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SIXTH ORDER CAHN-HILLIARD TYPE EQUATION

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ABSTRACT. In this article we are concerned with an initial-boundary-value problem for a sixth order Cahn-Hilliard type equation. The problem describes dynamics of phase transitions in ternary oil-water-surfactant systems in which three phases, microemulsion almost pure oil and almost pure water can coexist in equlibrium. We prove the existence and uniqueness of a strong, large time solution to such problem.

1. Introduction. In this article we are concerned with an initial-boundary-value problem for a sixth order Cahn-Hilliard type equation in 3-D. The problem describes dynamics of phase transitions in ternary oil-water-surfactant systems in which three phases, microemulsion, almost pure oil and almost pure water can coexist in equilibrium.

Such systems attract a lot of interest because of their unusual properties and important industrial and commercial applications. Surfactant is a surface active molecule which has amphiphilic character; one part of it is hydrophilic (water-loving) and the other lipophilic (fat-loving). Such molecule is called amphiphile.

When a small amount of amphiphilic molecules is added to a phase separated mixture of oil and water, a homogeneous microemulsion phase forms. Microemulsion is macroscopically a single-phase structured fluid. It consists of homogeneous regions of oil and water which form a complicated, intertwinned network with a typical length scale of a few hundred Å. This is possible because of their amphiphilic character, the surfactant molecules form a monolayer at the interface between oil

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and water regions and thereby reduce the interfacial tensions so that a phase with an extensive amount of internal interface can become stable.

In a line of their papers G. Gompper et al. [3]–[8] have proposed the following second order Landau-Ginzburg free energy reflecting the performed scattering experiments

$$\mathcal{F}\{\chi\} = \int_\Omega f(\chi, \nabla\chi, \nabla^2\chi)dx, \quad \Omega \subset \mathbb{R}^3, \tag{1}$$

with the density

$$f(\chi, \nabla\chi, \nabla^2\chi) = f_0(\chi) + \frac{1}{2}\kappa_1(\chi)|\nabla\chi|^2 + \frac{1}{2}\kappa_2(\Delta\chi)^2.$$

Here χ is the scalar order parameter which is proportional to the local difference of the oil and water concentrations. The properties of the amphiphile and its concentration enter the model (1) implicitly via the form of functions $f_0(\chi)$ and $\kappa_1(\chi)$ as well as the magnitude of constant $\kappa_2 > 0$.

The function $f_0(\chi)$ is the volumetric free energy density with three local minima at $\chi = \chi_0$, $\chi = \chi_w$ and $\chi = 0$ corresponding to oil-rich, water-rich and microemulsion phases. In case of oil-water symmetry $-\chi_0 = \chi_w = \chi_{bulk} = 1$, the following sixth order polynomial approximation has been used

$$f_0(\chi) = (\chi+1)^2(\chi^2+h_0)(\chi-1)^2 \tag{2}$$

with parameter $h_o \in \mathbb{R}$ corresponding to a deviation from oil-water-microemulsion coexistence.

The coefficient $\kappa_1(\chi)$ has been approximated by the quadratic function

$$\kappa_1(\chi) = g_0 + g_2\chi^2 \tag{3}$$

with constants $g_0 \in \mathbb{R}$ (negative in the microemulsion phase) and $g_2 > 0$. To reflect the scattering experiments the coefficient κ_2 has to be positive, assumed constant.

The free energy functional (1) has been studied in detail in the above mentioned papers, in particular stationary solutions to the corresponding Euler-Lagrange equation have been analysed.

2. **The dynamical model. The main result.** In the present paper we introduce a dynamical model for a conserved order parameter χ, governed by free energy (1). The model is a direct extension of the Cahn-Hilliard theory to a second order free energy (1). We assume the conservation of mass

$$\chi_t + \nabla\cdot\mathbf{j} = 0 \tag{4}$$

together with the following constitutive law for the mass flux $\mathbf{j}$

$$\mathbf{j} = -M\nabla\mu \tag{5}$$

where $M > 0$ is a constant mobility and μ is the chemical potential (driving force). In accord with the Cahn-Hilliard theory the chemical potential is given as the first variation of the free energy with respect to the order parameter

$$\mu = \frac{\delta f}{\delta\chi} \tag{6}$$

where $\delta f/\delta\chi$ is defined by the condition that

$$\frac{d}{d\lambda}\mathcal{F}\{\chi+\lambda\zeta\}|_{\lambda=0} =: \int_\Omega \frac{\delta f}{\delta\chi}\zeta dx$$

is to hold for all test functions $\zeta \in C_0^\infty(\Omega)$. In case of (1) this gives

$$\begin{aligned}\mu &= f_{0,\chi}(\chi) + \frac{1}{2}\kappa_{1,\chi}(\chi)|\nabla\chi|^2 - \nabla\cdot(\kappa_1(\chi)\nabla\chi) + \kappa_2\Delta^2\chi \\ &= f_{0,\chi}(\chi) - \frac{1}{2}\kappa_{1,\chi}(\chi)|\nabla\chi|^2 - \kappa_1(\chi)\Delta\chi + \kappa_2\Delta^2\chi,\end{aligned} \tag{7}$$

where $f_{,\chi}(\chi) = df(\chi)/d\chi$.
In result, equations (4)-(7) lead to the system

$$\begin{aligned}&\chi_t - \nabla\cdot(M\nabla\mu) = 0 && \text{in } \Omega^T = \Omega\times(0,T), \\ &\mu = f_{0,\chi} + \frac{1}{2}\kappa_{1,\chi}|\nabla\chi|^2 - \nabla\cdot(\kappa_1\nabla\chi) + \kappa_2\Delta^2\chi && \text{in } \Omega^T,\end{aligned} \tag{8}$$

which is supplemented by the initial and boundary conditions

$$\chi|_{t=0} = \chi_0 \quad \text{in } \Omega, \tag{9}$$

$$\mathbf{n}\cdot\nabla\chi = 0, \quad \mathbf{n}\cdot\nabla\Delta\chi = 0, \quad \mathbf{n}\cdot\nabla\mu = 0 \quad \text{on } S^T = S\times(0,T). \tag{10}$$

Here $\Omega \subset \mathbb{R}^3$ is assumed to be a bounded domain with a smooth boundary S, occupied by a ternary mixture, $T > 0$ is a time horizon, and $\mathbf{n}$ is the outward unit normal to S.

The boundary conditions $(10)_{1,2}$ are "natural" for the functional (1), used in the derivation of the energy identity (see (18) below). The condition $(10)_3$ represents the mass isolation at the boundary S.

Combining equations $(8)_1$ and $(8)_2$, and taking into account that by $(10)_{1,2}$,

$$\mathbf{n}\cdot\nabla\mu = \mathbf{n}\cdot\left[-\frac{1}{2}\kappa_{1,\chi}\nabla(|\nabla\chi|^2) + \kappa_2\nabla\Delta^2\chi\right],$$

we get the following equivalent formulation of system (8)–(10) in the form of an initial-boundary value problem for a sixth order Cahn-Hilliard type equation:

$$\chi_t - M\kappa_2\Delta^3\chi = M\Delta\left[f_{0,\chi}(\chi) + \frac{1}{2}\kappa_{1,\chi}(\chi)|\nabla\chi|^2 - \nabla\cdot(\kappa_{1,\chi}(\chi)\nabla\chi)\right] \quad \text{in } \Omega^T, \tag{11}$$

$$\chi|_{t=0} = \chi_0 \quad \text{in } \Omega, \tag{12}$$

$$\mathbf{n}\cdot\nabla\chi = 0, \quad \mathbf{n}\cdot\nabla\Delta\chi = 0, \quad \kappa_2\mathbf{n}\cdot\nabla\Delta^2\chi = \frac{1}{2}\kappa_{1,\chi}(\chi)\mathbf{n}\cdot\nabla(|\nabla\chi|^2) \quad \text{on } S^T. \tag{13}$$

In contrast to the classical Cahn-Hilliard problem (the case $\kappa_2 = 0$ and $\kappa_1 = \text{const} > 0$) system (11)–(13) includes the nonlinear boundary condition.

The main result of this paper concerns the existence and uniqueness of the strong, large time solution to problem (11)–(13).
Higher order generalizations of the Cahn-Hilliard equation attract recently some mathematical interest.

In [9] stationary solutions to one-dimensional sixth order convective Cahn-Hilliard type equation arising in epitaxially growing nano-structures have been analysed. moreover other related references have been indicated.

We mention also recent reference [1] where a dynamical sixts order Cahn-Hilliard equation in 2-D, arising in a similar context as in [9], has been studied from the point of view of the existence of global weak solutions.

We state now our result.

Theorem 2.1. *Assume $f(\chi, \nabla\chi, \nabla^2\chi)$ is given by (1) where $f_0(\chi)$ is a sixth order polynomial satisfying the condition*

$$f_0(\chi) \geq c\chi^6 - \bar{c} \quad \text{for all } \chi \in \mathbb{R}, \quad c > 0, \quad \bar{c} \geq 0, \tag{14}$$

$\kappa_1(\chi)$ has the form (3) and κ_2, M are positive constants. Let

$$\chi(0) = \chi_0 \in H^3(\Omega), \tag{15}$$

and $\chi_t(0)$ satisfies

$$\chi_t(0) = M\kappa_2\Delta^3\chi_0 + M\Delta\Big[f_{0,\chi_0}(\chi_0) + \frac{1}{2}\kappa_{1,\chi_0}(\chi_0)|\nabla\chi_0|^2 \\ -\nabla\cdot(\kappa_{1,\chi_0}(\chi_0)\nabla\chi_0)\Big] \in L_2(\Omega).$$

Moreover, the compatibility conditions hold on S:

$$\mathbf{n}\cdot\nabla\chi_0 = 0, \quad \mathbf{n}\cdot\nabla\Delta\chi_0 = 0, \quad \kappa_2\mathbf{n}\cdot\nabla\Delta^2\chi_0 = \frac{1}{2}\kappa_{1,\chi_0}(\chi_0)\mathbf{n}\cdot\nabla(|\nabla\chi_0|^2).$$

Then for any $T > 0$ problem (11)–(13) has a unique strong solution $\chi \in W_2^{6,1}(\Omega^T)$ satisfying the estimate

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \leq c \tag{16}$$

with a constant $c = \varphi(\|\chi_0\|_{H^3(\Omega)}, \|\chi_t(0)\|_{L_2(\Omega)}, T)$, where $\varphi(\cdot)$ is an increasing, positive function of its arguments.

Above and hereafter we use the following notation:

$W_2^k(\Omega) = H^k(\Omega)$, $k \in \mathbb{N}$;

$W_p^{kl,l}(\Omega^T) = L_p(0,T; W_p^{kl}(\Omega)) \cap W_p^l(0,T; L_p(\Omega))$, $k \in \mathbb{N}$, $l \in \mathbb{N}$, $p \in [1,\infty)$– the Sobolev space with the norm

$$\|u\|_{W_p^{kl,l}(\Omega^T)} = \left(\sum_{|\alpha|+ka\leq kl}\int_{\Omega^T}|D_x^\alpha\partial_t^a u|^p dxdt\right)^{1/p};$$

$W_p^{ks,s}(\Omega^T) = L_p(0,T; W_p^{ks}(\Omega)) \cap W_p^s(0,T; L_p(\Omega))$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$, $p \in [1,\infty)$ – the Sobolev-Slobodecki space with the norm

$$\|u\|_{W_p^{ks,s}(\Omega^T)} = \Bigg(\sum_{|\alpha|+ka\leq[ks]}\int_{\Omega^T}|D_x^\alpha\partial_t^a u|^p dxdt \\ + \sum_{|\alpha|=[ks]}\int_0^T\int_\Omega\int_\Omega\frac{|D_x^\alpha u(x,t) - D_{x'}^\alpha u(x',t)|^p}{|x-x'|^{n+p(ks-[ks])}}dxdx'dt \\ + \int_\Omega\int_0^T\int_0^T\frac{|\partial_t^{[s]}u(x,t) - \partial_{t'}^{[s]}u(x,t')|^p}{|t-t'|^{1+p(s-[s])}}dtdt'dx\Bigg)^{1/p}$$

where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$.

By $\varphi(\cdot)$ we denote a generic function which is an increasing, positive function of its arguments.

The proof of Theorem 2.1 is based on the Leray-Schauder fixed point theorem. The outline of the proof is presented in Section 4. The crucial part of the proof constitute a priori estimates which are derived by a successive improving of the

basic energy estimates up to getting estimate (16). The main tool in this procedure is the classical parabolic theory due to Solonnikov [11].
The outline of the derivation of a priori estimates is given in Section 3. A detailed version of the presented results will appear in [10].

3. **A priori estimates.** We begin with noting the conservation property of system (8)–(10) which follows by integrating $(8)_1$ over Ω and using boundary condition $(10)_3$:

$$\frac{d}{dt}\int_\Omega \chi dx = 0.$$

This shows that the mean value of χ is preserved, i.e.,

$$⨍_\Omega \chi(t)dx = ⨍_\Omega \chi_0 dx \quad \text{for all} \;\; t \geq 0. \tag{17}$$

Next we derive the energy identity for (8)–(10).

Lemma 3.1. *If χ and μ satisfying (8)–(10) are sufficiently regular then the following identity holds*

$$\frac{d}{dt}\int_\Omega \left[f_0(\chi)+\frac{1}{2}\kappa_1(\chi)|\nabla\chi|^2+\frac{1}{2}\kappa_2(\Delta\chi)^2\right]dx + M\int_\Omega |\nabla\mu|^2dx = 0 \quad for \;\; t\geq 0. \tag{18}$$

Proof. Multiplying $(8)_1$ by μ, integrating over Ω and by parts, using boundary condition $(10)_3$ leads to

$$\int_\Omega \chi_t\mu dx + M\int_\Omega |\nabla\mu|^2dx = 0.$$

Further, multiplying $(8)_2$ by $-\chi_t$ and integrating over Ω gives

$$-\int_\Omega \mu\chi_t dx + \int_\Omega \Big[f_{0,\chi}(\chi)\chi_t + \frac{1}{2}\kappa_{1,\chi}(\chi)|\nabla\chi|^2\chi_t$$
$$-\nabla\cdot(\kappa_1(\chi)\nabla\chi)\chi_t + \kappa_2\Delta^2\chi\chi_t\Big]dx = 0.$$

Adding the above identities by sides and noting the following relations, resulting on account of boundary conditions $(10)_{1,2}$,

$$\int_\Omega f_{0,\chi}(\chi)\chi_t dx = \frac{d}{dt}\int_\Omega f_0(\chi)dx,$$

$$\int_\Omega \left[\frac{1}{2}\kappa_{1,\chi}(\chi)|\nabla\chi|^2\chi_t - \nabla\cdot(\kappa_1(\chi)\nabla\chi)\chi_t\right]dx$$
$$= \int_\Omega \left[\frac{1}{2}\kappa_{1,\chi}(\chi)|\nabla\chi|^2\chi_t + \kappa_1(\chi)\nabla\chi\cdot\nabla\chi_t\right]dx$$
$$= \frac{1}{2}\frac{d}{dt}\int_\Omega \kappa_1(\chi)|\nabla\chi|^2,$$

$$\kappa_2 \int_\Omega \Delta^2\chi\chi_t dx = -\kappa_2 \int_\Omega \nabla\Delta\chi \cdot \nabla\chi_t dx$$
$$= \kappa_2 \int_\Omega \Delta\chi\Delta\chi_t dx = \frac{\kappa_2}{2}\frac{d}{dt}\int_\Omega (\Delta\chi)^2 dx,$$

we conclude the equality (18). □

The identity (18) shows that the total energy (1) is the Lyapunov functional for the system (8)–(10). Further, by the assumption on $f(\chi, \nabla\chi, \nabla^2\chi)$ we deduce from (18) the following energy estimate

$$\|\chi\|_{L_\infty(0,T;L_6(\Omega))} + \|\chi\|_{L_\infty(0,T;H^2(\Omega))} + \|\nabla\mu\|_{L_2(\Omega^T)} \le c_1 = \varphi(\|\chi_0\|_{H^2(\Omega)}). \quad (19)$$

Hence,

$$\|\chi\|_{L_\infty(\Omega^T)} + \|\nabla\chi\|_{L_\infty(0,T;L_6(\Omega))} \le c_1. \quad (20)$$

Moreover, since $\left|\int_\Omega \mu dx\right| \le c_1$, the Poincaré inequality implies that

$$\|\mu\|_{L_2(0,T;H^1(\Omega))} \le c_1. \quad (21)$$

The next lemma provides the bounds for the separate third order and fifth order terms in the formula for $\nabla\mu$:

$$\nabla\mu = f_{0,\chi\chi}\nabla\chi - \frac{1}{2}\kappa_{1,\chi\chi}|\nabla\chi|^2\nabla\chi - \frac{1}{2}\kappa_{1,\chi}\nabla(|\nabla\chi|^2)$$
$$- \kappa_{1,\chi}\Delta\chi\nabla\chi - \kappa_1\nabla\Delta\chi + \kappa_2\nabla\Delta^2\chi.$$

Lemma 3.2. *There is a positive constant $c_2 = \varphi(\|\chi_0\|_{H^2(\Omega)}, T)$ such that*

$$\|\chi\|_{L_2(0,T;H^3(\Omega))} \le c_2, \quad (22)$$
$$\|\chi\|_{L_2(0,T;H^5(\Omega))} \le c_2.$$

Proof. (outline). Multiplying equation $(8)_1$ by χ, integrating over Ω and by parts and then using equation $(8)_2$ for μ we deduce estimate $\|\nabla\Delta\chi\|_{L_2(\Omega^T)} \le c_2$ which by the elliptic regularity yields $(22)_1$. Estimate $(22)_2$ results directly from the formula for $\nabla\mu$, the bound $(22)_1$ and the elliptic regularity. □

Estimates on time derivative χ_t are stated in

Lemma 3.3. *Assume that the condition on $\chi_t(0)$ specified in (15) is satisfied. Then there is a positive constant $c_3 = \varphi(\|\chi_0\|_{H^2(\Omega)}, \|\chi_t(0)\|_{L_2(\Omega)}, T)$ such that*

$$\|\chi_t\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi_t\|_{L_2(0,T;H^3(\Omega))} \le c_3. \quad (23)$$

Proof. (outline). We differentiate equation (11) with respect to time, multiply by χ_t and integrate over Ω and by parts to arrive after using several interpolations to the inequality

$$\frac{d}{dt}\int_\Omega \chi_t^2 dx + \int_\Omega |\nabla\Delta\chi_t|^2 dx + \int_\Omega \chi^2(\Delta\chi_t)^2 dx \le c_1 \int_\Omega \chi_t^2 dx.$$

Hence, by the Gronwall lemma and the elliptic regularity estimate (23) follows. □

In view of $(22)_2$ and (23) it follows that

$$\|\chi\|_{W_2^{5,1}(\Omega^T)} \le c_3. \quad (24)$$

To derive the final estimate we apply the parabolic theory by Solonnikov [11] which implies in particular the following solvability result

Lemma 3.4. *[11] Let us consider the linear IBVP of the sixth order*

$$\begin{cases} \chi_t - \Delta^3\chi = F & in\ \Omega^T, \\ \chi|_{t=0} = \chi_0 & in\ \Omega, \\ \mathbf{n}\cdot\nabla\chi = 0, \quad \mathbf{n}\cdot\nabla\Delta\chi = 0 & on\ S^T, \\ \mathbf{n}\cdot\nabla\Delta^2\chi = G & on\ S^T. \end{cases}$$

If $F \in L_2(\Omega^T)$, $G \in W_2^{1-1/2,1/6-1/12}(S^T)$, $\chi_0 \in_2^{6-6/2}(\Omega)$ *as well as the compatibility conditons*

$$\mathbf{n}\cdot\nabla\chi_0 = 0, \quad \mathbf{n}\cdot\nabla\Delta\chi_0 = 0, \quad \mathbf{n}\cdot\nabla\Delta^2\chi_0 = G(0)$$

are satisfied on S, *then the above problem has the unique solution* $\chi \in W_2^{6,1}(\Omega^T)$ *and the following estimate holds*

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \le c(\|F\|_{L_2(\Omega^T)} + \|G\|_{W_2^{1/2,1/12}(S^T)} + \|\chi_0\|_{W_2^3(\Omega)}).$$

With the help of the above lemma we prove

Lemma 3.5. *There is a positive constant* $c_4 = \varphi(\|\chi_0\|_{H^3(\Omega)}, \|\chi_t(0)\|_{L_2(\Omega)}, T)$ *such that*

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \le c_4. \tag{25}$$

Proof. (outline). We apply Lemma 3.4 to problem (11)–(13). Denoting

$$\begin{aligned} F(\chi) &\equiv M\Delta\left[f_{0,\chi}(\chi) + \frac{1}{2}\kappa_{1,\chi}(\chi)|\nabla\chi|^2 - \nabla\cdot(\kappa_{1,\chi}(\chi)\nabla\chi)\right], \\ G(\chi) &\equiv \frac{1}{2\kappa_2}\kappa_{1,\chi}(\chi)\mathbf{n}\cdot\nabla(|\nabla\chi|^2), \end{aligned} \tag{26}$$

it follows that if $F(\chi) \in L_2(\Omega^T)$, $G(\chi) \in W_2^{1/2,1/12}(S^T)$ and $\chi_0 \in W_2^3(\Omega)(= H^3(\Omega))$ then problem (11)–(13) has the unique solution $\chi \in W_2^{6,1}(\Omega^T)$ satisfying the estimate

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \le c(\|F(\chi)\|_{L_2(\Omega^T)} + \|G(\chi)\|_{W_2^{1/2,1/12}(S^T)} + \|\chi_0\|_{W_2^3(\Omega)}. \tag{27}$$

On account of assumptions on $f_0(\chi)$ and $\kappa_1(\chi)$, using the imbeddings theorems, we deduce

$$\begin{aligned} &\|F(\chi)\|_{L_2(\Omega^T)} \le \varphi(\|\chi)\|_{W_2^{6,1}(\Omega^T)}), \\ &\|G(\chi)\|_{W_2^{1/2,1/12}(S^T)} \le c\|G(\chi)\|_{W_2^{1,1/6}(\Omega^T)} \le \varphi(\|\chi\|_{W_2^{6,1}(\Omega^T)}) \end{aligned} \tag{28}$$

where $W_2^{1/2,1/12}(S^T)$ is the space of traces of functions from the Sobolev-Slobodecki space $W_2^{1,1/6}(\Omega^T)$. From (27), (28) and (24) estimate (25) follows. □

4. **The proof of Theorem 1 (outline).** We apply the Leray-Schauder fixed point theorem, see e.g. [2]. Let

$$\Phi(\tau,\cdot): W_2^{6s,s}(\Omega^T) \ni \tilde{\chi} \mapsto \chi \in W_2^{6,1}(\Omega^T) \subset W_2^{6s,s}(\Omega^T), \quad s \in (11/12, 1), \quad \tau \in [0,1], \tag{29}$$

be the map defined by the following IBVP:

$$\begin{aligned} &\chi_t - M\kappa_2\Delta^3\chi = \tau F(\tilde{\chi}) && \text{in } \Omega^T, \\ &\tilde{\chi}|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\mathbf{n}\cdot\nabla\chi = 0, \quad \mathbf{n}\cdot\nabla\Delta\chi = 0, \quad \mathbf{n}\cdot\nabla\Delta^2\chi = \tau G(\tilde{\chi}) && \text{on } S^T \end{aligned} \tag{30}$$

where $F(\cdot)$ and $G(\cdot)$ are defined by (26). With the help of the Solonnikov theory, similarly as in Lemma 3.5, we prove that the map $\Phi(\tau,\cdot)$ is well-defined, i.e., for any $\bar\chi \in W_2^{6s,s}(\Omega^T)$ with $s \in (11/12,1)$, $\chi_0 \in W_2^3(\Omega)$ and $\tau \in [0,1]$ there exists a unique solution $\chi \in W_2^{6,1}(\Omega^T)$ to problem (30), satisfying the estimate

$$\|\chi\|_{W_2^{6,1}(\Omega^T)} \le \varphi(\|\bar\chi\|_{W_2^{6s,s}(\Omega^T)}, \|\chi_0\|_{W_2^3(\Omega)}). \tag{31}$$

We check that the map $\Phi(\tau,\cdot)$ satisfies the assumptions of the Leray-Schauder fixed point theorem, namely has the following properties:

(i) for any fixed $\tau \in [0,1]$ the map is completely continuous;
(ii) for every bounded subset B of the solution space $X \equiv W_2^{6s,s}(\Omega^T)$, $s \in (11/12,1)$, the family of maps $\Phi(\cdot,\xi) : [0,1] \to X$, $\xi \in B$, is uniformly equicontinuous;
(iii) $\Phi(0,\cdot)$ has precisely one fixed point in X;
(iv) there is a bounded subset B of X such that any fixed point in X of $\Phi(\tau,\cdot)$ is contained in B for every $\tau \in [0,1]$.

If (i)–(iv) are satisfied then the map $\Phi(1,\cdot)$ has at least one fixed point in $X = W_2^{6s,s}(\Omega^T)$. By the regularity properties (31) of problem (30) it follows that the fixed point belongs to the space $W_2^{6,1}(\Omega^T)$. Clearly, in view of the definition of the map $\Phi(1,\cdot)$ this means that the IBVP (11)–(13) has a solution in $W_2^{6,1}(\Omega^T)$.

A priori estimate (25) proves that any fixed point of $\Phi(1,\cdot)$ is contained in a bounded subset $B = \{\chi : \|\chi\|_{W_2^{6,1}(\Omega^T)} \le c_4\}$ of $X = W_2^{6s,s}(\Omega^T)$. It is clear that similar estimate holds for any $\tau \in [0,1]$, so assumption (iv) is satisfied.

In view of the compact imbedding

$$W_2^{6,1}(\Omega^T) \subset W_2^{6s,s}(\Omega^T) \quad \text{for } s < 1, \tag{32}$$

it follows that the map $\Phi(\tau,\cdot)$ takes the bounded subsets into precompact subsets in $W_2^{6s,s}(\Omega^T)$. To prove the complete continuity of $\Phi(\tau,\cdot)$ one needs to show its continuity. To this purpose we consider problem (30) corresponding to two functions $\bar\chi_i \in W_2^{6s,s}(\Omega^T)$, $i = 1,2$.

Denoting the differences

$$K = \chi_1 - \chi_2, \quad \bar K = \bar\chi_1 - \bar\chi_2,$$

we have

$$\begin{aligned}
&K_t - M\kappa_2\Delta^3 K = \tau M\Delta\Big[f_{0,\bar\chi_1} - f_{0,\bar\chi_2} - \frac{1}{2}(\kappa_{1,\bar\chi_1}|\nabla\bar\chi_1|^2 \\
&\quad - \kappa_{1,\bar\chi_2}|\nabla\bar\chi_2|^2) - (\kappa_1(\bar\chi_1)\Delta\bar\chi_1 - \kappa_1(\bar\chi_2)\Delta\bar\chi_2)\Big] \equiv \tau\bar F(\bar\chi_1,\bar\chi_2) && \text{in } \Omega^T,\\
&K|_{t=0} = 0 && \text{in } \Omega,\\
&\mathbf{n}\cdot\nabla K = 0, \quad \mathbf{n}\cdot\nabla\Delta K = 0 && \text{on } S^T,\\
&\mathbf{n}\cdot\nabla\Delta^2 K = \tau\frac{1}{2\kappa_2}\Big[\kappa_{1,\bar\chi_1}\mathbf{n}\cdot\nabla(|\nabla\bar\chi_1|^2) - \kappa_{1,\bar\chi_2}\mathbf{n}\cdot\nabla(|\nabla\bar\chi_2|^2)\Big] \\
&\quad \equiv \tau\bar G(\bar\chi_1,\bar\chi_2) && \text{on } S^T.
\end{aligned} \tag{33}$$

The folloiwng lemma proves the continuity of the map $\Phi(\tau,\cdot)$ which together with the compactness (32) establishes the complete continuity, i.e., assumption (i).

Lemma 4.1. *For any $\tilde{\chi}_1, \tilde{\chi}_2 \in W_2^{6s,s}(\Omega^T)$, $s \in (11/12, 1)$ such that $\|\tilde{\chi}_i\|_{W_2^{6s,s}(\Omega^T)} \leq \tilde{A}$, $i = 1, 2$, and any $\tau \in [0,1]$, there exists a unique solution $K \in W_2^{6,1}(\Omega^T)$ to problem (33) satisfying the estimate*

$$\|K\|_{W_2^{6,1}(\Omega^T)} \leq \tau\varphi(\tilde{A})\|\tilde{K}\|_{W_2^{6s,s}(\Omega^T)}. \tag{34}$$

Proof. (outline). Similarly as in Lemma 3.4 we derive the estimates

$$\|\tilde{F}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{L_2(\Omega^T)} \leq \varphi(\tilde{A})\|\tilde{K}\|_{W_2^{6s,s}(\Omega^T)},$$

$$\|\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{W_2^{1/2,1/12}(S^T)} \leq c\|\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{W_2^{1,1/6}(\Omega^T)} \leq \varphi(\tilde{A})\|\tilde{K}\|_{W_2^{6s,s}(\Omega)}$$

which in view of the Solonnikov theory (compare (27))

$$\|K\|_{W_2^{6,1}(\Omega^T)} \leq c(\tau\|\tilde{F}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{L_2(\Omega^T)} + \tau\|\tilde{G}(\tilde{\chi}_1, \tilde{\chi}_2)\|_{W_2^{1/2,1/12}(S^T)})$$

imply (34). □

The remaining assumptions (ii) and (iii) of the Leray-Schauder fixed point theorem are obviously satisfied.

In conclusion, we conclude the existence of a fixed point in the space $W_2^{6s,s}(\Omega^T)$. By the regularity properties (31) this fixed point belongs to $W_2^{6,1}(\Omega^T)$.

The uniqueness of the solution follows directly by considering the difference of two solutions and applying the Gronwall inequality.

REFERENCES

[1] A. Bodzenta, M. Korzec, P. Nayar, P. Rybka, *Global weak solutions to a sixth order Cahn-Hilliard type equation*, to appear.

[2] C. M. Dafermos, L. Hsiao, *Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity*, Nonlinear Anal. **6** (1982), 435–454.

[3] G. Gompper, J. Goss, *Fluctuating interfaces in microemulsion and sponge phases*, Phys. Rev. E **50**, No 2 (1994), 1325–1335.

[4] G. Gomper, M. Kraus, *Ginzburg-Landau theory of ternary amphiphilic systems. I. Gaussian interface fluctuations*, Phys. Rev. E **47**, No 6 (1993), 4289–4300.

[5] G. Gompper, M. Kraus, *Ginzburg-Landau theory of ternary amphiphilic systems. II. Monte Carlo simulations*, Phys. Rev. E **47**, No 6 (1993), 4301–4312.

[6] G. Gompper, M. Schick, *Correlation between structural and interfacial properites of amphiphilic systems*, Phys. Rev. Lett. **65**, No 9 (1990), 1116–1119.

[7] G. Gompper, M. Schick, Self-assembling amphiphilic systems, in: C. Domb and J. Lebowitz, Eds., Phase Transitions and Critical Phenomena, vol. 16, pages 1–176, London, 1994, Academic Press.

[8] G. Gompper, S. Zschocke, *Ginzburg-Landau theory of oil-water-surfactant mixtures*, Phys. Rev. A **46**, No 8 (1992), 4836–4851.

[9] M. D. Korzec, P. L. Evans, A. Münch, B. Wagner, *Stationary solutions of driven fourth-and sixth-order Cahn-Hilliard type equations*, SIAM J. Appl. Math. **69** (2008), 348–374.

[10] I. Pawlow, W. M. Zajączkowski, *Initial-boundary-value problem for a sixth order Cahn-Hilliard type equation*, in preparation.

[11] V. A. Solonnikov, *boundary value problems for linear parabolic systems of differential equaitons of general type*, Trudy Mat. Inst. Stieklov **83** (1965), 1–162 (in Russian).

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