POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

Redaktor techniczny
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PREFACE

The present volume comprises the papers from the 3rd Italian-Polish Conference which was held in [illegible] from [illegible] to 11. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of various scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied to the economics, management, data processing and technological systems.

From the past, such meetings, owing to the exchange of views and experiences between the specialists from both countries what is the essential advantage to both sides, ensure the essential gains, the systematization of the notions and [illegible] contribute to the essential development of the quantitative analysis and the popularization of large computing systems.

The meaning of such meetings, however, goes far beyond that, what is happening during the conference because the established contacts are also fruitful in form of common scientific research works and — last but not least — foster the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory,
2. System Theory in Economics,
3. Technological, Managerial and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

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DESIGN OF LINEAR TIME-VARYING TRACKING SYSTEMS WITH STOCHASTIC DISTURBANCES

1. INTRODUCTION

Recently a number of papers [1, 5] have been published which are devoted to the design of linear time-invariant tracking systems for polynomial command inputs. In the papers [2, 3, 4] the method has been generalized for linear time-varying systems and for linear time-invariant systems with delays.

The purpose of this paper is to propose a new designing method of linear time-varying tracking systems with stochastic disturbances for polynomial command inputs.

The proposed method can be considered as a generalization (for systems with stochastic disturbances) of the method given in [2] or as a generalization (for time-varying systems) of the method given in [6].

2. STATEMENT OF THE PROBLEM

Consider a linear time-varying plant described by nth order differential equation of the form

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = u+\eta \qquad (a_n = 1) \tag{1}$$

where $u=u(t) \in R^1$ is the input, $y=y(t) \in R^1$ is the output, $a_i = a_i(t)$ are the coefficients depending on time t and $\eta=\eta(t)$ is the disturbance. We assume that the disturbance is a stochastic process which fulfils the following conditions

$$\frac{d^r m_\eta}{dt^r} = 0 \tag{2}$$

and

$$\frac{\partial^{2r} R_\eta(t_1, t_2)}{\partial t_1^r \, \partial t_2^r} = 0 \tag{3a}$$

$$E\left\{\left(\frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix}_{|t=0}\right)\left(\frac{d^r\eta(t)}{dt^r}_{|t=t_2}\right)^T\right\}=0,$$
$$E\left\{\left(\frac{d^r\eta(t)}{dt^r}_{|t=t_1}\right)\left(\frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix}_{|t=0}\right)^T\right\}=0 \tag{3b}$$

where $m_\eta = m_\eta(t)$ is the mean of η and $R_\eta = R_\eta(t_1, t_2)$ is the second order joint moment of η.

Introducing the state vector

$$x^T = [x_1, x_2, \dots, x_n] \tag{4}$$

where $x_1 = y$, $x_2 = \dot{y}, \dots, x_n = \frac{d^{n-1}y}{dt^{n-1}}$, the equation (1) can be written as follows

$$\dot{x} = Ax + B(u+\eta) \tag{5}$$

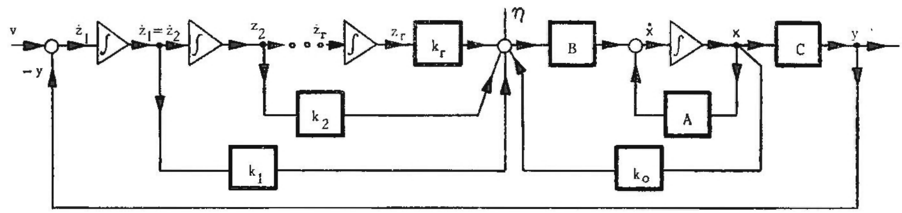
$$y = Cx \tag{6}$$

where

$$A = A(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7}$$

Let us consider the closed-loop tracking system (Fig.), which consists of the plant, r integrators described by the equations

$$\begin{aligned} \dot{z}_1 &= v - y \\ \dot{z}_2 &= z_1 \\ \dot{z}_3 &= z_2 \\ &\dots\dots \\ \dot{z}_r &= z_{r-1} \end{aligned} \tag{8}$$



and $r+1$ gain elements depending on time described by the matrix

$$k = [k_0, k_1, k_2, \dots, k_r] \tag{9}$$

The problem can be formulated as follows.
Choose the matrix (9) in such a way that the output y of the closed-loop system will track the command input of the form

$$v = v(t) = \sum_{i=0}^{r-1} \alpha_i t^i \tag{10}$$

so that

$$\lim_{t\to\infty} E\{[v(t)-y(t)]^k\} = 0 \quad \text{for} \quad k = 1, 2 \tag{11}$$

where E is the expectation operator.

3. SOLUTION OF THE PROBLEM

The system which consists of the plant and r integrators is described by the equation

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A_0 \begin{bmatrix} x \\ z \end{bmatrix} + B_0(u+\eta) + B_C v \tag{12}$$

where

$$z^T = [z_1, z_2, \dots, z_r] \tag{13}$$

$$A_0 = \begin{bmatrix} A & 0 & 0 & \dots & 0 & 0 \\ -C & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{14}$$

Substituting the relation

$$u = k \begin{bmatrix} x \\ z \end{bmatrix} = k_0 x + \sum_{i=1}^{r} k_i z_i \tag{15}$$

nto equation (12) we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A_C \begin{bmatrix} x \\ z \end{bmatrix} + B_C v + B_0 \eta \tag{16}$$

where

$$A_C = A_0 + B_0 k = \begin{bmatrix} A+Bk_0 & Bk_1 & Bk_2 & \dots & Bk_{r-1} & Bk_r \\ -C & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \tag{17}$$

Equation (16) describes the closed-loop tracking system.

Theorem

The output y of closed-loop system tracks the command input (10) so that the condition (11) is satisfied, if the matrix (9) has the following form

$$k = [a_0 - b_r, a_1 - b_{r+1}, \dots, a_{n-1} - b_{n+r-1}, b_{r-1}, \dots, b_1, b_0] \tag{18}$$

where $b_0, b_1, \dots, b_{n+r-1}$ are constant coefficients of the characteristic polynomial

$$\det[sI - A_C] = s^{n+r} + b_{n+r-1} s^{n+r-1} + \dots + b_1 s + b_0 \tag{19}$$

of the matrix (17) which has all eigenvalues in the open left-half plane.

Proof

Note that for

$$k_0 = [a_0 - b_r, a_1 - b_{r+1}, \dots, a_{n-1} - b_{n+r-1}] \tag{20}$$

and

$$k_i = b_{r-i} \quad \text{for} \quad i = 1, 2, \dots, r \tag{21}$$

we have

$$Bk_0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ a_0 - b_r & a_1 - b_{r+1} & \dots & a_{n-1} - b_{n+r-1} \end{bmatrix} \tag{22}$$

and

$$Bk_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_{r-i} \end{bmatrix} \quad \text{for} \quad i = 1, 2, \dots, r \tag{23}$$

Therefore

$$A_C = \begin{bmatrix} A+Bk_0 & Bk_1 & Bk_2 & \dots & Bk_{r-1} & Bk_r \\ -C & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ -b_r & -b_{r+1} & -b_{r+2} & \dots & -b_{n+r-1} & b_{r-1} & b_{r-2} & \dots & b_1 & b_0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (24)$$

By simple calculations can be verified that

$$\det[sI - A_C] = \begin{bmatrix} s & -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & s & -1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 0 & 0 & \dots & 0 & 0 \\ b_r & b_{r+1} & b_{r+2} & \dots & s+b_{n+r-1} & -b_{r-1} & -b_{r-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & 0 & \dots & 0 & s & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & s & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & s \end{bmatrix}$$

$$= s^{n+r} + b_{n+r-1} s^{n+r-1} + \dots + b_1 s + b_0 \quad (25)$$

Note that the matrix (24) has constant elements. Therefore, differentiating equation (16) r times, for command inputs of the form (10), we obtain

$$\frac{d^{r+1}}{dt^{r+1}}\begin{bmatrix} x \\ z \end{bmatrix} = A_C \frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix} + B_0 \frac{d^r \eta}{dt^r} \tag{26}$$

and

$$\frac{d^{r+1}}{dt^{r+1}} E\begin{bmatrix} x \\ z \end{bmatrix} = A_C \frac{d^r}{dt^r} E\begin{bmatrix} x \\ z \end{bmatrix} + B_0 \frac{d^r}{dt^r} E[\eta] \tag{27}$$

If all eigenvalues of the matrix (24) are located in the open left-half plane and the condition (2) is satisfied, it follows from equation (27) that

$$\lim_{t\to\infty} \frac{d^r E[z_r]}{dt^r} = 0 \qquad (28$$

From the equations (8) we have

$$\frac{d^r z_r}{dt^r} = v - y \tag{29}$$

Therefore

$$\lim_{t\to\infty} \frac{d^r E[z_r]}{dt^r} = \lim_{t\to\infty} E[v-y] = 0 \tag{30}$$

Taking into considerations the solution

$$\frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix} = e^{A_C t} \frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix}_{|t=0} + \int_0^t e^{A_C(t-\tau)} B_0 \frac{d^r \eta(\tau)}{d\tau^r} d\tau \tag{31}$$

of equation (26) and the conditions (3a), (3b) we can write

$$E\left\{\left(\frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix}_{|t=t_1}\right)\left(\frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix}_{|t=t_2}\right)^T\right\} = e^{A_C t_1} R_0 e^{A_C t_2} +$$

$$+ \int_0^{t_1}\int_0^{t_2} e^{A_C(t_1-\tau_1)} B_0 \frac{\partial^{2r} R_\eta(\tau_1, \tau_2)}{\partial\tau_1^r \, \partial\tau_2^r} B_0^T e^{A_C(t_2-\tau_2)} d\tau_1 \, d\tau_2 = e^{A_C t_1} R_0 e^{A_C t_2} \tag{32}$$

where

$$R_0 = E\left\{\left(\frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix}_{|t=0}\right)\left(\frac{d^r}{dt^r}\begin{bmatrix} x \\ z \end{bmatrix}_{|t=0}\right)^T\right\} \tag{33}$$

If all eigenvalues of the matrix (24) are located in the open left-half plane, it follows from equations (32), (29) that

$$\lim_{t\to\infty} E\left\{\left(\frac{d^r z_r}{dt^r}\right)^2\right\} = \lim_{t\to\infty} E\{(v-y)^2\} = 0 \tag{34}$$

This completes the proof.

From the above considerations the following designing procedure of the closed-loop tracking systems follows.

1. Choose the eigenvalues $s_1, s_2, \ldots, s_{n+r}$ of the matrix (24), which are located in the open left-half plane, so that the closed-loop tracking system will have prescribed dynamical characteristics.

2. For the given $s_1, s_2, \ldots, s_{n+r}$ calculate the coefficients $b_0, b_1, \ldots, b_{n+r-1}$ of the characteristic polynomial (19).

3. Using the formula (18) calculate the matrix k for the given coefficients $b_0, b_1, \ldots, b_{n+r-1}$ and $a_0, a_1, \ldots, a_{n-1}$.

These considerations can be generalized for higher order moments ($k \geqslant 3$) and for multivariable systems.

4. EXAMPLE

Consider a linear plant described by the equations (5), (6) for

$$A = \begin{bmatrix} 0, & 1 \\ e^{-t}, & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{35}$$

and $\eta(t)$ is a stochastic process with $m_\eta(t) = E[\eta(t)] = t$, $R_\eta(t_1, t_2) = E[\eta(t_1)\,\eta(t_2)] = t_1 t_2$ which fulfils the condition (3b).

Choose the matrix

$$\boldsymbol{k} = [\boldsymbol{k}_1\, \boldsymbol{k}_2\, \boldsymbol{k}_3\, \boldsymbol{k}_4] \tag{36}$$

so that the output y of the closed-loop system will track the command input of the form

$$v = v(t) = t \tag{37}$$

with

$$\lim_{t\to\infty} E\{[v-y]^k\} = 0 \quad \text{for} \quad k = 1, 2 \tag{38}$$

In this case we have $r = 2$ and it is easy to verify that the conditions (2), (3a) are satisfied.

Let the eigenvalues of the closed-loop matrix be $s_1 = s_2 = s_3 = s_4 = -2$. Therefore

$$\det[s\boldsymbol{I} - \boldsymbol{A}_C] = (s+2)^4 = s^4 + 8s^3 + 24s^2 + 32s + 16 \tag{39}$$

and

$$b_0 = 16, \quad b_1 = 32, \quad b_2 = 24, \quad b_3 = 8$$

Using the formula (18) we obtain

$$k = [a_0 - b_2, a_1 - b_3, b_1, b_0] = [e^{-t} - 24, -9, 32, 16] \tag{40}$$

REFERENCES

[1] Bradshaw A., Porter B., Design of linear multivariable discrete-time tracking systems. Intern. J. Systems Science. vol. 6 No 2 pp. 117—125 (1975).

[2] Kaczorek T., Design of linear time-varying tracking systems. Bull. Acad. Polon., Ser. Sci. Techn. vol. 24 No 4 (1976).

[3] Kaczorek T., Design of multivariable time-varying tracking systems. Bull. Acad. Polon., Ser. Polon., Ser. Sci. Techn. vol. 24 No 4 (1976).

[4] Kaczorek T., Kozera W., Projektowanie śledzących wielowymiarowych układów liniowych stacjonarnych z opóźnieniami. Archiwum Automatyki i Telemechaniki, vol. XX No 2 (1976).

[5] Porter B. Bradshaw A. Design of linear multivariable continuous-time tracking systems. Intern. J. Systems Science. vol. 5, No 12 pp. 1155—1164 (1974).

[6] Pusz J. Projektowanie śledzących wielowymiarowych układów liniowych stacjonarnych poddanych losowym zakłóceniom. Archiwum Automatyki i Telemechaniki, vol. XXI No 3 (1977).

SUMMARY

A new designing method of single-input single-output linear time-varying tracking systems with stochastic disturbances for polynomial command inputs is proposed. It has been shown that it is possible to choose the gain matrix k of the closed-loop system so that the output $y(t)$ tracks the command input $v(t)$ with $\lim_{t \to \infty} E\{[v(t)-y(t)]^k\} = 0$ for $k = 1, 2$ (E — expectation operator).

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