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Mathematical Model of Generalized Grey Linear Programming And Its Solution

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Abstract — On the basis of studying grey characteristics of linear programming problem and through analysing instances, we establish generalized grey linear programming model (Model G-WZ) and prove its solution. A theoretical evidence and a wide sense solution are given for studying generalized grey linear programming problem.

Keywords: grey linear programming, interval grey number, the objective function, constraint condition, the maximum value range, the narrow sense minimum value range.

Because of the existence of grey characteristics of objects, many linear programming problems are difficult (or not able) to be solved by classical linear programming method. In artical [1], professor Deng Julong discussed the linear programming problem which includes grey elements, such as predicted linear programming problem and drifted one. In order to give the solution of generalized grey linear programming problems, under the enlightenment of the thought of prof.Deng, and according to the conception of grey number and its operational rules [2], we establish generalized grey linear programming model and give general solution through some practical examples.

1. The Establishment of Model G-WZ

Guiding Example—There are 1000 chickens raised in a chicken farm, and they are raised with two kinds of forage soya and millet. It is known that each chickens eats 1~1.3 kilogram of mixed forage every day, and that it needs 0.21~0.23 kg of protein and 0.004~0.006 kg of calcium at least every day. Per kilogram of soya contains 48~52% protein and 0.5~0.8% calcium, its price is 0.38~0.42 yuan. And per kilogram of millet contains 8.5~11.5% protein and 0.3% calcium, its price is 0.20 yuan. How should the forage be mixed in order to pay the least expense of the forage? What is the least expense?

Obviously, the parameters in the Guiding Example are mostly grey numbers. This kind of problems are countless in practice, but it is difficult to find the solution of them using directly the classical linear programming method. To solve this problem, we establish the linear programming model as follow:

Suppose that X_1 kilograms of soya and X_2 kilograms of millet are needed in the whole chicken farm every day. Then we have:

The objective function: $\text{Min}Z = [0.38, 0.42]X_1 + 0.20X_2$

Constraint condition:

$$X_1 + X_2 = [1, 1.3] \times 1000$$
$$[0.48, 0.52]X_1 + [0.085, 0.115]X_2 > [0.21, 0.23] \times 1000$$
$$[0.005, 0.008]X_1 + 0.003X_2 > [0.004, 0.006] \times 1000$$
$$X_1, X_2 > 0$$

Where, a number such as $[0.38, 0.42]$ is called interval grey number, and mark $[0.2, 0.2] = 0.2$. This modle is called Grey Linear Programming Model. Such that, we can establish generalized grey linear programming model.

Definition 1. Let $g(G)$ be grey numbers set [2]. $\forall [a_{ij}, b_{ij}], [c_j, d_j], [e_i, f_i], Z, X_j \in g(G)$. $i=1,2,\ldots,m; j=1,2,\ldots,n$. Then the model as follows:

$$\begin{cases} \mathrm{Min(Max)}\ Z = \sum_{j=1}^{n}[c_j, d_j]X_i \\ \sum_{j=1}^{n}[a_{ij}, b_{ij}]X_j \geq (=, \leq)\ [e_i, f_i] \quad (\ i=1,2,\ldots,m\ ) \\ X_1, X_2, \ldots, X_n \geq 0 \end{cases}$$

is called generalized grey linear programming model, and called Model G-WZ for short.

2. Solution of Model G-WZ

Giving the solution of the guiding example as an example, we explain the solution process of Model G-WZ.

2.1. Standardization of the Model

The model in the above-mentioned Definition 1 can be standardized as follows:

$$\begin{cases} \mathrm{Min}Z = \sum_{j=1}^{n}[c_j, d_j]X_j \\ \sum_{j=1}^{n}[a_{ij}, b_{ij}]X_j \geq [e_i, f_i] \quad (\ i=1,2,\ldots,m\ ) \\ X_j \geq 0 \quad (\ j=1,2,\ldots,n) \end{cases}$$

2.2. Determine the Maximum Value Range and the Narrow Sense Minimum Value Range on the Constraint Condition.

At first, if we have a constraint condition: $[1,2]X_1+[1,4]X_2 \geq [2, 4]$, it can be transformed into a group of inequalities: $1X_1+1X_2 \geq 2$, $1X_1+1X_2 \geq 4$, $2X_1+4X_2 \geq 2$, $2X_1+4X_2 \geq 4$, $2X_1+1X_2 \geq 2$, $2X_1+1X_2 \geq 4$, $1X_1+4X_2 \geq 2$, $1X_1+4X_2 \geq 4$. According to graphic solution we can find clearly that, in the range of $X_1, X_2 \geq 0$, the range of inequality $2X_1+4X_2 \geq 2$ includes ones of other inequalities, and the range of $1X_1+1X_2 \geq 4$ is the minimum one among them. When the coefficients X_1 and X_2 change separately in the range of $[1,2]$ and $[1,4]$, and the constant term of the inequality changes in the range of $[2,4]$, the above-mentioned conclusoin can be achieved similarly. Therefore, we call that $2X_1+4X_2 \geq 2$ is the maximum value range on the constraint condition $[1,2]X_1+[1,4]X_2 \geq [2,4]$, and that $1X_1+1X_2 \geq 4$ is the narrow sense minimum valve range on that constraint condition. Generally, we have:

Definition 2. Let
$$\sum_{j=1}^{n}[a_1^{(j)}, a_2^{(j)}]X_j \geq [e_1, e_2], \quad (\ X_j \geq 0, j=1,2,\ldots,n\ ) \quad \ldots\ldots\ldots\ldots(1)$$

Then
$$\sum_{j=1}^{n} a_2^{(j)} X_j \geq e \quad \ldots\ldots\ldots\ldots(2)$$

is called the characteristic formula of(1). Where, $a^{(j)} \in [a_1^{(j)}, a_2^{(j)}], e \in [e_1, e_2], (j=1,2,\ldots,n)$.

Definition 3. If the value range of a constraint inequality can include ones of all other constraint inequalities, the value range of this inequality is called the maximum value range of all constraint inequalities. If the value range of a constraint inequality is included in one of any other constraint inequalities, the valus range of this inequality is called the narrow sense minimum value range of all constraint inequalities.

Theorem. Let (2) be characteristic formula of constraint condition (1).

then
$$\sum_{j=1}^{n} a_2^{(j)} X_j \geq e_1 \quad \ldots\ldots\ldots\ldots(3)$$

is the maximum value range inequality of (1).

$$\sum_{j=1}^{n} a_1^{(j)} X_j \geq e_2 \quad \ldots\ldots\ldots\ldots(4)$$

is the narrow sense minimum value range inequality of (1).

Proof:

(i) In formula (2), let only one coefficient $a^{(k)}$ change. When $a^{(k)}$ equals $a_1^{(k)}$ and $a_2^{(k)}$ separately, (2) is transformed into:

$$a^{(1)}X_1+\ldots+a^{(k-1)}X_{k-1}+a_1^{(k)}X_k+a^{(k+1)}X_{k+1}+\ldots+a^{(n)}X_n \geqslant e \qquad \ldots\ldots(5)$$

$$a^{(1)}X_1+\ldots+a^{(k-1)}X_{k-1}+a_2^{(k)}X_k+a^{(k+1)}X_{k+1}+\ldots+a^{(n)}X_n \geqslant e \qquad \ldots\ldots(6)$$

As $a^{(k)} \in [a_1^{(k)}, a_2^{(k)}]$, $a_1^{(k)} \leqslant a_2^{(k)}$, $X_k \geqslant 0$, however, we can let $(a_2^{(k)}-a^{(k)})X_k=\varepsilon \geqslant 0$. (6) can be resolved into the sum of value ranges of two inequalities: $a^{(1)}X_1+\ldots+a^{(k)}X_k+\ldots+a^{(n)}X_n \geqslant e$ and $e > a^{(1)}X_1+\ldots+a^{(k)}X_k+\ldots+a^{(n)}X_n \geqslant e-(a_2^{(k)}-a^{(k)})X_k=e-\varepsilon$

This means the value range of (6) includes one of (2). Therefore, in formula (2), when only one coefficient $a^{(k)}$ changes in the range of $[a_1^{(k)}, a_2^{(k)}]$, (6) is the maximum value range of (2). Similarly, in formula (2), when only one coefficient $a^{(k)}$ changes in the range of $[a_1^{(k)}, a_2^{(k)}]$, the value range of (2) includes one of (5). (5) is the narrow sense minimum value range of (2). According to afore-said conclusion, we can get, when $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$ all change simulstaneously (e doesn't change), the maximum value range of (2) is:

$$\sum_{j=1}^{n} a_2^{(j)} X_j \geqslant e \qquad \ldots\ldots(7)$$

Similarly, we can get, when $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$ all change simultaneously (e doesn't change), the narrow sense minimum value range of (2) is: $\sum_{j=1}^{n} a_1^{(j)} X_j \geqslant e$

(ii) If $e=e_1$, (7) is transformed into (3). As $e \in [e_1, e_2]$, $e_1 \leqslant e_2$, (3) can be resolved into the sum of value ranges of two inequalities: $\sum_{j=1}^{n} a_2^{(j)} X_j \geqslant e$ and $e > \sum_{j=1}^{n} a_2^{(j)} X_j \geqslant e_1$

This means the value range of (3) includes one of (7). So (3) is the maximum value range of (7). According to afore-said (i), we know that (7) is the maximum value range of (2) when e doesn't change. Therefore, (3) is the maximum value range of (2).
Similarly, we can get, (4) is the narrow sense minimum value range of (2).
Thus, the theorem is all right.
If constraint condition of standardized model is made up of a group of inequalities, every inequality can get one maximum value range inequality and one narrow sense minimum value range inequality. Then all maximum value range inequalities merge into the maximum value range of total constraint condition, and all narrow sense minimum value range inequalities merge into the narrow sense minimum value range of total constraint condition.

2.3. Transform Model G-WZ into Two Generalized Classical Linear Programming Problems.

Since coefficients of Model G-WZ include grey number, the final operational result-objective value MinZ is grey number too. Let $MinZ=[z_1, z_2]$, $z_1 \leqslant z_2$.

To the objective function: $MinZ=\sum_{j=1}^{n} [c_j, d_j] X_j$

since, $X_1, X_2, \ldots, X_n \geqslant 0$, lower limit z_1 and upper limit z_2 of MinZ separately are

$$z_1=MinZ_1=\sum_{j=1}^{n} c_j X_j \qquad z_2=MinZ_2=\sum_{j=1}^{n} d_j X_j$$

And, z_1 should be objective value of $MinZ_1$ in the maximum value range, z_2 should be objective value of $MinZ_2$ in the narrow sense minimum value range. Therefore, Model G-WZ is finally transformed into the resolution of two generalized linear programming problems.

$$\begin{cases} MinZ_1=\sum_{j=1}^{n} c_j X_j \\ \sum_{j=1}^{n} b_{ij} X_j \geqslant e_i & (i=1,2,\ldots,m) \\ X_j \geqslant 0 & (j=1,2,\ldots,n) \end{cases} \quad \text{and} \quad \begin{cases} MinZ_2=\sum_{j=1}^{n} d_j X_j \\ \sum_{j=1}^{n} a_{ij} X_j \geqslant f_i & (i=1,2,\ldots,m) \\ X_j \geqslant 0 & (j=1,2,\ldots,n) \end{cases}$$

To the guiding example, they are:

$$\begin{cases} MinZ_1=0.38X_1+0.20X_2 \\ 1300\geq X_1+X_2\geq 1000 \\ 0.52X_1+0.115X_2\geq 0.21\times 1000 \\ 0.008X_1+0.003X_2\geq 0.004\times 1000 \\ X_1, X_2\geq 0 \end{cases} \quad \text{and} \quad \begin{cases} MinZ_2=0.42X_1+0.20X_2 \\ 1300\geq X_1+X_2\geq 1000 \\ 0.48X_1+0.085X_2\geq 0.23\times 1000 \\ 0.005X_1+0.003X_2\geq 0.006\times 1000 \\ X_1, X_2\geq 0 \end{cases}$$

2.4. Solve It Using Classical Linear Programming Method and Get the Final Results.

According to classical linear programming method, solve the two linear programming problems above, we can get z_1 and z_2 separately (may be not resolutive). Corresponding programming value are $X_1', X_2', \ldots, X_n'$ and $X_1'', X_2'', \ldots, X_n''$. So the objective value is: $MinZ = [z_1, z_2]$ programming value can be marked as:

$$\begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_n \end{pmatrix} = \left[\begin{pmatrix} X_1' \\ X_2' \\ \cdot \\ \cdot \\ X_n' \end{pmatrix}, \begin{pmatrix} X_1'' \\ X_2'' \\ \cdot \\ \cdot \\ X_n'' \end{pmatrix} \right]$$

If z_1 is not resolutive, z_2 must be not resolutive. Then the objective function MinZ is not resolutive too.

If z_2 is not resolutive and z_1 is resolutive, we can mark $MinZ=[z_1, \infty^+]$. Programming value $X_1'', X_2'', \ldots, X_n''$ doesn't exist, and then don't write it.

To the guiding example, we can get:

$X_1=234.57$, $X_2=765.43$, $z_1=242.22$; $X_1=1050$, $X_2=250$, $z_2=491$

Therefore, the objective value of the guiding example is: $MinZ = [242.22, 491]$

the programming value is: $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \left[\begin{pmatrix} 234.57 \\ 765.43 \end{pmatrix}, \begin{pmatrix} 1050 \\ 250 \end{pmatrix} \right]$

Conclution shows that, the objective value MinZ is grey number [242.22, 491], $\begin{pmatrix} 234.57 \\ 765.43 \end{pmatrix}$ and $\begin{pmatrix} 1050 \\ 250 \end{pmatrix}$ are separately programming value when objective value separately lower limit and upper limit of grey number [242.22, 491].

In the end, to compare with solution of drifted grey linear programming and pseudo-solution of grey linear programming, we solve one example of article[1] using Method G-WZ.

Example—Solve the follow grey linear programming problem:

$$\begin{cases} MaxZ = [1,7]X_1+[4,12]X_2 \\ [1,21]X_1+[4,10]X_2\leq 380 \\ 3X_1+10X_2\leq 300 \\ 4X_1+5X_2\leq 198 \\ X_1, X_2\geq 0 \end{cases}$$

Solution: Using Method G-WZ, we can get, the objective value of the example is:

$$MaxZ=[119.34, 425.28]$$

coresponding programming value is: $\begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \left[\begin{pmatrix} 3.34 \\ 29 \end{pmatrix}, \begin{pmatrix} 19.2 \\ 24.24 \end{pmatrix} \right]$

Analysis: In article [1], the programming solution of above-mentioned example is given. Where, drifted optimum solution is: $X_1=3.34$, $X_2=29$, and its objective value is f=371.18, which is a definite value of the above-gotten objective value range (that is MaxZ). Such is the pseudo-solution of grey linear prigramming. However, we give definitely the range of objective value using Method G-WZ, and provide decision-makers with scientific data available. And it is of great practical value.

References:

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