Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics Volume II: Applications

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Systems Research Institute
Polish Academy of Sciences

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The Kemeny median for partial rankings. Binary pairwise comparisons matrix approach

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Abstract

Problem of determining group judgement is to be solved when an aggregated opinion should be formulated on the basis of individual preferences. In general it is assumed that individual preferences are given in the order scale and that all the alternatives can be compared, i.e. linear orders are considered. However the latter assumption is not always satisfied. In such a case one can make use of partial rankings. Pairwise comparisons of alternatives provide a practical framework for acquiring information on ordinal preferences.

In the paper the concept of binary comparisons matrix is presented. It can be applied to the problem of determining Kemeny's median in the case when there are some incomparable alternatives in experts' rankings. An integer optimization problem is formulated and some numerical examples are given. When experts provide their opinions in the form of linear orders the results obtained with classic as well as binary comparisons matrices are the same.

Keywords: preferences aggregation, partial rankings, pairwise comparisons, Kemeny's median, discrete optimization.

1 Introduction

Problem of determining group judgement arises when an aggregated opinion should be formulated on the basis of individual preferences of e.g. experts, project

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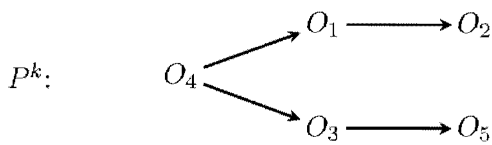
reviewers, consumers, etc. The way in which the preferences over a set of alternatives are expressed depends on quality and quantity of data available. Hence some assumptions on the form of data are to be taken. In many applications it is assumed that individual preferences are given in the order scale. In most methods of group judgement it is also assumed that all the alternatives can be compared, i.e. linear orders are considered, tied preferences allowed or not. However this assumption is not always satisfied. In such a case one can make use of partial rankings. Pairwise comparisons of alternatives provide a practical framework for obtaining information on preferences in the order scale. Kemeny and Snell [6] considered the problem of deriving a consensus ranking in the case when individual preferences are given as pairwise comparisons. To derive the aggregate opinion it was assumed that one has to minimize the number of differences between the group judgement and individual rankings. The resulting aggregate preference ranking is called Kemeny's median. It is shown (e.g. Nurmi [8], Truchon [10]) that Kemeny's median has good properties i.e. it is Condorcet consistent and it satisfies a weak version of local independence of irrelevant alternatives (Truchon [10, 11], Davenport and Kalagnanam [5]). It has also a good "physical" interpretation as an aggregated preference ranking being the closest one - in the sens of the distance defined - to the set of individual rankings. However, determining Kemeny's median is a problem of discrete optimization and is computationally NP-hard (Davenport and Kalagnanam [5]), therefore its practical use is limited. Some algorithms using branch and bound technique (Conitzer at al.[3], Davenport and Kalagnanam [5], Cook et al. [4]) have been proposed to determine Kemeny's median. However, for small n (number of ranked alternatives), n = 3, 4, 5, it is possible to solve the problem using brute-force algorithm (all possible linear preference orders, tied alternatives allowed, are analyzed, Bury and Wagner [1]). The development of integer optimization algorithms makes it possible to practically apply Kemeny's method. It can be assumed - as an extension of generally adopted assumptions - that individual rankings as well as the group judgement may be partial. An interesting suggestion how to take into account partial rankings given by experts was given by Litvak [7]. In the paper the problem of deriving Kemeny's median for a general case of partial rankings as a solution of an integer optimization problem is formulated and some numerical examples are shown.

2 The form of experts' rankings. Binary pairwise comparisons matrices

Let $\mathcal{O}$ denotes the set of alternatives, $\mathcal{O} = \{O_1, \ldots, O_n\}$ to be ranked by experts and $\mathcal{K} = \{1, \ldots, K\}$ is the set of experts. In the paper we assume that preference rankings are given in the order scale. The following notation is used.
$O_i \succ^k O_j$, if the $k-th$ expert regards alternative O_i to be better than O_j,
$O_i \approx^k O_j$, if the $k-th$ expert regards alternatives O_i and O_j to be equivalent,
$O_i \prec^k O_j$, if the $k-th$ expert regards alternative O_j to be better than O_i, $O_i \perp^k O_j$, if the $k-th$ expert cannot compare alternatives O_i and O_j.

All the comparisons are accomplished with regard to the adopted criterion (set of criteria).

When the $k-th$ expert judgement P^k is given as linear preference order it is generally assumed that an alternative regarded as the best one - in the sense of a criterion/ criteria considered - takes the first position and that regarded as the worst takes the last one. Tied alternatives are given in brackets, e.g. $P^k = \{O_1, (O_2, O_3), O_4, O_5\}$ i.e. $O_1 \succ O_2 \approx O_3 \succ O_4 \succ O_5$. When partial orders are considered, P^k may be given as follows



i.e. O_4 is better ($\succ$) than all the other alternatives, $O_1 \succ O_2, O_3 \succ O_5$ but O_1 and O_2 are not comparable with either O_3 or O_5.

Experts may also present their preference rankings in the form of pairwise comparisons matrices.

Let us define a binary pairwise comparisons matrix B as follows (for simplicity the index k denoting the number of expert has been omitted).

$$\text{if} \begin{cases} O_i \succ O_j \\ O_i \approx O_j \\ O_i \prec O_j \\ O_i \perp O_j \end{cases} \quad \text{then} \quad \begin{array}{ll} b_{ij} = 1, & b_{ji} = 0 \\ b_{ij} = 1, & b_{ji} = 1 \\ b_{ij} = 0, & b_{ji} = 1 \\ b_{ij} = 0, & b_{ji} = 0 \end{array} ; \quad b_{ii} = 1 \tag{1}$$

The $B = [b_{ij}]$ matrix defines a binary relation $\mathcal{B}(\succ, \approx, \perp)$ on the set of alternatives $\mathcal{O}$.

An individual ranking P is transitive iff its binary matrix B fulfills the following condition (Ross and Wright[9])

$$B * B \leq B \tag{2}$$

where * denotes Boolean product of binary matrices. The (i, j) component of the Boolean product $B * B$ is of the form

$$b_{i1} \cdot b_{1j} \vee b_{i2} \cdot b_{2j} \vee \ldots \vee b_{ik} \cdot b_{kj} \vee \ldots \vee b_{in} \cdot b_{nj}.$$

It follows from (2) that the following inequalities hold

$$b_{i1} \cdot b_{1j} \vee b_{i2} \cdot b_{2j} \vee \ldots \vee b_{ik} \cdot b_{kj} \vee \ldots \vee b_{in} \cdot b_{nj} \leq b_{ij}, \text{ for } i, j = 1, \ldots, n \quad (3)$$

Binary pairwise comparisons matrices can be used to analyze properties of experts' rankings. In the case under consideration one has to impose transitivity conditions on the solution of the distance minimization problem in order to determine the group judgement.

The outranking matrix L_B, where $l_{B\,ij}$ - denotes the number of experts in whose opinions alternative $O_i \succeq O_j$, (Bury and Wagner[1]) is defined in the same way as classic outranking matrix L (Nurmi[8]). From (1) one has $b_{ij}^k = 1$, for $O_i \succ^k O_j$ or $O_i \approx^k O_j$. Hence

$$l_{B\,ij} = \sum_{k=1}^{K} b_{ij}^k \quad (4)$$

It is worth to note that when individual rankings are linear orders with no tied alternatives the L_B matrix is the same (except elements on the main diagonal) as the classic outranking matrix L.

3 Distance between rankings

Let us introduce the following definitions. They are analogous to those formulated for classic pairwise comparisons matrices.

Definition 1
Distance between a pair of alternatives (O_i, O_j) in an opinion P^{k_1} and a pair of alternatives (O_i, O_j) in an opinion P^{k_2}, expressed in terms of elements of binary matrix B, is of the form

$$d_{Bij}\left(P^{k_1}, P^{k_2}\right) = \left|b_{ij}^{k_1} - b_{ij}^{k_2}\right| + \left|b_{ji}^{k_1} - b_{ji}^{k_2}\right| \quad (5)$$

It follows from (1) that in general the binary comparisons matrix B is not symmetric, hence the second component of the distance (5) is necessary.

Definition 2
Distance between opinions P^{k_1} and P^{k_2}, expressed in terms of elements of binary matrix B, is of the form

$$\begin{aligned} d_B\left(P^{k_1}, P^{k_2}\right) &= \sum_{i=1}^{n}\sum_{j>i}^{n}\left(\left|b_{ij}^{k_1} - b_{ij}^{k_2}\right| + \left|b_{ji}^{k_1} - b_{ji}^{k_2}\right|\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|b_{ij}^{k_1} - b_{ij}^{k_2}\right| + \left|b_{ji}^{k_1} - b_{ji}^{k_2}\right|\right) \end{aligned} \tag{6}$$

Definition 3
The distance d_B of a given opinion P from the set $P^{(k)} = \{P^1, P^2, \ldots, P^K\}$ of experts' rankings is

$$d_B\left(P^{(k)}, P\right) = \sum_{k=1}^{K} d_B\left(P^k, P\right) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{K}\left(\left|b_{ij}^{k} - b_{ij}\right| + \left|b_{ji}^{k} - b_{ji}\right|\right) \tag{7}$$

Definition 4
The Kemeny median (defined with respect to binary comparisons matrix) is such an opinion P_B, that is the closest to the set $P^{(k)}$ i.e. it minimizes the distance $d_B\left(P^{(k)}, P\right)$.

The problem of determining Kemeny's median can be formulated as follows: find an opinion P_B (expressed in terms of elements of B matrix) that

$$\min_B d_B\left(P^{(k)}, P\right) \rightarrow P_B \tag{8}$$

When solving the problem (8) it is necessary to formulate the transitivity conditions for the opinion to be searched. This will be discussed in details in the next Section.

Let us assume that in an opinion P some alternatives are tied or not comparable. The distance (6) between P^k and P can be written as

$$\begin{aligned} d_B\left(P^{(k)}, P\right) = \frac{1}{2}\Bigg[&\sum_{i,j\in I_1}\left(\left|b_{ij}^{k} - 1\right| + \left|b_{ji}^{k}\right|\right) + \sum_{i,j\in I_1^*}\left(\left|b_{ij}^{k}\right| + \left|b_{ji}^{k} - 1\right|\right) \\ &+ \sum_{i,j\in I_2}\left(\left|b_{ij}^{k} - 1\right| + \left|b_{ji}^{k} - 1\right|\right) + \sum_{i,j\in I_3}\left(\left|b_{ij}^{k}\right| + \left|b_{ji}^{k}\right|\right)\Bigg] \end{aligned} \tag{9}$$

where
I_1 - a set of indices (i,j) such that in $P \quad O_i \succ O_j$,
I_1^* - a set of indices (i,j) such that in $P \quad O_i \prec O_j$,
I_2 - a set of indices (i,j) such that in $P \quad O_i \approx O_j$,
I_3 - a set of indices (i,j) such that in $P \quad O_i \perp O_j$.

It follows from the definition of the sets I_1 and I_1^* that if $(i,j) \in I_1$, then $(j,i) \in I_1^*$. Hence

$$\sum_{i,j \in I_1^*} \left(\left| b_{ij}^k \right| + \left| b_{ji}^k - 1 \right| \right) = \sum_{i,j \in I_1} \left(\left| b_{ji}^k \right| + \left| b_{ij}^k - 1 \right| \right) \tag{10}$$

Making use of (10) the distance (9) can be written as follows

$$\begin{aligned} d_B\left(P^{(k)}, P\right) &= \frac{1}{2}\left[\sum_{i,j\in I_1} \left(\left| b_{ij}^k - 1 \right| + \left| b_{ji}^k \right| \right) + \sum_{i,j\in I_1} \left(\left| b_{ij}^k - 1 \right| + \left| b_{ji}^k \right| \right) \right] \\ &+ \frac{1}{2}\left[\sum_{i,j\in I_2} \left(\left| b_{ij}^k - 1 \right| + \left| b_{ji}^k - 1 \right| \right) + \sum_{i,j\in I_3} \left(\left| b_{ij}^k \right| + \left| b_{ji}^k \right| \right) \right] \\ &= \sum_{i,j\in I_1} \left(\left| b_{ij}^k - 1 \right| + \left| b_{ji}^k \right| \right) + \frac{1}{2} \sum_{i,j\in I_2} \left(\left| b_{ij}^k - 1 \right| + \left| b_{ji}^k - 1 \right| \right) \\ &+ \frac{1}{2} \sum_{i,j\in I_3} \left(\left| b_{ij}^k \right| + \left| b_{ji}^k \right| \right) \end{aligned} \tag{11}$$

To simplify the form of (11) the following auxiliary binary variables x_{ij}, z_{ij} and y_{ij} are introduced.

$$x_{ij} = \begin{cases} 1 & \text{if in opinion } P \quad O_i \succ O_j \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

$$z_{ij} = \begin{cases} 1 & \text{if in opinion } P \quad O_i \approx O_j \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

$$y_{ij} = \begin{cases} 1 & \text{if in opinion } P \quad O_i \perp O_j \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Using these variables the distance $d_B(P^{(k)}, P)$ can be written as follows

$$
\begin{aligned}
d_B(P^{(k)}, P) &= \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|b_{ij}^k - 1\right| + \left|b_{ji}^k\right|\right) x_{ij} \\
&+ \frac{1}{2}\sum_{k=1}^{K}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|b_{ij}^k - 1\right| + \left|b_{ji}^k - 1\right|\right) z_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|b_{ij}^k\right| + \left|b_{ji}^k\right|\right) y_{ij}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} br_{ij}x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} be_{ij}z_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} bc_{ij}y_{ij}
\end{aligned} \tag{15}
$$

The coefficients $br_{ij}, be_{ij}, bc_{ij}$ are defined as follows:

$$br_{ij} = \sum_{k=1}^{K}\left(\left|b_{ij}^k - 1\right| + \left|b_{ji}^k\right|\right) \tag{16}$$

$$be_{ij} = \frac{1}{2}\sum_{k=1}^{K}\left(\left|b_{ij}^k - 1\right| + \left|b_{ji}^k - 1\right|\right), \tag{17}$$

$$bc_{ij} = \frac{1}{2}\sum_{k=1}^{K}\left(\left|b_{ij}^k\right| + \left|b_{ji}^k\right|\right) \tag{18}$$

The $BR = [br_{ij}], BE = [be_{ij}], BC = [bc_{ij}]$ matrices correspond to the loss matrices $[r_{ij}], [e_{ij}]$ defined for classic pairwise comparisons matrices (Bury and Wagner [1]).

4 Optimization problem

Making use of (15) the problem of distance minimization can be written as follows:

$$\min_{x_{ij}, z_{ij}, y_{ij}} d_B(p^{(k)}, P) = \sum_{i=1}^{n}\sum_{j=1}^{n} br_{ij}x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} be_{ij}z_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} bc_{ij}y_{ij} \tag{19}$$

$\forall_{i,j=1,\dots,n,\ j\neq i}$ the b_{ij} elements of B matrix and the variables x_{ij}, z_{ij}, y_{ij} should satisfy the following:

$$b_{ij} = x_{ij} + z_{ij}, \quad x_{ij} + x_{ji} \le 1, \quad z_{ij} = z_{ji} \tag{20}$$

$$x_{ij} + x_{ji} + z_{ij} + y_{ij} = 1 \tag{21}$$

$$y_{ij} = y_{ji}, \quad x_{ii} = 0, \quad y_{ii} = 0, \quad z_{ii} = 1, \quad b_{ii} = 1 \tag{22}$$

Equality (21) means that only one of the following relations holds:

$$O_i \succ O_j, O_i \prec O_j, O_i \approx O_j \quad \text{or} \quad O_i \perp O_j.$$

In order to determine constraints to which the optimization problem (19) is subjected one has to take into account that for each b_{ij}, $i, j = 1, \ldots, n$, there are n^2 transitivity conditions (3). For $i = j$ (main diagonal of the B matrix) $b_{ii} = 1$ and (3) always holds, so we have to consider $n^2 - n$ inequalities. The inequality (3) is to be fullfilled for each b_{ij} (specially for $b_{ij} = 0$) hence the following condition is to be satisfied

$$\forall_{i,j=1,\ldots,n,\, j\neq i}\; b_{ih} \cdot b_{hj} \leq b_{ij},\; h = 1, \ldots, n \tag{23}$$

For $h = i$ (and for $h = j$ analogously) inequality $b_{ii} \cdot b_{ij} \leq b_{ij}$ always holds, hence there are (n-2) conditions (23), $h = 1, \ldots, n,\; h \neq i,\; h \neq j$ to be satisfied. For each of $n(n-1)$ elements b_{ij} out of main diagonal one has $(n-2)$ inequalities, hence there are $n(n-1)(n-2)$ transitivity conditions to be fulfilled. $\forall_{i,j=1,\ldots,n,\, j\neq i}\forall_{h=1,\ldots,n,\, h\neq i, h\neq j}$ the product $b_{ih} \cdot b_{hj}$ can be replaced with the linear expression $(b_{ih} + b_{hj} - 1)$. The values of the product $b_{ih} \cdot b_{hj}$ and of the sum $(b_{ih} + b_{hj} - 1)$ respectively are presented in the Tables 1 and 2.

Table 1. The product $b_{ih} \cdot b_{hj}$

$b_{hj} \backslash b_{ih}$	0	1
0	0	0
1	0	1

Table 2. The sum $(b_{ih} + b_{hj} - 1)$

$b_{hj} \backslash b_{ih}$	0	1
0	-1	0
1	0	1

It follows from these tables that the given product and the sum differ only for $b_{ih} = b_{hj} = 0$. However, in this case inequality $b_{ih} \cdot b_{hj} \leq b_{ij}$ holds for every b_{ij}.

The problem of determining Kemeny's median can be formulated as follows

$$\min_B d_B(P^{(k)}, P) = \min_{B, x_{ij}, z_{ij}, y_{ij}} d_B(P^{(k)}, P) =$$
$$\min_{B, x_{ij}, z_{ij}, y_{ij}} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} br_{ij} x_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} be_{ij} z_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} bc_{ij} y_{ij} \right) \tag{24}$$

subject to

$$\forall_{i,j=1,\ldots,n,\, j\neq i} \quad b_{ih} + b_{hj} - 1 \leq b_{ij},\; h = 1, \ldots, n \tag{25}$$

and (20) $\div$ (22). Such a discrete linear optimization problem can be solved with the use of CPLEX software. Similar optimization problem - for the classic pairwise comparisons matrix - is hard to accomplish due to the difficulty of proper formulation of transitivity conditions of the solution. It is worth to note that if there are no incomparable alternatives in individual preference rankings, preference orders and values of distances obtained with the use of classic as well as binary pairwise comparisons matrices are equal (Bury and Wagner [2]). However, it should be mentioned that formulation of an integer optimization problem (24) with constraints (20) $\div$ (22) and (25) - by means of binary comparisons matrix B - makes it possible to impose some additional constraints on the form of group judgement, e.g. only linear orders can be taken into account $\forall_{i,j=1,\ldots,n}\ y_{ij} = 0$ or no tied alternatives are allowed $\forall_{i,j=1,\ldots,n}\ z_{ij} = 0$.

5 Numerical examples

Example 5.1.
This example is taken from Cook [4]. It presents partial rankings of six research projects given by five experts:

$P^1 : O_1, O_3, O_2, O_5$
$P^2 : O_2, O_1, O_4, O_6$
$P^3 : O_4, O_3, O_5, O_6$
$P^4 : O_6, O_1, O_4, O_5$
$P^5 : O_6, O_2, O_3, O_1$

Pairwise comparisons matrices B^k (corresponding to rankings of five experts k = 1, 2, 3, 4, 5) are as follows

B^1	O_1	O_2	O_3	O_4	O_5	O_6
O_1	1	1	1	0	1	0
O_2	0	1	0	0	1	0
O_3	0	1	1	0	1	0
O_4	0	0	0	1	0	0
O_5	0	0	0	0	1	0
O_6	0	0	0	0	0	1

B^2	O_1	O_2	O_3	O_4	O_5	O_6
O_1	1	0	0	1	0	1
O_2	1	1	0	1	0	1
O_3	0	0	1	0	0	0
O_4	0	0	0	1	0	1
O_5	0	0	0	0	1	0
O_6	0	0	0	0	0	1

B^3	O_1	O_2	O_3	O_4	O_5	O_6
O_1	1	0	0	0	0	0
O_2	0	1	0	0	0	0
O_3	0	0	1	0	1	1
O_4	0	0	1	1	1	0
O_5	0	0	0	0	1	1
O_6	0	0	0	0	0	1

B^4	O_1	O_2	O_3	O_4	O_5	O_6
O_1	1	0	0	1	1	0
O_2	0	1	0	0	0	0
O_3	0	0	1	0	0	0
O_4	0	0	0	1	1	0
O_5	0	0	0	0	1	0
O_6	1	0	0	1	1	1

B^5	O_1	O_2	O_3	O_4	O_5	O_6
O_1	1	0	0	0	0	0
O_2	1	1	1	0	0	0
O_3	1	0	1	0	0	0
O_4	0	0	0	1	0	0
O_5	0	0	0	0	1	0
O_6	1	1	1	0	0	1

The loss matrices $BR = [br_{ij}], BE = [be_{ij}], BC = [bc_{ij}]$ are as follows

$$BR = \begin{bmatrix} 6 & 6 & 5 & 3 & 3 & 6 \\ 6 & 6 & 5 & 4 & 4 & 5 \\ 5 & 5 & 6 & 6 & 3 & 5 \\ 7 & 6 & 4 & 6 & 3 & 4 \\ 7 & 6 & 7 & 7 & 6 & 3 \\ 4 & 5 & 5 & 6 & 5 & 6 \end{bmatrix} \qquad BE = \begin{bmatrix} 0 & 4 & 4 & 3 & 3 & 4 \\ 3 & 0 & 4 & 4 & 4 & 4 \\ 4 & 4 & 0 & 5 & 3 & 4 \\ 5 & 5 & 4 & 0 & 3 & 3 \\ 5 & 5 & 5 & 5 & 0 & 4 \\ 3 & 4 & 4 & 4 & 4 & 0 \end{bmatrix}$$

$$BC = \begin{bmatrix} 5 & 1 & 1 & 2 & 2 & 1 \\ 2 & 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 & 2 & 2 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 2 & 1 & 1 & 1 & 1 & 5 \end{bmatrix}$$

The solutions to the optimization problem (24) with constraints (20) ÷ (22) and (25) are as follows:

$O_2, O_1, O_4, O_3, O_6, O_5$

$O_2, O_6, O_1, O_4, O_3, O_5$

$O_2, O_1, O_4, O_6, O_3, O_5.$

Their distance from the set of experts' opinions is 63. The latter solution (and distance) are the same as those obtained by Cook et al. [4]. It should be noted that Cook et al. applied branch and bound algorithm to solve the problem and the distance function they used was formulated in terms of different pairwise comparisons coefficients, tied rankings excluded. However, it can be easily shown, that for the case of no ties both distances are the same.

Example 5.2.

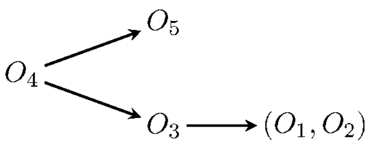
The preference rankings of five alternatives given by five experts are as follows, tied alternatives are given in brackets.

P^1 : O_2, O_1, O_3, O_4, O_5

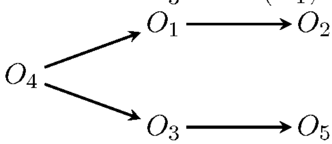
P^2 : $(O_1, O_5), O_3, O_2, O_4$

P^3 : O_1, O_2, O_3, O_4, O_5

P^4:



P^5:

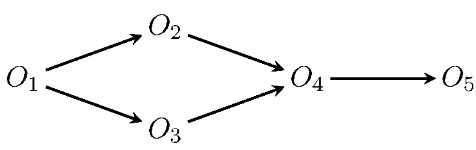


The Kemeny median - derived with brute-force algorithm in the set of all linear orders possible is as follows:

O_1, O_2, O_3, O_4, O_5 or O_1, O_3, O_2, O_4, O_5.

The distance from the set of experts' rankings is equal to 35.

The solution of the optimization problem (24) with constraints (20) $\div$ (22) and (25) in the set of partial rankings is of the form



Its distance from the set of experts' rankings is equal to 34.

The next example concerns the case of tied alternatives in experts' rankings, the presence of the Condorcet winner and a cycle (resulted from the simple majority rule) in the group judgement.

Example 5.3.
The preference rankings of seven alternatives given by seven experts are as follows, tied alternatives are given in brackets.

$P^1 : O_1, (O_2, O_3), (O_4, O_5), O_6, O_7$
$P^2 : O_4, (O_1, O_2), O_5, (O_3, O_7), O_6$
$P^3 : O_4, O_2, O_1, O_5, O_7, O_3, O_6$
$P^4 : O_1, O_7, O_3, O_4, O_6, O_2, O_5$
$P^5 : O_1, O_6, O_2, O_5, O_7, O_3, O_4$
$P^6 : O_6, O_1, O_2, O_7, O_4, O_5, O_3$
$P^7 : O_5, O_1, O_2, O_6, O_4, O_7, O_3$

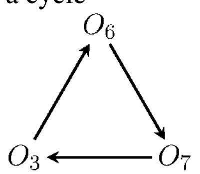
The outranking matrix L_B (determined with the use of binary comparisons matrices $B^1, \ldots, B^7$) is as follows

$LB =$

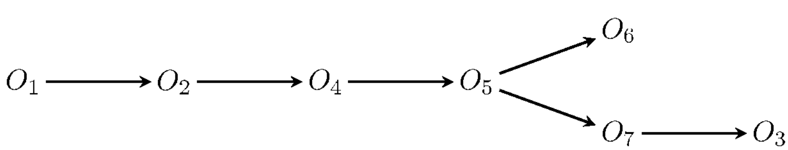
	O_1	O_2	O_3	O_4	O_5	O_6	O_7
O_1	7	6	7	5	6	6	7
O_2	2	7	6	4	6	4	6
O_3	0	2	7	3	2	4	2
O_4	2	3	4	7	5	4	4
O_5	1	1	5	3	7	4	5
O_6	1	3	3	3	3	7	4
O_7	0	1	6	3	2	3	7

Alternative O_1 is the Condorcet winner.

It follows from the outranking matrix that $O_1 \succ O_2 \succ O_4 \succ O_5$ and that there is a cycle



The solution of optimization problem (24) subject to the constraints (20) ÷ (22) and (25) in the set of partial rankings is an opinion



Its distance from the set of experts' rankings is equal to 84.
This solution is consistent with the property formulated by Truchon [11, 10] as the extended Condorcet criterion, when the Condorcet winner exists, there is a cycle in experts' opinions (resulted from the simply majority rule) and due to the cycle the group judgement may be a partial order.

In the set of linear orders the solutions of (24) with constraints (20) $\div$ (22) and (25) are the following preference orders:
$O_1, O_2, O_4, O_5, O_7, O_3, O_6$ or $O_1, O_2, O_4, O_5, O_6, O_7, O_3$.
Their distance from the set of the experts' rankings is equal to 84 too.

6 Conclusions

In the paper the concept of the binary comparisons matrix is presented. It was applied to the problem of determining Kemeny's median in the case when there are some incomparable alternatives in experts' rankings. In the case when experts give their opinions in the form of linear orders the results obtained with classic as well as binary comparisons matrices are the same (Bury and Wagner [2]. Moreover, application of the binary comparisons matrices to the description of preference rankings makes it possible to formulate the transitivity conditions necessary for determining Kemeny's median as the solution of the distance minimization problem.

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The papers presented in this Volume 2 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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