)-(12) let us introduce the set U_{ad}^{ρ} of admissible fraction functions. This set has the form

$$U_{ad}^{\rho} = \{\rho \in L^{\infty}(\Omega; R^{s-1}) : 1 - \beta_s \leq \sum_{i=1}^{s-1} \rho_i \leq 1 - \alpha_s$$

$$\alpha_i \leq \rho_i \leq \beta_i, \quad \int_{\Omega} \rho_i dx = m_i \mid \Omega \mid \quad \text{for } i = 1.,,,, s-1.\} \tag{13}$$

The set U_{ad}^{ρ} is assumed to be nonempty. Recall from [12] the cost functional approximating the normal contact stress on the contact boundary

$$J_{\eta}(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u) \eta_N(x) ds, \tag{14}$$

depending on the auxiliary given bounded function $\eta(x) \in M^{st}$. The auxiliary set $M^{st} = \{\eta = (\eta_1, \eta_2) \in H^1(D; R^2) : \eta_i \leq 0 \text{ on } D, \ i = 1, 2, \quad \| \eta \|_{[H^1(D)]^2} \leq 1\}$. Functions σ_N and η_N are the normal components of the stress field σ corresponding to a solution u satisfying system (5) - (8) and the function η, respectively. The cost functional (14) approximates the normal contact stress and is associated with the elastic energy functional [7]. Let us introduce the regularized cost functional $J(\rho, u)$ in the form:

$$J(\rho, u) = J_{\eta}(u) + E(\rho), \tag{15}$$

where the functional $J_{\eta}(u)$ is given by (14). The Ginzburg-Landau free energy functional $E(\rho)$ is expressed as

$$E(\rho) = \sum_{i=1}^{s-1} \int_{\Omega} \psi(\rho_i) d\Omega, \quad \psi(\rho_i) = \frac{\gamma\epsilon}{2} \mid \nabla \rho_i \mid^2 + \frac{\gamma}{\epsilon} \psi_B(\rho_i), \tag{16}$$

where $\epsilon > 0$ is a constant, $\gamma > 0$ is a parameter related to the interfacial energy density. Function $\psi_B(\rho_i) = \rho_i^2(1 - \rho_i^2)$ is a double-well potential [8] which characterizes the phases [18].

The structural optimization problem for system (11)-(12) takes the form: *find* $\rho^{\star} \in U_{ad}^{\rho}$ *such that*

$$J(\rho^{\star}, u^{\star}) = \min_{\rho \in U_{ad}^{\rho}} J(\rho, u), \tag{17}$$

where $u^{\star} = u(\rho^{\star})$ denotes a solution to the state system (11)-(12) depending on $\rho^{\star}$ and U_{ad}^{ρ} is given by (13). The existence of an optimal solution $\rho^{\star} \in U_{ad}^{\rho}$ to the problem (17) follows by classical arguments (see [4–6, 17]).

4 Necessary optimality condition

In order to compute the first variation of the cost functional (15) we apply a Lagrangian approach combined with Allen–Cahn approach [2, 4, 12, 18]. Let us introduce the Lagrangian $L(\rho) = L(\rho, u, \lambda, p^a, q^a, \mu) : L^\infty(\Omega; R^{s-1}) \cap U^\rho_{ad} \times H^1(\Omega; R^2) \times L^2(\Gamma_2; R^2) \times K_1 \times \Lambda_1 \times R^{s-1}$ associated to the problem (17):

$$L(\rho, u, \lambda, p^a, q^a, \mu) = J_\eta(u) + E(\rho) + \sum_{i=1}^{s-1} \int_\Omega g(\rho_i) a_{mjkl} e_{mj}(u) e_{kl}(p^a) dx - \int_\Omega f_i p_i^a dx - \int_{\Gamma_1} p_i p_i^a ds + \int_{\Gamma_2} \lambda p_T^a ds + \int_{\Gamma_2} q^a u_T ds + \sum_{i=1}^{s-1} \mu_i \Big(\int_\Omega \rho_i(x) dx - m_i \mid \Omega \mid \Big), \tag{18}$$

where $(p^a, q^a) \in K_1 \times \Lambda_1$ denotes an adjoint state defined as follows:

$$\sum_{i=1}^{s-1} \int_\Omega g(\rho_i) a_{mjkl} e_{mj}(\eta + p^a) e_{kl}(\varphi) dx + \int_{\Gamma_2} q^a \varphi_T ds = 0 \quad \forall \varphi \in K_1, \tag{19}$$

$m, j, k, l = 1, 2$ and

$$\int_{\Gamma_2} \zeta (p_T^a + \eta_T) ds = 0 \ \ \forall \zeta \in \Lambda_1. \tag{20}$$

The sets K_1 and Λ_1 are given by

$$K_1 = \{ \xi \in V_{sp} \ : \ \xi_N = 0 \ \text{ on } A^{st} \ \}, \tag{21}$$

$$\Lambda_1 = \{ \zeta \in \Lambda \ : \ \zeta(x) = 0 \text{ on } B_1 \cup B_2 \cup B_1^+ \cup B_2^+ \}, \tag{22}$$

while the coincidence set $A^{st} = \{ x \in \Gamma_2 \ : \ u_N + v = 0 \}$. Moreover $B_1 = \{ x \in \Gamma_2 : \lambda(x) = -1 \}$, $B_2 = \{ x \in \Gamma_2 : \lambda(x) = +1 \}$, $\tilde{B}_i = \{ x \in B_i : u_N(x) + v = 0 \}$, $i = 1, 2$, $B_i^+ = B_i \setminus \tilde{B}_i$, $i = 1, 2$. The derivative of the Lagrangian L with respect to ρ has the form for all $\zeta \in H^1(\Omega; R^{s-1})$

$$\int_\Omega \frac{\partial J}{\partial \rho}(\rho, u) \zeta dx = \int_\Omega \frac{\partial L}{\partial \rho}(\rho, u, \lambda, p^a, q^a, \mu) \zeta dx = \sum_{i=1}^{s-1} \int_\Omega [\gamma \epsilon \nabla \rho_i \cdot \nabla \zeta + \frac{\gamma}{\epsilon} \psi_B'(\rho_i) \zeta + \mu_i \zeta] dx + \sum_{i=1}^{s-1} \int_\Omega [g'(\rho_i) a_{mjkl} e_{mj}(u_\epsilon) e_{kl}(p^a + \eta) - f(p^a + \eta)] \zeta dx. \tag{23}$$

By standard arguments [2, 11, 16] the necessary optimality condition to the optimization problem (17) has the form:

Theorem 1. *Let* $(\rho^\star, u^\star, \lambda^\star, p^{a\star}, q^{a\star}, \mu^\star) \in L^\infty(\Omega; R^{s-1}) \cap U^\rho_{ad} \times H^1(\Omega; R^2) \times L^2(\Gamma_2; R) \times K_1 \times \Lambda_1 \times R^{s-1}$ *be an optimal solution to structural optimization* *problem (17). Than it satisfies (5)-(8), (19)-(20) and for all* $(\rho, u, \lambda, p^a, q^a, \mu) \in L^\infty(\Omega, R^{s-1}) \cap U_{ad}^\rho \times H^1(\Omega; R^2) \times L^2(\Gamma_2; R^2) \times K_1 \times \Lambda_1 \times R^{s-1}$ *it holds*

$$L(\rho^\star, u^\star, \lambda^\star, p^{a\star}, q^{a\star}, \mu) \leq L(\rho^\star, u^\star, \lambda^\star, p^{a\star}, q^{a\star}, \mu^\star) \leq L(\rho, u, \lambda, p^a, q^a, \mu^\star). \tag{24}$$

Based on (24) an optimal solution to problem (17) can be found using Uzawa method combined with gradient flow equation in primal step (see [12] for details). Here the projection technique on the set of admissible fraction functions is used rather than Uzawa approach. Assume $\rho = \rho(x,t), t \in [0,T]$. Denote by $P_{U_{ad}^\rho}$ the orthogonal projection operator on the set U_{ad}^ρ and by ρ_{0i}, $i = 1, ..., s$ a given functions. Using the projection of the cost functional derivative (23) on the admissible set (13) the necessary optimality condition to problem (17) takes the form: *find sufficiently regular* $(\rho^\star, u^\star, \lambda^\star, p^{a\star}, q^{a\star})$ *satisfying (5)-(8), (19)-(20) as well as*

$$\frac{\partial \rho(t)}{\partial t} = P_{U_{ad}^\rho}(\rho - \frac{\partial J(\rho,u)}{\partial \rho}) - \rho \quad \text{in } \Omega,\ \forall t \in [0,T), \tag{25}$$

$$\nabla \rho_i \cdot n = 0 \quad \text{on } \partial\Omega,\ \forall t \in [0,T),\ i = 1, ..., s-1, \tag{26}$$

$$\rho_i(0,x) = \rho_{i0}(x) \quad \text{in } \Omega,\ t = 0,\ i = 1, ..., s-1. \tag{27}$$

Remark, for $\rho(x,t) = \rho^\star(x,t)$ the right hand side of equation (25) vanishes, i.e., $\frac{\partial \rho(t)}{\partial t} = 0$ and $\rho^\star$ is an optimal solution to the problem (17). Therefore the system (25)-(27) is the constrained gradient flow equation for the cost functional (15) of Allen–Cahn type.

4.1 Operator splitting approach

For the sake of numerical calculations we reformulate the optimality condition (25)-(27) using the operator splitting approach [18]. The cost functional (15) is sum of two functionals

$$J(\rho, u) = J_1(\rho, u) + J_2(\rho), \tag{28}$$

$$J_1(\rho, u) = J_\eta(u) + \sum_{i=1}^{s-1} \int_\Omega \frac{\gamma}{\epsilon} \psi_B(\rho_i) d\Omega, \quad J_2(\rho) = \sum_{i=1}^{s-1} \int_\Omega \frac{\gamma\epsilon}{2} \mid \nabla \rho_i \mid^2 d\Omega. \tag{29}$$

Based on (23) one can calculate the derivatives $\frac{\partial J_1}{\partial \rho}$ and $\frac{\partial J_2}{\partial \rho}$ of the functionals J_1 and J_2, respectively. Assume the time interval $[0,T]$ is divided into N subintervals with step $\Delta t = t_{k+1} - t_k$, $k = 0, ..., N$ and $\rho_k = \rho(t_k)$ is known. The control variable ρ_{k+1} at the next time step t_{k+1} is calculated in two steps. First the trial value $\tilde{\rho}$ is calculated from the gradient flow equation (25) for the functional J_1:

$$\frac{d\tilde{\rho}}{dt} = P_{U_{ad}^\rho}(\tilde{\rho} - \frac{\partial J_1}{\partial \rho}) - \tilde{\rho}, \quad \tilde{\rho}(t_k) = \rho_k, \quad t_k < t \leq t_{k+1}. \tag{30}$$

Next this solution is updated by solving the gradient flow equation for the functional J_2 with the boundary condition (26)

$$\frac{d\rho}{dt} = -\frac{\partial J_2}{\partial \rho}, \quad \rho(t_k) = \tilde{\rho}_{k+1}, \quad t_k < t \leq t_{k+1}. \tag{31}$$

Therefore the conceptual algorithm for solving the system (25)-(27) has the form

Step 1: Choose $\rho_0 \in U^{\rho}_{ad}$ as well as constants $tol > 0$ and $k_{max} > 0$. Set $k = 0$.
Step 2: For given ρ_k find (u_k, λ_k) satisfying (11)-(12) and
(p^a_k, q^a_k) satisfying (19)-(20).
Step 3: Compute $\frac{\partial J_{1k}}{\partial \rho}$ using (23). Next compute $\tilde{\rho}_k$ using (30), i.e.,
$\tilde{\rho}_k = \rho_k + \Delta t_k \Big(P_{U^{\rho}_{ad}}[\rho_k - \frac{\partial J_{1k}}{\partial \rho}] - \rho_k \Big)$.
Step 4: For given $\tilde{\rho}_k$ calculate ρ_{k+1} solving (31), i.e.,
$\rho_{k+1} = \tilde{\rho}_k + \Delta t_k \gamma \epsilon \Delta \rho_{k+1}$.
Step 5: If $\| \rho_{k+1} - \rho_k \|_{L^\infty(\Omega; R^{s-1})} \leq tol$ or $k \geq k_{max}$ Stop.
Otherwise go to Step 2.

5 Numerical results

The discretized structural optimization problem (17) is solved numerically. Time derivatives are approximated by the forward finite difference. Piecewise constant and piecewise linear finite element method is used as discretization method in space variables. The derivative of the double well potential is linearized with respect to ρ_i. Primal-dual active set method has been used to solve state and adjoint systems (5)-(8) and (19)-(20). Scheme (30)-(31) has been used to solve (25)-(27). The algorithms are programmed in Matlab environment. As an example a body occupying 2D domain

$$\Omega = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ 0 < v(x_1) \leq x_2 \leq 4\}, \tag{32}$$

is considered. The boundary Γ of the domain Ω is divided into three pieces

$$\begin{aligned} \Gamma_0 &= \{(x_1, x_2) \in R^2 : x_1 = 0, 8 \ \wedge \ 0 < v(x_1) \leq x_2 \leq 4\}, \\ \Gamma_1 &= \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ x_2 = 4\}, \\ \Gamma_2 &= \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ v(x_1) = x_2\}. \end{aligned} \tag{33}$$

The domain Ω and the boundary Γ_2 depend on the function $v(x_1) = 0.125(x_1 - 4)^2$. The body is loaded by boundary traction $p_1 = 0$, $p_2 = -5.6 \cdot 10^6 N$ along Γ_1, body forces $f_i = 0$, $i = 1, 2$. Auxiliary function η is selected as piecewise constant (or linear) on D and is approximated by a piecewise constant (or bilinear) functions. The computational domain $D = [0, 8] \times [0, 4]$ is selected. Domain D is discretized with a fixed rectangular mesh of 80×40.

Fig. 2 presents the optimal domain obtained by solving structural optimization problem (17) in the computational domain D using the optimality condition (25)-(27). The areas with weak phases appear in the central part of the body and

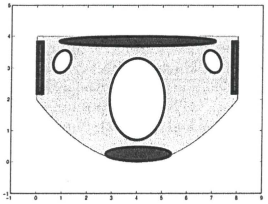

Fig. 2. Optimal material distribution in domain $\Omega^{\star}$.

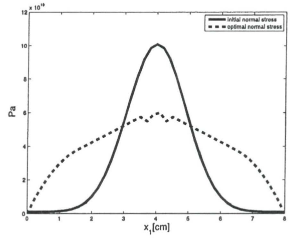

Fig. 3. Initial and optimal normal contact stress.

near the fixed edges. The areas with strong phases appear closed to the contact zone and along the edges. The rest of the domain is covered with intermediate phase. The obtained normal contact stress is almost constant along the optimal shape boundary and has been significantly reduced comparing to the initial one (see Fig. 3).

6 Conclusions

The structural optimization problem for elastic contact problem with Tresca friction has been solved numerically in the paper. Obtained numerical results indicate that the optimal topologies are qualitatively comparable to the results reported in other phase-field topology optimization methods. Since the optimization problem is non-convex it has possibly many local solutions dependent on initial estimate. Gradient flow method employed in H^1 space is more regular and efficient than standard Allen-Cahn approach.

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