Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics Volume II: Applications

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On the choice of membership function in a fuzzy model of compliance with highly uncertain emissions

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Abstract

Limitation of pollutant emissions by a central planner is an often applied tool in environmental policy. In this paper we consider a situation of very uncertain emission estimates. An approach to verification of compliance in such cases was proposed by the present authors. The method makes use of the fuzzy numbers to represent emission uncertainty. In this paper we consider the problem of choice of an analytical membership function for the fuzzy number, where information on an emission uncertainty distribution is provided by the Monte Carlo method. To assess accuracy of decision parameters for different analytical membership functions, we perform simulations for a few national greenhouse gases emission inventories. The investigation results show that the function proposed in the earlier publications is the most robust and convenient in practical applications.

Keywords: uncertain emissions, compliance, fuzzy numbers, membership function, practical cases, estimation, accuracy of a decision parameter.

1 Introduction

Compliance verification under uncertainty is an often encountered problem in environmental policy. This applies to a situation where highly uncertain observations of pollutant emissions are compared with a given upper limit imposed for

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some safety reasons. Recently, a method for checking compliance in such a situation has been proposed by the present authors [9, 10, 11, 12], with intervals or fuzzy numbers representing uncertain values. The situation considered in this paper is when the variables cannot be measured directly, like, for example, emissions of the greenhouse gases, whose values are estimated by inventory of all emitted sources. Uncertainty of inventories can be assessed either by calculation of their standard deviation using the variance summation rule for the random variables (Tier 1 method) or by the Monte Carlo simulations (Tier 2 method). The latter method gives more information on the distribution of errors and therefore it is of our interest in this paper. Thus we assume that the membership functions of the fuzzy numbers can be acquired from estimates of the uncertainty distributions obtained by the Monte Carlo method. To get analytical solutions for the limit under uncertainty, an approximation of these estimates by a specific class of functions has been proposed.

The idea of the proposed verification method lies in determining the condition for the examined emission values to assure small enough risk of noncompliance. It requires that the approximating function has to reproduce adequately both the area of the distribution as well as an appropriate area of its tail. Another important advantage of the approximation function is that it enables getting the final solution in a closed form, which is useful in practical applications. Taking into account the fuzzy approach proposed, it should be noted that the uncertainty distribution estimates are usually asymmetric with very diversified shapes. Therefore it is difficult to find a probabilistic distribution matching well all the cases, as is shown with the Gumbel distribution chosen for examination in the sequel. Moreover, the algebra of simple operations, e.g. addition or subtraction of variables, is easier for the fuzzy variables than for the probabilistic ones.

Our earlier choice of the fuzzy number membership function approximations to represent the uncertainty followed this rationale. To check this intuitive choice, in this paper we compare and discuss influence of a few selected approximations on accuracy of determining the compliance area.

The problem of approximating the membership functions of fuzzy numbers has been already discussed in the literature. The discussion started with the interval [4, 5] and trapezoidal [1, 6] approximations. Later, also more general approximating functions have been considered [2, 15, 13, 7]. These latter studies use families of functions dependent on parameters, which enable them to approximate a wide class of membership functions. Similar approach is also applied in the present paper. However, in the references mentioned above the main criterion for approximation is good overall fit. Usually, preservation of some properties of the fuzzy numbers considered is additionally required. Thus, the approximating functions usually become complicated enough to enable flexible fit to the very shape-variable membership functions. In this paper, we require not only good overall approximations, but also good approximations of the right tails, as well as simple representations allowing for analytical solutions to the considered problems.

The idea of this paper arose from a comment of an undisclosed reviewer of the paper [12], who suggested that other approximations are worth considering besides the one used in [12], which is called the power membership function in the sequel. A specific comment of the reviewer pointed to the polynomial functions, considered in this paper among other membership functions.

The content of the paper is as follows. In Section 2 we explain our interpretation of compliance under uncertainty and present basic formulae for compliance limit with the risk α. In Section 3 we introduce the examined membership functions and derive the compliance limit in each case. Section 4 contains numerical results for a few greenhouse gases emission inventories in the Netherlands and Austria, and Section 5 concludes.

2 Managing the compliance risk for uncertain emission estimates

Le us denote by x an uncertain variable to be compared with an imposed limit L. Thus, we have to decide whether

$$x \leq L \tag{1}$$

There may be another formulation of the problem: to check whether the uncertain variable has been reduced by a specified fraction, usually in a specified period of time. Denoting by x_b the initial (basic) variable, by x_c the final (compliant) variable, and by δ the required reduction fraction, our problem is to decide if

$$x_c \leq (1 - \delta)x_b \tag{2}$$

This can be, however, written in the form

$$x_c - (1 - \delta)x_b \leq 0 \tag{3}$$

If the uncertainty distribution of the variable $x_c - (1 - \delta)x_b$ is calculated, then the problem (3) is exactly the same as (1). Moreover, if we transform (1) to

$$x - L \leq 0$$

and treat the left-hand side expression as a new variable, then the problem of deciding if the variable is non-positive can be considered. Thus, in the sequel we confine to a simple problem

$$x \leq 0 \tag{4}$$

The earlier mentioned difference between the probabilistic and fuzzy approaches can be easily demonstrated when considering the left-hand side of the inequality (3). In the probabilistic approach, in general, it is possible to compute a distribution of the difference $Dx = x_c - (1-\delta)x_b$, where x_b and x_c are independent continuous random variables. Although it may be easy for some special distributions, it may require numerical computations of the convolution integral for others. Moreover, x_c and x_b are seldom independent.

For the fuzzy approach, calculation of the distribution of Dx may be also cumbersome in a general case. However, it is fairly easy to find distributions for fuzzy numbers with LR membership functions which are, additionally, general enough in determining their shapes and, therefore, well suited to represent uncertain values with fairly different distributions. A way of introducing dependence between variables has been proposed in [9].

A fuzzy number B is the fuzzy set, which satisfies three conditions:

1. The membership functions of B attains the highest value 1 in only one point, called the core point.

2. The membership function does not increase starting from the core point toward both sides.

3. Every η-cut[1] is a (connected) close interval.

A weaker definition of a fuzzy number is sometimes used, where the value 1 may be attained in more than one point. In this paper we use the former stronger definition.

Two functions defined for non-negative arguments may be introduced, L and R, ([3]), such that they have the unique value 1 at 0, $L(0) = R(0) = 1$, equal zero for arguments greater or equal 1, $L(x) = R(x) = 0$ for $x \geq 1$, and are not increasing. Then, given that $\mu(m) = 1$, the membership function of a fuzzy number may be constructed using the above functions as its left and right branches

$$\mu_B^l(x) = L\Big(\frac{m-x}{d^l}\Big) \quad \text{for } x \leq m \tag{5}$$

[1]Here we name the η-cut of a fuzzy set B the notion usually called the α-cut, i.e. the set $B_\eta = \{x \in \text{supp } B | \mu_B(x) \geq \eta\}$, for $\eta \in (0, 1]$.

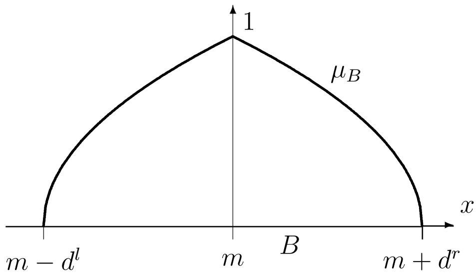

Figure 1: A membership functions of the set B.

$$\mu_B^r(x) = R\Big(\frac{x-m}{d^r}\Big) \quad \text{for } x \geq m \tag{6}$$

where d^l and d^r are support parameters, see Figure 1. Let us denote the fuzzy number constructed this way as $B(m, d^l, d^r)$.

Here operations on fuzzy numbers are restricted only to fuzzy numbers described in the above LR form. For two fuzzy numbers $B(m, d^l, d^r)$ and $C(n, q^l, q^r)$, the following operations are defined:

1. Addition
$$B + C = (m + n, d^l + q^l, d^r + q^r) \tag{7}$$

2. Multiplication by a positive real number c
$$cB = (cm, cd^l, cd^r) \tag{8}$$

3. Multiplication by a negative real number c
$$cB = (cm, |c|d^r, |c|d^l) \tag{9}$$
with interchange of the function L and R in (5) and (6)
$$\mu_{cB}^l(x) = R\Big(\frac{cm-x}{|c|d^r}\Big) \quad \text{for } x \leq cm$$
$$\mu_{cB}^r(x) = L\Big(\frac{x-cm}{|c|d^l}\Big) \quad \text{for } x \geq cm$$

The above gives simple rules for summing/subtracting fuzzy numbers.

Let us now look at a simple membership function $\mu(x)$ in Figure 2. It presents an example that taking decision on compliance based only on the modal value $\hat{x}$

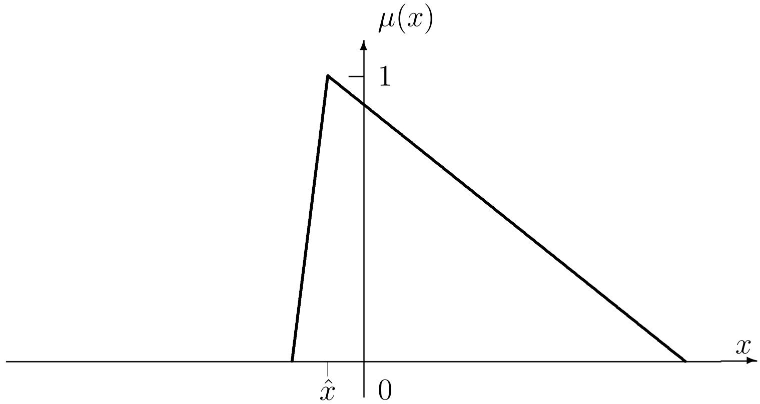

Figure 2: A distribution of a fuzzy number: compliant or non-compliant?

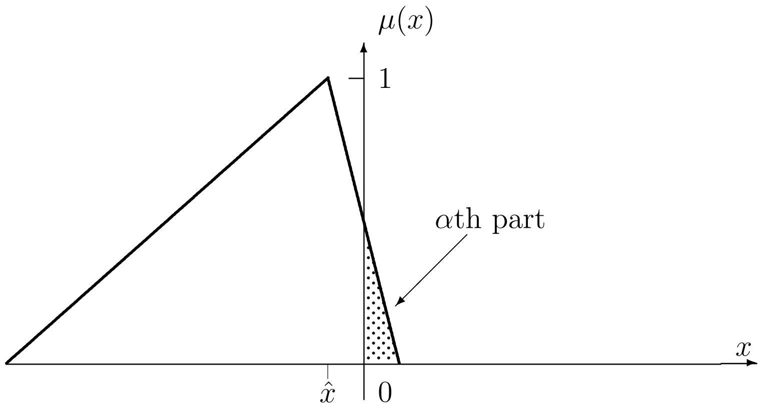

Figure 3: A distribution of a non-positive fuzzy number with the risk α.

may be questionable. Although the modal value $\hat{x}$ in this figure is negative, most of the distribution lies in the positive semi-axis. Thus, it is more likely that the real value is positive than non-positive.

To ensure that (4) is satisfied in any case, we could require that the whole support of the fuzzy number is located in the nonnegative semi-axis. However, this condition could be too demanding in many instances, as the right tails of distributions may be long or even infinite, at least in the theory. Therefore we admit acceptance of the limit with some risk of noncompliance α. We say that *the variable satisfies the limit with the risk α, if the part of the distribution, which is located in the positive semi-axis, is not bigger than α*, see Figure 3.

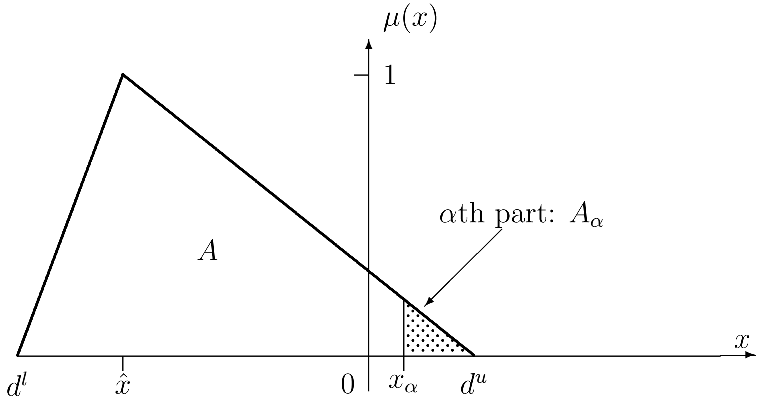

Figure 4: Calculation of x_α.

To calculate the αth part of the distribution, two integrals has to be found, see Figure 4

$$A = \int_{d^l}^{d^u} \mu(x)dx \qquad\qquad A_\alpha = \int_{x_\alpha}^{d^u} \mu(x)dx$$

where x_α is the value of x that cut the right αth part of the distribution. Thus, the following condition has to be satisfied

$$\alpha = \frac{A_\alpha}{A} = \frac{\int_{x_\alpha}^{d^u} \mu(x)dx}{\int_{d^l}^{d^u} \mu(x)dx} \tag{10}$$

where d^l is the left end of the support and d^u is the right end (both may be infinite, but the integrals must be finite). To check non-positivity with the risk α, the value x_α satisfying (10) has to be determined, and then the inequality

$$x_\alpha \leq 0 \tag{11}$$

has to be verified, see Figure 4.

The equation (10) can always be solved for x_α numerically. However, if an appropriate analytical function $\mu(x)$ matching well the experimental (e.g. Monte Carlo) distributions is found, the equation can be solved analytically.

3 Specification of membership functions

As a next step, we determine a membership function $\mu(x)$. Following the approach introduced in [12], we adopt as an approximant an analytical membership function being dependent on parameters. The approximant is fitted to the distribution of emission inventory errors applying the minimum mean squared error criterion.

Although the most popular membership functions are the triangular and trapezoidal ones, these functions provide poor approximations of the distribution tails, which is of great importance in the present application. Instead, we consider other candidates: four functions which belong to a class of LR membership functions, and the function based on the Gumbel distribution.

First, we focus on the LR functions, being formed of two branches: left μ_L and right μ_R, written in a general form as

$$\mu_{LR}(x;\boldsymbol{p}^l,\boldsymbol{p}^u,\boldsymbol{p}) = \begin{cases} \mu_L(x;\boldsymbol{p}^l,\boldsymbol{p}) & \text{for} \quad x<0 \\ \mu_R(x;\boldsymbol{p}^u,\boldsymbol{p}) & \text{for} \quad x\geq 0 \end{cases} \tag{12}$$

where the vector $\boldsymbol{p}^l$ contains selected parameters related to the left branch function μ_L, the vector $\boldsymbol{p}^u$ contains selected parameters related to the right branch functions μ_R, and $\boldsymbol{p}$ is the vector of remaining parameters. In what follows, we describe the procedure of fitting the function (12) to an empirical distribution, and then present particulate functions of this class.

Suppose that the distribution of emission inventory errors is given by a histogram of n bins of length k and of frequencies h_i, for $i=1,\ldots,n$, see Figures 5 - 8. Fitting the LR membership function to the histogram follows an iterative procedure. Let us denote the value of the vector parameters in the iteration t as $\boldsymbol{p}^l_{[t]},\boldsymbol{p}^u_{[t]}$ and $\boldsymbol{p}_{[t]}$. Each iteration consists of two steps. For every $t=0,1,2,\ldots$ we proceed as follows.

In the first step, the function (12) is fitted to the distribution for the left and right branches separately, starting with a given vector of parameters $\boldsymbol{p}_{[0]}$. For this reason, the histogram has to be split into two parts: the one of n_1 bins for $x<0$, and another one of n_2 bins for $x\geq 0$, where $n_1+n_2=n$. Then the optimization tasks are written as

$$\boldsymbol{p}^l_{[t+1]} = \text{arg min}_{\boldsymbol{p}^l} \sum_{i=1}^{n_1} \left(h_i k - \int_{x_i}^{x_{i+1}} \mu_L\left(x;\boldsymbol{p}^l,\boldsymbol{p}_{[t]}\right) dx \right)^2 \tag{13}$$

for the left branch, and as

$$\boldsymbol{p}^u_{[t+1]} = \text{arg min}_{\boldsymbol{p}^u} \sum_{i=n_1+1}^{n} \left(h_i k - \int_{x_i}^{x_{i+1}} \mu_R\left(x;\boldsymbol{p}^u,\boldsymbol{p}_{[t]}\right) dx \right)^2 \tag{14}$$

for the right branch, where x_i, x_{i+1} denote, respectively, the left and the right ends of the bin i.

In the second step, given the estimates $\boldsymbol{p}^l_{[t+1]}$ and $\boldsymbol{p}^u_{[t+1]}$, the function (12) is fitted again, this time for both the branches together, to calculate the estimate

$$\boldsymbol{p}_{[t+1]} = \arg\min_{\boldsymbol{p}} \sum_{i=1}^{n_1} \left(h_i k - \int_{x_i}^{x_{i+1}} \mu_L \left(x; \boldsymbol{p}^l_{[t+1]}, \boldsymbol{p} \right) dx \right)^2 + \sum_{i=n_1+1}^{n} \left(h_i k - \int_{x_i}^{x_{i+1}} \mu_R \left(x; \boldsymbol{p}^u_{[t+1]}, \boldsymbol{p} \right) dx \right)^2 \quad (15)$$

Both steps of the procedure are repeated until the change between two consecutive iterations reported for the estimated parameters becomes sufficiently low.

We analyze the following examples of a general class of LR function. They were chosen to represent three groups of functions: (i) complicated ones, like polynomial or rational functions [2, 15], here represented by the polynomial function of the 4th order and the polynomial-power function [7]; (ii) simple functions like the power function [11], considered also in the sequel, or the function $1-|x|^\gamma$, see [13]; and (iii) the functions inspired by the probabilistic distributions, here represented by two membership functions called Gauss and Gumbel functions.

The polynomial membership function. We consider the 4th order polynomial

$$\mu_M(x) = \begin{cases} a^l_4 x^4 + a^l_3 x^3 + a^l_2 x^2 + a^l_1 x + a_0 & \text{for} \quad d^l_M \le x < 0 \\ a^u_4 x^4 + a^u_3 x^3 + a^u_2 x^2 + a^u_1 x + a_0 & \text{for} \quad 0 \le x \le d^u_M \\ 0, & \qquad \text{otherwise} \end{cases} \quad (16)$$

with coefficients a^u_4, a^u_3, a^u_2, a^u_1, a_0, a^l_4, a^l_3, a^l_2, a^l_1, and the support (d^l_M, d^u_M). Here, we have

$$\begin{aligned} \boldsymbol{p}^l &= \left[a^l_4, a^l_3, a^l_2, a^l_1 \right] \\ \boldsymbol{p}^u &= \left[a^u_4, a^u_3, a^u_2, a^u_1 \right] \\ \boldsymbol{p} &= \left[a_0, d^l_M, d^u_M \right] \end{aligned}$$

In the case of the polynomial function, one has to ensure that its value is monotone increasing (non-decreasing) for negative arguments, and monotone decreasing (non-increasing) for positive arguments. Therefore, we impose additional constraints:

$$\frac{\partial}{\partial x} \mu_L \left(x; \boldsymbol{p}^l, \boldsymbol{p}_{[t]} \right) \Big|_{m_i} \ge 0 \qquad \text{for } i = 1, \ldots, n_1 \quad (17)$$

into the optimization task (13),

$$\frac{\partial}{\partial x} \mu_R \left(x; \boldsymbol{p}^u, \boldsymbol{p}_{[t]} \right) \Big|_{m_i} \le 0 \qquad \text{for } i = 1, \ldots, n_2 \quad (18)$$

into the optimization task (14), and analogously

$$\frac{\partial}{\partial x}\,\mu_L\left(x;\boldsymbol{p}^l_{[t+1]},\boldsymbol{p}\right)\Big|_{m_i}\geq 0 \qquad \text{for } i=1,\ldots,n_1$$

$$\frac{\partial}{\partial x}\,\mu_R\left(x;\boldsymbol{p}^u_{[t+1]},\boldsymbol{p}\right)\Big|_{m_i}\leq 0 \qquad \text{for } i=1,\ldots,n_2$$

into (15), with m_i denoting the middle of a bin i.

Furthermore, for $x=0$ we require that

$$\frac{\partial}{\partial x}\mu_L(x;\boldsymbol{p}^l,\boldsymbol{p})\Big|_0=0$$

and

$$\frac{\partial}{\partial x}\mu_R(x;\boldsymbol{p}^u,\boldsymbol{p})\Big|_0=0$$

which immediately provides $a^l_1=a^u_1=0$.

Final restrictions concern the support parameters d^l_M and d^u_M. In order to facilitate proper fit of the function (16) to the histograms, we require that

$$\mu_L(d^l_M;\boldsymbol{p}^l,\boldsymbol{p})=0$$

and

$$\mu_R(d^u_M;\boldsymbol{p}^u,\boldsymbol{p})=0$$

The polynomial membership function (16) does not allow us to find x_α analytically. While calculation of A is straightforward, the condition $A_\alpha=\alpha A$ takes the form

$$\begin{aligned}
&\frac{a^u_4}{5}\left(d^u_M\right)^5+\frac{a^u_3}{4}\left(d^u_M\right)^4+\frac{a^u_2}{3}\left(d^u_M\right)^3+a_0d^u_M-\frac{a^u_4}{5}x^5_\alpha-\frac{a^u_3}{4}x^4_\alpha-\frac{a^u_2}{3}x^3_\alpha-a_0x_\alpha\\
&=\alpha\left[\frac{a^u_4}{5}\left(d^u_M\right)^5+\frac{a^u_3}{4}\left(d^u_M\right)^4+\frac{a^u_2}{3}\left(d^u_M\right)^3+a_0d^u_M\right.\\
&\qquad\qquad\left.-\frac{a^l_4}{5}\left(d^l_M\right)^5-\frac{a^l_3}{4}\left(d^l_M\right)^4-\frac{a^l_2}{3}\left(d^l_M\right)^3-a_0d^l_M\right]
\end{aligned}$$

which is a polynomial of the 5th order in x_α. Finding its root x_α has to be performed numerically.

The cubic-power membership function. The following function was proposed in [15] and analysed in [7], see equations (2) and (9) therein,

$$\mu_C(x) = \begin{cases} \frac{a_C}{a_l}\left([t_l(x)]^2[3-2t_l(x)] + \beta_0^l - \beta_0^l[1-t_l(x)]^{a_l} + \beta_1^l[t_l(x)]^{a_l}\right) \\ \qquad\qquad\qquad\qquad \text{for} \quad -d_C^l \le x < 0 \\ \frac{a_C}{a_u}\left([t_u(x)]^2[3-2t_u(x)] + \beta_0^u - \beta_0^u[1-t_u(x)]^{a_u} + \beta_1^u[t_u(x)]^{a_u}\right) \\ \qquad\qquad\qquad\qquad \text{for} \quad 0 \le x \le d_C^u \\ 0, \qquad\qquad\qquad\qquad\quad \text{otherwise} \end{cases} \tag{19}$$

where

$$t_l(x) = 1 + \frac{x}{d_C^l}, \qquad a_l = 1 + \beta_0^l + \beta_1^l$$
$$t_u(x) = 1 - \frac{x}{d_C^u}, \qquad a_u = 1 + \beta_0^u + \beta_1^u$$

$\beta_0^l, \beta_1^l, \beta_0^u, \beta_1^u \ge 0$, with scaling factor a_C, and of the support $\left(-d_C^l, d_C^u\right)$. We refer to (19) as to the cubic-power membership function, despite its original name introduced in [15]. For this function we set

$$\begin{aligned} \boldsymbol{p}^l &= \left[\beta_0^l, \beta_1^l\right] \\ \boldsymbol{p}^u &= \left[\beta_0^u, \beta_1^u\right] \\ \boldsymbol{p} &= \left[a_C, d_C^l, d_C^u\right] \end{aligned}$$

Unfortunately, the formula for x_α cannot be determined analytically in this case. Although A is easily obtained as

$$\begin{aligned} A &= \int_{-d_C^l}^{d_C^u} \mu_C(x)dx \\ &= a_C \frac{d_C^l\left[2 + 3\beta_1^l + \beta_0^l\left(3 + 2\beta_0^l + 2\beta_1^l\right)\right]}{2a_l\left(2 + \beta_0^l + \beta_1^l\right)} + \\ &\qquad + a_C \frac{d_C^u\left[2 + 3\beta_1^u + \beta_0^u\left(3 + 2\beta_0^u + 2\beta_1^u\right)\right]}{2a_u\left(2 + \beta_0^u + \beta_1^u\right)} \end{aligned}$$

the condition $A_\alpha = \alpha A$ leads to an implicit function of x_α

$$\frac{a_C}{a_u}\left(\beta_0^u d_C^u + \frac{d_C^u}{2} - \frac{\beta_0^u d_C^u}{2 + \beta_0^u + \beta_1^u} - x_\alpha - \beta_0^u x_\alpha + \frac{x_\alpha^3}{\left(d_C^u\right)^2} - \frac{x_\alpha^4}{2\left(d_C^u\right)^3} \right.$$
$$\left. + \frac{\beta_0^u x_\alpha \left(\frac{x_\alpha}{d_C^u}\right)^{a_u}}{2 + \beta_0^u + \beta_1^u} + \beta_1^u \frac{2x_\alpha - d_C^u - \frac{x_\alpha^2}{d_C^u}}{2 + \beta_0^u + \beta_1^u}\left(1 - \frac{x_\alpha}{d_C^u}\right)^{\beta_0^u + \beta_1^u}\right) = \alpha A$$

thus x_α is further calculated numerically.

The power membership function. Proposed already in [12], the function has the form

$$\mu_P(x) = \begin{cases} a_P\left(1 + \frac{x}{d_P^l}\right)^{\gamma^l} & \text{for} \quad -d_P^l \le x < 0 \\ a_P\left(1 - \frac{x}{d_P^u}\right)^{\gamma^u} & \text{for} \quad 0 \le x \le d_P^u \\ 0 & \quad\quad\quad \text{otherwise} \end{cases} \qquad \gamma^l, \gamma^u \neq 0 \tag{20}$$

with parameters γ^l, γ^u, a_P being a scaling factor, and with the support $(-d_P^l, d_P^u)$. Regarding the above described procedure for parameter estimation, we have $\boldsymbol{p}^l = [\gamma^l]$, $\boldsymbol{p}^u = [\gamma^u]$, and $\boldsymbol{p} = \left[d_P^l, d_P^u, a_P\right]$.

In the case of the power function, the formula for x_α can be derived in a closed form (see also [12]). We start with calculating A, which is the sum of the areas under the left and right branches of $\mu_P(x)$

$$\begin{aligned} A &= \int_{-d_P^l}^{0} a_P\left(1 + \frac{x}{d_P^l}\right)^{\gamma^l} dx + \int_0^{d_P^u} a_P\left(1 - \frac{x}{d_P^u}\right)^{\gamma^u} dx \\ &= a_P \frac{d_P^l}{1+\gamma^l} + a_P \frac{d_P^u}{1+\gamma^u} \end{aligned}$$

The area A_α is

$$A_\alpha = \int_{x_\alpha}^{d_P^u} a_P\left(1 - \frac{x}{d_P^u}\right)^{\gamma^u} dx = a_P \frac{d_P^u}{1+\gamma^u}\left(1 - \frac{x_\alpha}{d_P^u}\right)^{1+\gamma^u}$$

and acknowledging the condition $A_\alpha = \alpha A$, we obtain the formula

$$x_\alpha = \left\{1 - \left[\left(1 + \frac{d_P^l}{d_P^u}\frac{1+\gamma^u}{1+\gamma^l}\right)\alpha\right]^{\frac{1}{1+\gamma^u}}\right\} d_P^u$$

The Gauss membership function. Based on the Gauss distribution, the following LR membership function can be constructed

$$\mu_G(x) = \begin{cases} a_G \exp\left(-\frac{x^2}{a^l}\right) & \text{for} \quad x < 0 \\ a_G \exp\left(-\frac{x^2}{a^u}\right) & \text{for} \quad x \ge 0 \end{cases} \tag{21}$$

with parameters a^l, a^u, scaling factor a_G, and an infinite support. In this case, $\boldsymbol{p}^l = [a^l]$, $\boldsymbol{p}^u = [a^u]$, and $\boldsymbol{p} = [a_G]$.

For this membership function we obtain

$$\begin{aligned} A &= \int_{-\infty}^{0} a_G \exp\left(-\frac{x^2}{a^l}\right) + \int_{0}^{+\infty} a_G \exp\left(-\frac{x^2}{a^u}\right) \\ &= \frac{a_G}{2}\sqrt{\pi}\left(\sqrt{a^l}+\sqrt{a^u}\right) \end{aligned}$$

and

$$A_\alpha = \int_{x_\alpha}^{+\infty} a_G \exp\left(-\frac{x^2}{a^u}\right) = a_G\sqrt{a^u\pi}\left[1-\Phi\left(\frac{\sqrt{2}}{\sqrt{a^u}}x_\alpha\right)\right]$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Therefore the equality $A_\alpha = \alpha A$ leads to the formula for x_α expressed by means of the ζth quantile of the standard normal distribution, denoted q_ζ. That is,

$$x_\alpha = \frac{\sqrt{a^u}}{\sqrt{2}} q_\zeta$$

with

$$\zeta = 1 - \frac{\alpha}{2}\left(\frac{\sqrt{a^l}}{\sqrt{a^u}} + 1\right)$$

The Gumbel membership function. Finally, we examine a function which is no longer of LR class, but the one based on the probabilistic Gumbel distribution. The function has the form

$$\mu_{Gumb}(x) = a_{Gumb}\frac{\exp\left\{-\exp\left(-\frac{x}{\beta}\right)-\frac{x}{\beta}\right\}}{\beta} \tag{22}$$

with the parameter β and the scaling factor a_{Gumb}.

Considering the parameter estimation, a natural counterpart of (15) is now a one shot minimization

$$\text{arg min}_{\beta,a_{Gumb}} \sum_{i=1}^{n}\left(h_i k - \int_{x_i}^{x_{i+1}} \mu_{Gumb}\left(x;\beta,a_{Gumb}\right)dx\right)^2 \tag{23}$$

where we apply the same notation regarding the histogram of emission inventory errors.

For the Gumbel function the formula for x_α can be determined analytically. The Gumbel membership function is simply the Gumbel distribution scaled with a_{Gumb}. Thus, the area A equals

$$A = \int_{-\infty}^{+\infty} a_{Gumb} \frac{\exp\left\{-\exp\left(-\frac{x}{\beta}\right) - \frac{x}{\beta}\right\}}{\beta} dx = a_{Gumb}$$

Next we compute A_α

$$\begin{aligned} A_\alpha &= \int_{x_\alpha}^{+\infty} a_{Gumb} \frac{\exp\left\{-\exp\left(-\frac{x}{\beta}\right) - \frac{x}{\beta}\right\}}{\beta} dx \\ &= a_{Gumb} \left\{1 - \exp\left[-\exp\left(-\frac{x_\alpha}{\beta}\right)\right]\right\} \end{aligned}$$

and the condition $A_\alpha = \alpha A$ yields the following formula for x_α

$$x_\alpha = -\beta \ln\left[-\ln\left(1-\alpha\right)\right]$$

4 Results for the greenhouse gas emission inventories in the Netherlands and Austria

Below we compare performance of the five membership functions for uncertainty distributions of selected greenhouse gas emission inventories in the Netherlands and Austria.

The International Panel on Climate Change (IPCC) in the report [8] specifies guidelines for preparation of uncertainty analysis in national greenhouse gas inventory reports. In particular, the Tier 2 method uses the Monte Carlo approach resulting in a distribution of inventory errors. This type of assessment has been already done for several countries, including the Netherlands and Austria. These estimates of emission inventory uncertainty (i.e. histograms), presented already in [14] for the Netherlands and in [16] for Austria, serve us as a basis to determine parameters of membership functions.

For the purpose of the present study we have selected inventories reflecting a possibly wide range of distributions arising in this kind of application. Specifically, we analyse the following national emission inventories: carbon dioxide (CO_2) in 1990, methane (CH_4) in 2004, and fluorine (F) in 1995 for the Netherlands; and the total greenhouse gas (GHG) emissions in 2005 for Austria.

In order to facilitate estimation of membership function parameters, each histogram of the inventory distribution was appropriately scaled so that the value of the mode equals 1, and shifted to get the modal value at 0 Tg; details of which are specified in Table 1.

Table 1: Shift parameter [Tg CO_2eq].

CO_2	CH_4	F	GHG
158.974	17.326	8.372	93.257

Each histogram is then fitted to the membership functions presented in Section 3. Tables 2 - 6 present estimated parameter values, along with the minimized criterion, i.e. (15) for the LR functions and (23) for the Gumbel membership function, denoted here by Q. For the Gauss function, we fixed parameter $a_G = 1$ for simplicity. Results are also displayed in the upper and two middle rows of Figures 5 - 8.

Table 2: Parameters estimated for the power membership function.

	a_P	d_P^l	γ^l	γ^u	d_P^u	Q
CO_2	1.167	3.42	1.74	1.68	3.73	0.0037
CH_4	1.172	3.30	1.38	3.12	5.74	0.0052
F	1.065	1.60	0.87	1.16	3.00	0.0003
GHG	1.127	8.08	1.06	4.92	33.91	0.0328

Table 3: Parameters estimated for the cubic-power membership function.

	d_C^l	β_0^l	β_1^l	a_C
CO_2	3.401	3.1E-7	1.423	1.129
CH_4	3.481	1.9E-6	1.168	1.159
F	1.976	6.3E-9	0.761	1.052
GHG	9.486	1.2E-7	0.982	1.131
	β_0^u	β_1^u	d_C^u	Q
CO_2	4.1E-6	1.309	3.697	0.0019
CH_4	5.3E-8	2.607	4.729	0.0041
F	0.0049	1.064	3.431	0.0003
GHG	1.3E-8	3.791	23.77	0.0383

Table 4: Parameters estimated for the polynomial membership function.

	d_M^l	a_4^l	a_3^l	a_2^l	a_0
CO_2	-3.69	-0.014	-0.147	-0.414	0.967
CH_4	-3.55	-0.011	-0.119	-0.361	0.977
F	-1.89	-0.018	-0.355	-0.879	0.976
GHG	-10.06	-4.8E-5	-0.002	-0.029	0.811
	a_4^u	a_3^u	a_2^u	d_M^u	Q
CO_2	-0.008	0.096	-0.310	4.74	0.0042
CH_4	-0.007	0.086	-0.290	4.06	0.0148
F	-0.024	0.186	-0.451	3.04	0.0003
GHG	-8.6E-6	5.5E-4	-0.009	17.48	0.3197

Table 5: Parameters estimated for the Gauss membership function.

	a^l	a^u	Q
CO_2	2.693	3.349	0.00016
CH_4	3.283	3.255	0.00428
F	1.113	2.940	0.00100
GHG	25.55	48.17	0.05841

Table 6: Parameters estimated for the Gumbel membership function.

	a_{Gumb}	β	Q
CO_2	3.16	1.165	0.0291
CH_4	3.33	1.264	0.0670
F	2.49	0.929	0.0013
GHG	10.96	4.153	0.0498

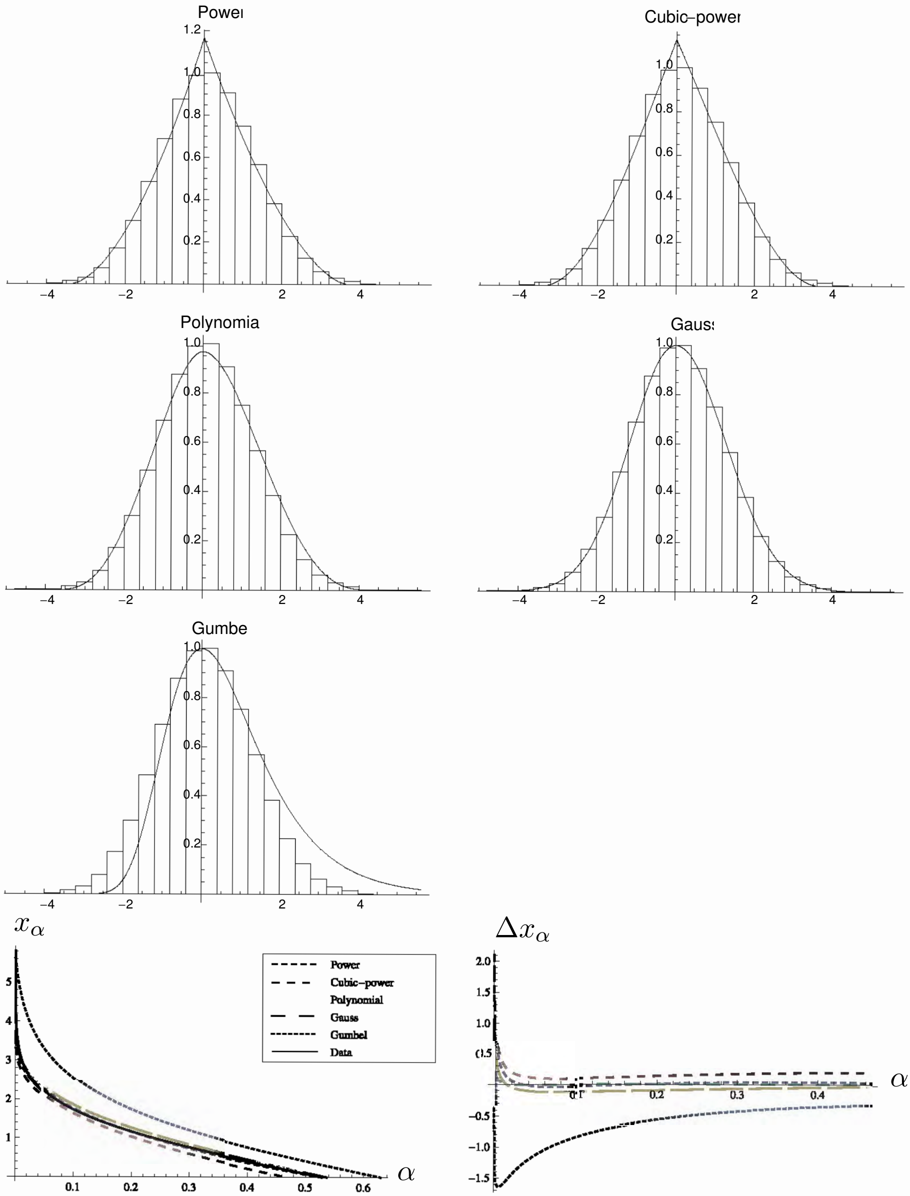

Figure 5: Membership functions fitted to histogram of CO_2 inventory (upper and two middle rows); dependence between x_α and α (bottom left); and a difference between the reference x_α and that of approximating functions (bottom right).

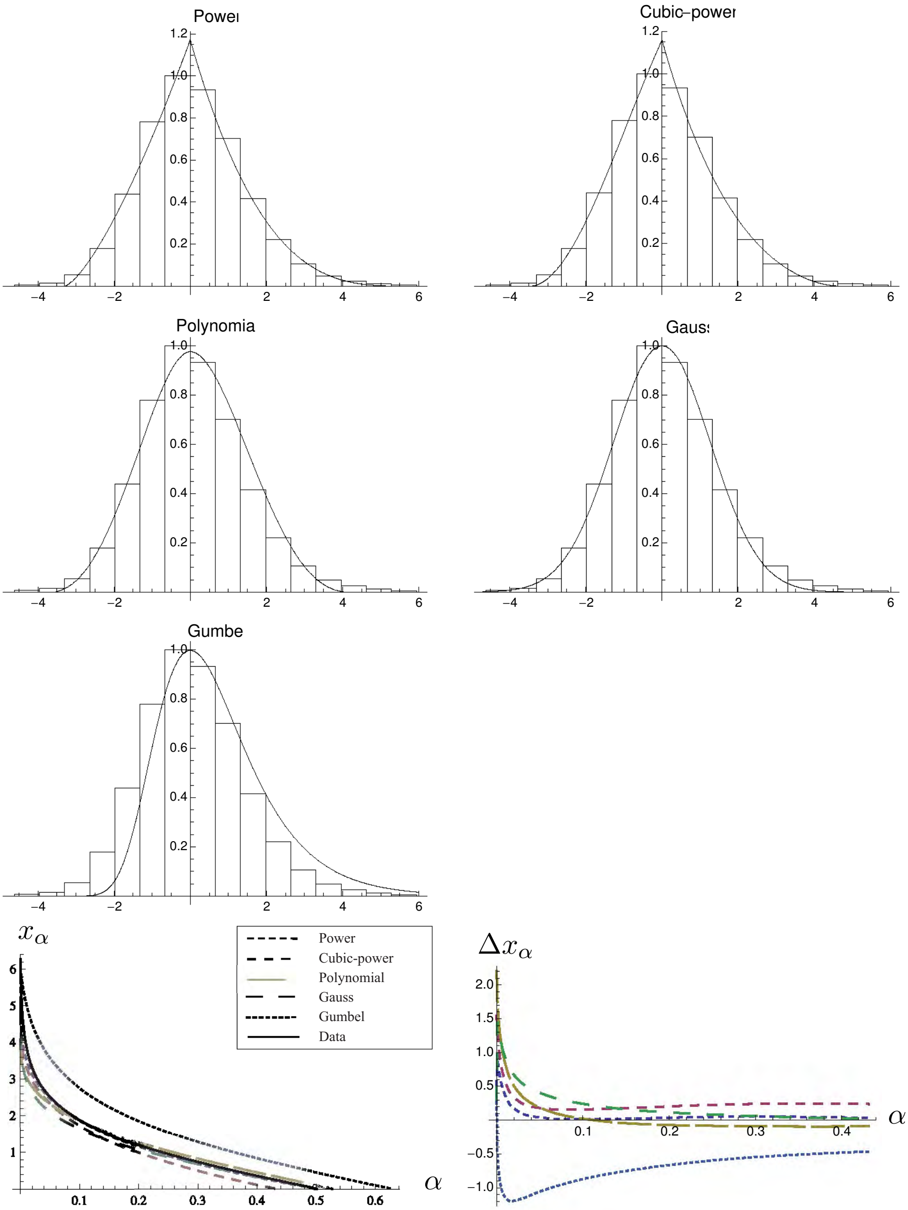

Figure 6: Membership functions fitted to histogram of CH_4 inventory (upper and two middle rows); dependence between x_α and α (bottom left); and a difference between the reference x_α and that of approximating functions (bottom right).

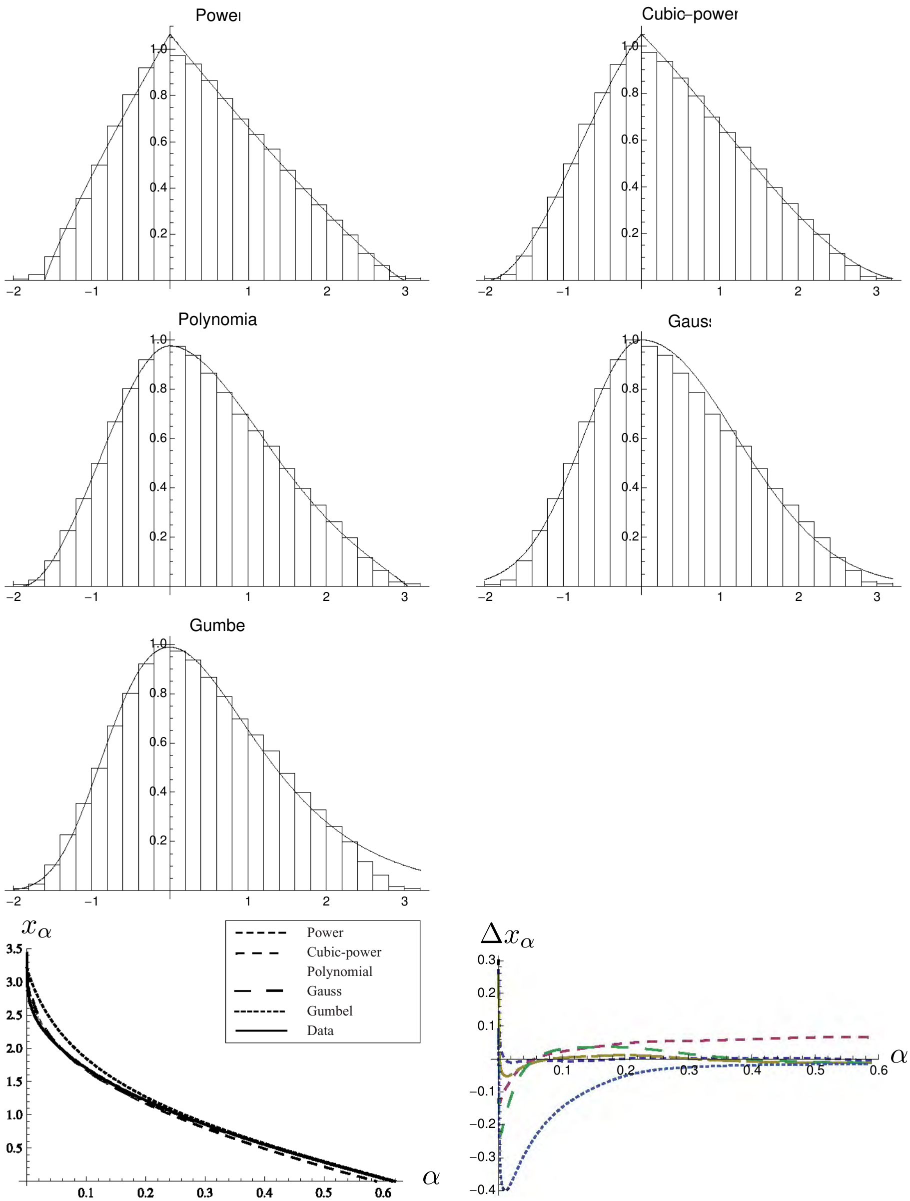

Figure 7: Membership functions fitted to histogram of F inventory (upper and two middle rows); dependence between x_α and α (bottom left); and a difference between the reference x_α and that of approximating functions (bottom right).

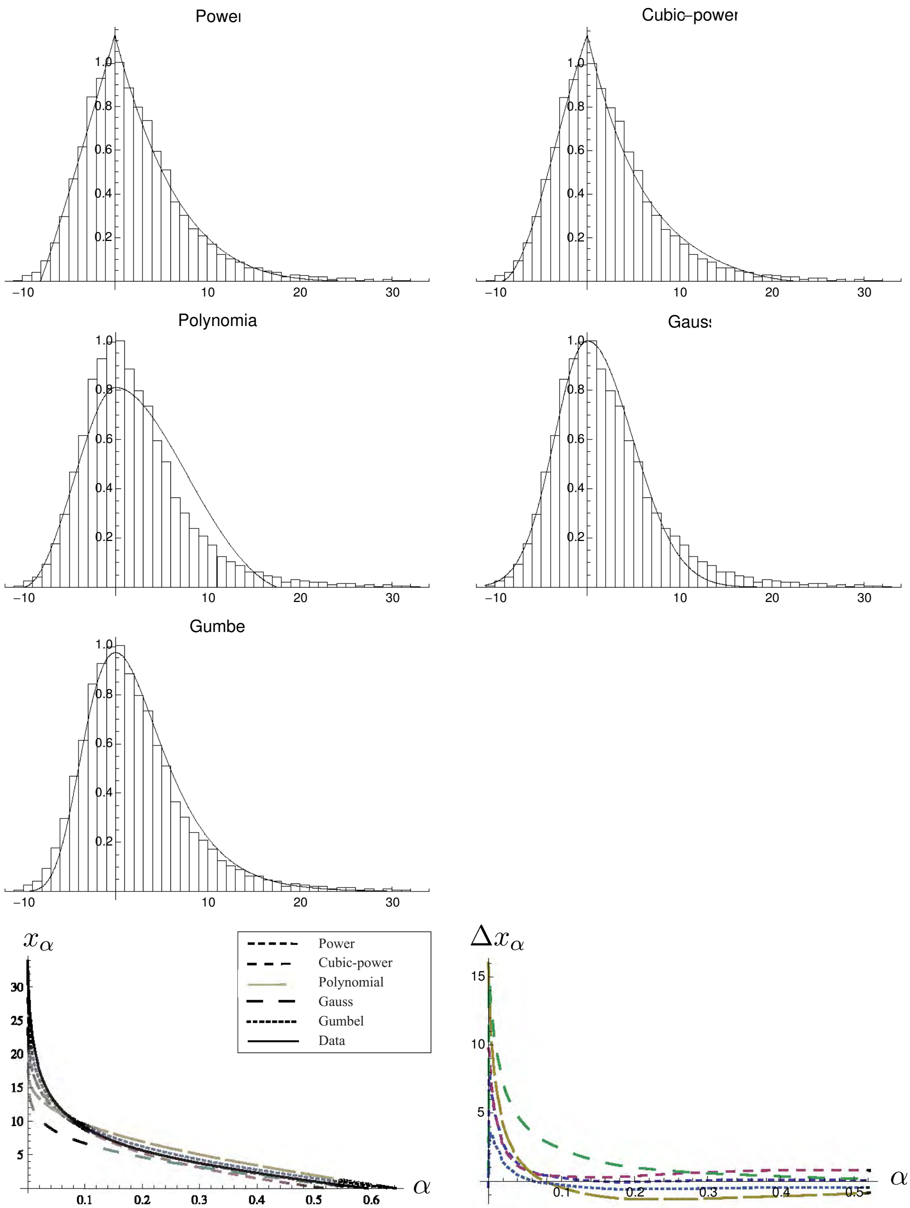

Figure 8: Membership functions fitted to histogram of GHG inventory (upper and two middle rows); dependence between x_α and α (bottom left); and a difference between the reference x_α and that of approximating functions (bottom right).

Quality of the overall fit of the approximating functions to the distributions is summarized with Q. For CO_2 inventory the lowest $Q = 0.00016$ is obtained for the Gauss function, and it is much lower than results of the remaining functions, except the cubic-power function, where it is only slightly higher. For CH_4 inventory, the best fit is reported for the cubic-power ($Q = 0.0041$), the Gauss ($Q = 0.0043$), and the power ($Q = 0.0052$) membership functions. Histogram of F inventory, having a triangular shape, is approximated very well with all the membership functions considered. Finally, a strongly asymmetric histogram of GHG inventory is best approximated with the power ($Q = 0.0328$) and the cubic-power ($Q = 0.0383$) functions.

We should, however, keep in mind that the most important issue in this model is to properly determine x_α given the risk α. This relationship for each inventory is illustrated in the bottom row plots of Figures 5 - 8. The reference value of x_α, denoted Data in the plots, is the one calculated based on the histogram. To facilitate comparison, we present also plots with a difference between the reference x_α and that of respective approximating functions; this difference is denoted Δx_α in the figures.

5 Conclusions

A good estimate of x_α depends on both quality of the overall fit of an approximant to an experimental distribution and the fit to its right tail. There exist big discrepancies in these fits among the examined cases. They are summarized in Table 7, where each approximating function is assessed for all the inventories within the following scale: very good (VG), good (G), fair (F), and bad (B). We take into account the overall fit (OF), fit of the tail (TF), and accuracy of x_α estimation. There is a good correlation of the latter mentioned accuracy with the former mentioned fits.

The complicated functions of the group (i) usually give good fit, although the polynomial function failed in the GHG case, and the cubic-power function provided comparatively worse x_α values for bigger αs than other functions. Anyway, parameter estimation is the most troublesome for all the groups. To achieve a proper solution, additional conditions to assure monotonicity of functions and good starting points are needed. The power function from the group (ii) is very robust: all the cases were assessed at least good. Although other approximations are better in some cases, they got only fair or even bad notes in others. The functions considered for the group (iii) give very good fits for some cases and bad for others.

Table 7: Assessment of accuracies. OF - overall fit, TF - tail fit.

case	fit	power	cubic-power	polynomial	Gauss	Gumbel
	OF	G	VG	G	VG	B
CO_2	TF	G	G	VG	VG	B
	x_α	G	F	VG	VG	B
	OF	G	VG	F	VG	B
CH_4	TF	VG	VG	G	G	B
	x_α	VG	F	G	F	B
	OF	VG	VG	VG	VG	VG
F	TF	VG	G	VG	F	B
	x_α	G	F	VG	F	B
	OF	VG	VG	B	G	VG
GHG	TF	VG	VG	B	B	VG
	x_α	G	F	F	B	VG

The group (i) functions are too complicated to provide closed analytical solution for x_α. The solution have to be found numerically and rounding errors may cause some problems here. The functions from the group (ii) give more chances to derive the analytical solution for x_α, as is examplified by the power function (20). This property, however, is not guarantied, as can be seen for the function proposed in [13]. The group (iii) functions do not cause big problems here, due to well developed procedures to deal with the probabilistic distributions.

Good approximations of x_α, much below 1%, can be obtained for $\alpha = 0.07 - 0.08$ or bigger. For smaller values the accuracy ranges from a few to even 20 - 30%, and for the group (i) functions numerical problems may be spotted. Thus, it is not recommended to use very small values of α, if accuracy is important. However, the authors believe that for most cases the risk $\alpha = 0.1$ is small enough, and the smaller ones will not be needed in practical applications.

To conclude, the group (i) functions are too complicated in practical use. The group (iii) functions are much more convenient in this respect, provided an analytical solution for x_α is available. The group (iii) functions are not flexible enough to be well fitted to all uncertainty distributions. Therefore, the group (ii) functions are recommended for use. If properly chosen, they can provide a good and robust fit. Moreover, they are very convenient in application whenever an analytical solution for x_α exists.

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The papers presented in this Volume 2 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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