POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

Redaktor techniczny
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PREFACE

The present volume comprises the papers from the 3rd Italian-Polish Conference which was held in Warsaw [illegible] to 11. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied to the economic, management, data processing and technological systems.

From the past, such meetings enrich the exchange of views and experiences between the specialists from both countries what is the benefit which applies to both sides. Above the general goal, the summarization of the systems and control theory results achieved by the scientists, these conferences enrich the essential development of the quantitative methods and the practical use of models of large complex systems.

The meaning of such meetings, however, goes far beyond that, what is happening during the conference because the established contacts are also fruitful in form of common scientific research works and — last but not least — mean the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory;
2. Systems Theory in Economics;
3. Technological, Management and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

G. Treccani
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AN ALGORITHM FOR CONVEX PROGRAMMING VIA FUNCTIONAL LINEAR PROGRAMMING*)

1. INTRODUCTION

This work is an application of functional linear programming (FLP) to the solution of convex programming problems. A simple-type method for solving FLP problems has been implemented by dr J. Mc Keown and dr J. Gomulka of the NOC of the Hatfield Polytechnic (UK) and in collaboration with them, our group (G. Treccani, G. Resta and E. Sideri) at the Mathematical Institute of the University of Genoa, is working to the implementation of an efficient algorithm for convex programming. Although the theoretical approach is quite different, the basic idea is similar to Wolfe's and Dantzig's "generalized linear programming", when applied to the solution of convex problems. Nevertheless, a much smaller amount of computational work seems to be involved at any iteration, and the convergence properties are different.

2. FUNCTIONAL LINEAR PROGRAMMING (FLP)

Let S^i, $i=1, \ldots, m$ be compact sets of some real euclidean space, and the measure χ^i be any positive regular G-additive real function defined on the algebra of Borel subsets of S^i.

Let $C^i(\vartheta^i)$, $\boldsymbol{b}^i(\vartheta^i)$ be such that $C^i : S^i \to R$, $\boldsymbol{b}^i : S^i \to R^M$, and $\boldsymbol{q} \in R^M$

A FLP problem is then the following:

$$\text{Minimize} \quad W = \sum_{i=1}^{m} \int_{S^i} C^i(\vartheta^i)\, d\chi^i \quad \text{over} \quad \chi^i$$

$$\text{subject to} \quad \sum_{i=1}^{m} \int_{S^i} \boldsymbol{b}^i(\vartheta^i)\, d\chi^i = \boldsymbol{q} \tag{2.1}$$

It has been proved by dr J. Gomulka that if $\{\chi^i, \ldots, \chi^m\}$ is a solution of this problem, then there exists another solution where measures are concentrated at a finite set of points $\{\vartheta_l^i\}$ such that $\sum_{i,l} \vartheta_l^i = M$.

*) This research has been supported by CNR, Comitato Science Matematiche.

This fact suggests the possibility of solving the FLP problem by a simplex--type algorithm, which has been implemented by dr J. Mc Keown. This method is quite similar to the revised simplex method for Ordinary Linear Programming (OLP), but it requires at every iteration the unconstrained minimization over ϑ^i of the "reduced gradients" $C^i(\vartheta^i)$, which are defined as a natural generalization to FLP of the corresponding quantities in OLP.

It can be proved that FLP problem can be dualized in the following:

$$\text{Maximize} \quad D = v^T p \quad \text{over} \quad v \in R^M \tag{2.2}$$

$$\text{subject to} \quad v^T b^i(\vartheta^i) \leqslant C^i(\vartheta^i); \; \vartheta \in S^i, \quad i = 1, \dots, m$$

Then for any feasible solution of 2.1. and any feasible solution of 2.2. we have $D \leqslant W$ and 2.1. has an optimal solution if and only if the dual problem 2.2. has an optimal solution, for which $D = W$. It follows that we can find the optimal solution of 2.2. by solving the dual FLP problem 2.1. by the simplex-type method.

3. CONVEX PROGRAMMING PROBLEM

The convex programming problem to be solved is the following:

$$\text{Maximize} \quad \mathbf{q}^T \boldsymbol{u} \quad \text{over} \quad \boldsymbol{u} \in R^M \tag{3.1}$$

$$\text{subject to} \quad \varphi_r(\boldsymbol{u}) \leqslant 0; \quad r = 1, \dots, m$$

where φ_r are convex continuously differentiable functions. The general idea is to replace the r-th convex constraint $C_r = \{\boldsymbol{u} \in R^M : \varphi_r(\boldsymbol{u}) \leqslant 0\}$ by an infinite family of linear constraints of hyperplanes tangent to the boundary surface $\varphi_r(\boldsymbol{u}) = 0$.

If we parametrize the boundary of C_r by a M-1 dimensional parameter ϑ^r, then the constraints in 3.1. can be written as:

$$\text{grad} \; \varphi_r[\boldsymbol{u}^r(\vartheta^r)]^T [\boldsymbol{u} - \boldsymbol{u}^r(\vartheta^r)] \leqslant 0, \quad \vartheta^r \in S^r$$

where $\varphi_r[\boldsymbol{u}^r(\vartheta^r)] = 0$ for $\vartheta^r \in S^r$, and if we set:

$$\text{grad} \; \varphi_r = \boldsymbol{b}^r, \quad \text{grad} \quad \varphi_r[\boldsymbol{u}^r(\vartheta^r)]^T \boldsymbol{u}^r(\vartheta^r) = C^r(\vartheta^r)$$

then 3.1. becomes:

$$\text{Maximize} \quad \mathbf{q}^T \boldsymbol{u} \quad \text{over} \quad \boldsymbol{u} \in R^M \tag{3.2}$$

$$\text{subject to} \quad \boldsymbol{u}^T \boldsymbol{b}^r(\vartheta^r) \leqslant C^r(\vartheta^r); \; \vartheta^r \in S^r, \quad r = 1, \dots, m.$$

This is the dual problem of the FLP problem 2.1.

In the following we shall consider a slightly different convex programming problem:

$$\text{Minimize} \quad \varphi_0(\boldsymbol{x}) \quad \text{over} \quad \boldsymbol{x} \in R^{M-1} \tag{3.3}$$

$$\text{subject to} \quad \varphi_i(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{a}_i \leqslant \beta_i, \quad i = 1, \dots, s-1$$

$$\varphi_i(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{a}_i)^T A_i (\boldsymbol{x} - \boldsymbol{a}_i) \leqslant \beta_i, \quad i = s, \dots, m.$$

where $s < m$, $a_i \in R^{M-1}$ are linearly independent for $i=1, \ldots, s-1$ and the matrices A_i are symmetric and positive definite, while φ_0 is convex continuously differentiable.

This problem can be written in the form 3.1. and 3.2., setting $\boldsymbol{u}=(x, u_M) \in R^M$ $\boldsymbol{q}=(0, 0, \ldots, 0, -1) \in R^M$, and adding a convex constraint

$$C_0 = \{\boldsymbol{u} \in R^M : x = (u_1, \ldots, u_{M-1}),\ \varphi_0(x) - u_M \leqslant 0\}$$

The dual FLP problem can be formulated as follows:

$$\text{Maximize} \quad \boldsymbol{l}^T \boldsymbol{d} \quad \text{over} \quad \boldsymbol{l}, \boldsymbol{d} \text{ and } A \tag{3.4}$$

$$\text{subject to} \quad \boldsymbol{l} = A^{-1} \boldsymbol{q}$$

where: (i) A is a $M \times M$ nonsingular matrix whose columns are M vectors of the type $\boldsymbol{b}^r(\vartheta^r)$; (ii) $\boldsymbol{d}$ is the M-vector having $-C^r(\vartheta^r)$ as components for ϑ^r chosen as in A; (iii) $\boldsymbol{l}$ is an M-vector with nonnegative components.

Setting $Z = -(A^T)^{-1} d$ the problem can be written as follows:

$$\text{Minimize} \quad Z^T \boldsymbol{q} \quad \text{over} \quad Z \in R^M, \ \boldsymbol{d} \text{ and } A \tag{3.5}$$

$$\text{subject to} \quad Z = -(A^T)^{-1} d$$

The algorithm is iterative and changes at every iteration one column of the matrix A and the corresponding component of the vector $\boldsymbol{d}$ in such a way that improves the solution.

It is easy to see that introducing one column from the J-th constraint a possible choice is the one corresponding the value ϑ^J_{min} which minimizes the reduced gradient

$$C'_J = C^J_J(\vartheta^J) - Z^T \boldsymbol{b}^J(\vartheta^J) \quad \text{over} \quad \vartheta^J$$

If $imin$ is the index corresponding to $C'_{imin} = \min C'_J$ over J, the algorithm substitutes one column of A and one component of $\boldsymbol{d}$ with $\boldsymbol{b}^{imin}(\vartheta^{imin}_{min})$ and $-C^{imin}(\vartheta^{imin}_{min})$, in such a way that A is still nonsingular, $\boldsymbol{l} = A^{-1}\boldsymbol{q}$ with nonnegative components and $Z^T\boldsymbol{q}$ is smaller.

We can observe that $-C' = \max \boldsymbol{b}^J(\vartheta^J)^T[\boldsymbol{u}^J(\vartheta^J) - Z] \geqslant \boldsymbol{b}^J(\vartheta^J)^T[\boldsymbol{u}^J(\vartheta^J) - Z]$ for every $\vartheta^J \in S^J$.

In particular if $\boldsymbol{u}$ is the projection of the point $\boldsymbol{Z}$ on the convex set C^J, and $\boldsymbol{b}$ is the corresponding gradient, then we have $-C'_J \geqslant \|\boldsymbol{b}\| \, \|\boldsymbol{u} - Z\|$, which implies $-\|\boldsymbol{b}\| \geqslant \text{dist}\,(Z, C)$. If now we assume $0 \neq \alpha = \min \|\boldsymbol{b}^J(\vartheta^J)\|$ over $\vartheta^J \in S^J$, then we have $-C'_J \geqslant \alpha \,\text{dist}\,(\boldsymbol{Z}, C^J)$.

If now $\boldsymbol{A}_n, \boldsymbol{l}_n, \boldsymbol{d}_n$ and $\boldsymbol{Z}_n$ are the matrix and the vectors at the n-th interation we can observe that:

$$\min \lim_{n \to \infty} (C'_{i\,\min})_n = 0 \ \text{implies} \ \min \lim_{n \to \infty} \text{dist}\,(Z_n, \bigcap_J C^J) = 0.$$

This implies that for some subsequence $\boldsymbol{Z}_{n_k}$ we have

$$\lim \boldsymbol{Z}_{n_k} = \boldsymbol{u}^* \in \bigcap_J C^J$$

By the duality theorem this implies that u^* is an optimal solution of 3.2. and the first $M-1$ components and the last component are respective by the components of the optimal solution and the optimal cost of problem 3.3. For the case $M=2$ we have proved that it must be $\min \lim_{n\to\infty} (C'_{i\,\min})_n = 0$, so that the method must converge.

4. MAIN FEATURE OF THE ALGORITHM

The algorithm solves the problem 3.4. following the revised simplex algorithm.

The main differences between the proposed method and the revised simplex are the choice of the initial basis and the choice of the column to be introduced in the basis.

4.1. INITIAL BASIS

For a problem where some of the convex constraints C^r may be unbounded and the gradients of φ_r may be orthogonal to $\boldsymbol{q}$, the problem of introducing an initial feasible basis is rather difficult. We overcome this difficulty by introducing as an additional constraint a sphere containing the optimal solution in the interior. The initial basis is then formed by M vectors which are gradients of $\boldsymbol{M}$ points on the spherical surface such that they are linearly independent and $\boldsymbol{q}$ is a linear combination with positive coefficient of them.

4.2. COLUMN TO BE INTRODUCED

From a theoretical point of view the computation of the column to be introduced in the basis involves two difficulties:

(i) the evaluation of C'_j,

(ii) the computation of a point $\boldsymbol{u} \in C^{i\,min}$ such that the gradient $\boldsymbol{b} = \text{grad}\, \varphi_{i\,min}(\boldsymbol{u})$, which is the column to be introduced, satisfies $\boldsymbol{b}^T(\boldsymbol{u}-\boldsymbol{Z}) = C'_{i\,min}$.

We have proved that in the case of the problem 3.3. both steps (i) and (ii) require only computation of one value of the functions φ for every constraint, so that there is practically no extra computational work respect to the one which is required by the revised simplex for ordinary linear programming problem.

REFERENCE

J. Gomulka, J. J. Mc Keown and G. Treccani: Functional Linear Programming — Techn. Rep. No 70 — Numerical Optimisation Centre — The Hatfield Polytechnic — Hatfield (UK) November 1975.

SUMMARY

This work presents an application of functional linear programming (FLP) to the solution of convex programming problems. The basic idea is similar to Wolfe's and Dantzig's "generalized linear programming" when applied to the solution of convex problems, although the theoretical approach is quite different. Nevertheless, a much smaller amount of computational work seems to be involved at any iteration, and the convergence properties are different.

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