Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics Volume II: Applications

Editors

Krassimir T. Atanassov
Michał Baczyński
Józef Drewniak
Janusz Kacprzyk
Maciej Krawczak
Eulalia Szmidt
Maciej Wygralak
Sławomir Zadrożny

Editors

Krassimir T. Atanassov
Władysław Homenda
Olgierd Hryniewicz
Janusz Kacprzyk
Maciej Krawczak
Zbigniew Nahorski
Eulalia Szmidt
Sławomir Zadrożny

SRI PAS IBS PAN

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics Volume II: Applications

Systems Research Institute
Polish Academy of Sciences

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
Volume II: Applications

Editors

Krassimir T. Atanassov
Władysław Homenda
Olgierd Hryniewicz
Janusz Kacprzyk
Maciej Krawczak
Zbigniew Nahorski
Eulalia Szmidt
Sławomir Zadrożny

IBS PAN **SRI PAS**

Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
www.ibspan.waw.pl

ISBN 9788389475305

Fuzzy approach to evaluation of portfolio of financial and insurance instruments

Piotr Nowak and Maciej Romaniuk
Systems Research Institute PAS,
ul. Newelska 6, 01–447 Warszawa, Poland
pnowak@ibspan.waw.pl, mroman@ibspan.waw.pl

Abstract

In this paper we evaluate the model of portfolio which consists of a few layers of insurance and financial instruments. The approach based on neutral martingale method and Monte Carlo simulations is used. In order to price the catastrophe bond we use fuzzy parameters and apply Vasicek model under assumption of independence between catastrophe occurrence and behaviour of financial market. Then the simulations based on the obtained fuzzy pricing formula are carried out. The presented fuzzy sets approach may incorporate expertise knowledge to overcome lack of precise data in the discussed case.

Keywords: catastrophe bonds, insurance portfolio, Vasicek model, neutral martingale method, Monte Carlo simulations, fuzzy sets.

1 Introduction

The insurance industry face overwhelming risks caused by natural catastrophes. Losses from single catastrophic event could reach 30 billion $ (Hurricane Andrew, 1992) or even 40 – 60 billion $ (Hurricane Katrina, 2005, see [16]). To cope with dramatic consequences of such extreme events integrated policy based on applying a whole set of financial and insurance instruments is required.

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume II: Applications *(K.T. Atanassow, W. Homenda, O. Hryniewicz, J. Kacprzyk, M. Krawczak, Z. Nahorski E. Szmidt, S. Zadrożny, Eds.), IBS PAN - SRI PAS, Warsaw, 2009.*

The classical insurance mechanisms are not prepared to deal with extreme losses caused by natural catastrophes. Even one, single catastrophe could cause problems with reserves for many insurers or even bankruptcy of these enterprises. For example, after Hurricane Andrew more than 60 insurance companies became insolvent (see [16]). The traditional insurance models (see [2]) deal with independent, rather small risks like car accidents. In such case there is a simple strategy of selecting an insurance contract portfolio: the greater the number of risks, the better (see [2, 6]). But catastrophic risks require new strategies for an insurance company. The sources of losses from natural catastrophes are strongly dependent in terms of time and localization, e.g. single hurricane could start fire in many houses. Therefore the traditional strategy of portfolio construction can only increase the probability of bankruptcy of insurer (see [6]). Additionally, classical insurance mechanisms are often criticized because of serious problems with adverse selection and moral hazard. Also the primary insurers rely on classical reinsurance markets which are affected by pricing cycles.

As it was mentioned, the single catastrophic event could result in damages measured in billions of dollars. Becuase daily fluctuations on worldwide financial markets reach also tens of billion \$, securization of losses may be helpful for dealing with results of extreme natural catastrophes (see e.g. [4, 9, 10, 12]). One of possible types of ”packaging” the losses is known as catastrophe bonds (see e.g. [8, 19, 23]).

In this paper we evaluate the model of portfolio which consists of a few layers of insurance and financial instruments and may be constructed by the insurer or other entity like government. We use approach based on neutral martingale method, Monte Carlo simulations and fuzzy sets. We price the catastrophe bond applying Vasicek model used for zero-coupon bound under assumption of independence between catastrophe occurrence and behaviour of financial market. The fuzzy parameters are applied for the obtained pricing formula. Then appropriate simulations are carried out.

The constructed portfolio should be ”fair”, i.e. it should fulfil needs of both insureds and insurers. In order to achieve this goal, we set the limits for the probability of insurer bankruptcy and the probability of overpayment for insureds.

There is a need to take into account possible errors and uncertainties which arise from estimation of rare events with serious, catastrophic consequences like natural catastrophes. Therefore we apply the approach based on fuzzy sets which may also incorporate expertise knowledge to overcome lack of precise, historical data.

This paper is organized as follows. In Section 2 we present some preliminaries for fuzzy sets. In Section 3 we discuss properties of catastrophe bonds, layers of the constructed portfolio, evaluation of such portfolio, consequences of fuzzy approach for cat bond pricing and obtained results of Monte Carlo simulations. We conclude the paper in Section 4.

2 Preliminaries

In this section we recall some basic facts about fuzzy sets and numbers.

Let X be a universal set and $\tilde{A}$ be a fuzzy subset of X. We denote by $\mu_{\tilde{A}}$ its membership function $\mu_{\tilde{A}} : X \to [0,1]$, and by $\tilde{A}_\alpha = \{x : \mu_{\tilde{A}} \geq \alpha\}$ the α-level set of $\tilde{A}$, where $\tilde{A}_0$ is the closure of the set $\{x : \mu_{\tilde{A}} \neq 0\}$.

In our paper we assume that $X = \mathbb{R}$.

Let $\tilde{a}$ be a fuzzy number. Then the α-level set $\tilde{a}_\alpha$ is a closed interval, which can be denoted by $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ (see e.g. [29]).

We can now introduce the arithmetic of any two fuzzy numbers. Let $\odot$ be a binary operator $\oplus$, $\ominus$, $\otimes$ or $\oslash$ between fuzzy numbers $\tilde{a}$ and $\tilde{b}$, where the binary operators correspond to $\circ$: $+$, $-$, $\times$ or $/$, according to the "Extension Principle" in [29]. Then the membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\mu_{\tilde{a}\odot\tilde{b}}(z) = \sup_{(x,y):x\circ y=z} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\} \ . \tag{1}$$

Let $\odot_{int}$ be a binary operator $\oplus_{int}$, $\ominus_{int}$, $\otimes_{int}$ or $\oslash_{int}$ between two closed intervals $[a,b]$ and $[c,d]$. Then

$$[a,b] \odot_{int} [c,d] = \{z \in \mathbb{R} : z = x \circ y, \forall x \in [a,b], \forall y \in [c,d]\} \ , \tag{2}$$

where $\circ$ is an usual operation $+, -, \times$ and $/$, if the interval $[c,d]$ does not contain zero in the last case.

Therefore, if $\tilde{a}, \tilde{b}$ are fuzzy numbers, then $\tilde{a} \odot \tilde{b}$ is also the fuzzy number and its α-level set is given by

$$(\tilde{a} \oplus \tilde{b})_\alpha = \tilde{a}_\alpha \oplus_{int} \tilde{b}_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha] \ ,$$

$$(\tilde{a} \ominus \tilde{b})_\alpha = \tilde{a}_\alpha \ominus_{int} \tilde{b}_\alpha = [\tilde{a}^L_\alpha - \tilde{b}^U_\alpha, \tilde{a}^U_\alpha - \tilde{b}^L_\alpha] \ ,$$

$$(\tilde{a} \otimes \tilde{b})_\alpha = \tilde{a}_\alpha \otimes_{int} \tilde{b}_\alpha =$$

$$= [\min\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\}, \max\{\tilde{a}^L_\alpha\tilde{b}^L_\alpha, \tilde{a}^L_\alpha\tilde{b}^U_\alpha, \tilde{a}^U_\alpha\tilde{b}^L_\alpha, \tilde{a}^U_\alpha\tilde{b}^U_\alpha\}] \ ,$$

$$(\tilde{a} \oslash \tilde{b})_\alpha = \tilde{a}_\alpha \oslash_{int} \tilde{b}_\alpha =$$

$$= [\min\{\tilde{a}^L_\alpha/\tilde{b}^L_\alpha, \tilde{a}^L_\alpha/\tilde{b}^U_\alpha, \tilde{a}^U_\alpha/\tilde{b}^L_\alpha, \tilde{a}^U_\alpha/\tilde{b}^U_\alpha\}, \max\{\tilde{a}^L_\alpha/\tilde{b}^L_\alpha, \tilde{a}^L_\alpha/\tilde{b}^U_\alpha, \tilde{a}^U_\alpha/\tilde{b}^L_\alpha, \tilde{a}^U_\alpha/\tilde{b}^U_\alpha\}] \ ,$$

if α-level set $\tilde{b}_\alpha$ does not contain zero for all $\alpha \in [0,1]$ in the case of $\oslash$.

Triangular fuzzy number $\tilde{a}$ with membership function $\mu_{\tilde{a}}(x)$ is defined as

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{for} \quad a_1 \le x \le a_2 \\ \frac{x-a_3}{a_2-a_3} & \text{for} \quad a_2 \le x \le a_3 \\ 0 & \text{otherwise} \end{cases} , \tag{3}$$

where $[a_1, a_3]$ is the supporting interval and the membership function has peak in a_2. Triangular fuzzy number $\tilde{a}$ is denoted as

$$\tilde{a} = (a_1, a_2, a_3) \ .$$

Triangular fuzzy numbers are special case of Left-Right (or L-R) fuzzy numbers (e.g. see [1, 5]), where linear functions used in the definition are replaced by monotonic functions, i.e.

Definition 1. *A fuzzy set $\tilde{A}$ on the set of real numbers is called L-R number if the membership function may be calculated as*

$$\mu_{\tilde{a}}(x) = \begin{cases} L\left(\frac{a_2-x}{a_2-a_1}\right) & \textit{for} \quad a_1 \le x \le a_2 \\ R\left(\frac{x-a_2}{a_3-a_2}\right) & \textit{for} \quad a_2 \le x \le a_3 \\ 0 & \textit{otherwise} \end{cases} , \tag{4}$$

where L and R are continuous strictly decreasing function defined on $[0,1]$ with values in $[0,1]$ satisfying the conditions

$$L(x) = R(x) = 1 \ \textit{if} \ x = 0 \ , L(x) = R(x) = 0 \ \ if \ x = 1 \ .$$

The L-R fuzzy number $\tilde{a}$ is denoted as

$$\tilde{a} = (a_1, a_2, a_3)_{LR} \ .$$

Next we turn to fuzzy estimation based on statistical approach (see [3]). This approach may be seen as a way to obtain L-R numbers based on statistical data.

Let X be a random variable with probability density function $f_\theta(.)$. Assume that parameter θ is unknown and are to be estimated from a sample $X_1, X_2, \ldots X_n$. Let $\hat{\theta}$ be a statistics based on $X_1, X_2, \ldots X_n$ which is used for such estimation. Then for the given confidence level $0 \le \beta \le 1$ we have the $\beta \cdot 100\%$ confidence interval $[\theta_L(\beta), \theta_R(\beta)]$ for θ which is established by the condition

$$\mathrm{P}_{f_\theta}(\theta_L(\beta) \le \theta \le \theta_R(\beta)) = \beta \ . \tag{5}$$

If we suppose that $[\theta_L(0), \theta_R(0)] = [\hat{\theta}, \hat{\theta}]$ then we could construct fuzzy estimator $\hat{\tilde{\theta}}$ of $\tilde{\theta}$. We place the confidence intervals, one on top of the other, to produce a triangular shaped fuzzy $\tilde{\theta}$ whose α-cuts are the confidence intervals on $\beta = (1-\alpha)$ confidence levels. Therefore we have

$$\hat{\tilde{\theta}}_\alpha = [\theta_L(1-\alpha), \theta_R(1-\alpha)] \tag{6}$$

for e.g. $0.01 \leq \alpha \leq 1$. In order to "finish" construction of the fuzzy estimator, we suppose that

$$\hat{\tilde{\theta}}_\alpha = [\theta_L(0.99), \theta_R(0.99)] \tag{7}$$

for $0 \leq \alpha < 0.01$. It means that we drop the graph of $\hat{\tilde{\theta}}$ straight down to complete its α-cuts (see [3] for additional details).

3 Portfolio construction

3.1 Catastrophe bonds

Because of problems with coverage of losses by insurance enterprises, dependencies among sources of risks, potentially unlimited losses, problems with adverse selection, moral hazard and reinsurance pricing cycles applying alternative financial or insurance instruments may be profitable. The problem is to "package" natural disasters risk and appropriate losses into classical forms of tradable financial assets, like bonds or options. The most popular catastrophe-linked security is the catastrophe bond, known also as *cat bond* or *Act-of-God* bond (see [8, 11, 19, 23]). Cat bonds were introduced in 1992, and become wider known in April 1997, when USAA, an insurer from Texas, initiated two new classes of cat bonds: A-1 and A-2. There is one important difference between cat bonds and other financial instruments – the premiums from cat bond are always connected with additional random variable, i.e. occurrence of some natural catastrophe in specified region and fixed time interval. Such event is called *triggering point* (see [11]). For example, the A-1 USAA bond was connected with hurricane on the east coast of USA between July 15, 1997 and December 31, 1997. If there had been a hurricane in mentioned above region with more than $1 billion loses against USAA, the coupon of the bond would have been lost. As usually, the structure of payments for cat bonds depends also on some primary underlying asset. In case of A-1 USAA bond, the payment equalled LIBOR plus 282 basis points. As we can see from this example, the triggering point changes the structure of payments for the cat bond. The triggering points may be related to various kinds of random events — e.g. to magnitude of earthquake, the losses from flood, etc. (see e.g.

[17, 24]). The main aim of cat bonds is to transfer *risk* from insurance markets or governmental budgets to financial markets.

3.2 Layers of portfolio

We construct the portfolio which has a few layers of financial and insurance instruments. This model or other similar portfolio may be used by insurer or other organization (like government) in order to evaluate parameters and possible scenarios, e.g. to calculate the probability of ruin, the value of maximum losses for given probability level, the necessity of using the considered instrument in the evaluated portfolio, etc.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We consider trading horizon $[0, T']$, $T' > 0$. We fix some $T \in [0, T']$. The structure of catastrophe model is based on subdividing the considered region into m nodes (or cells). Applying this structure, the scenario of catastrophe and losses arising from this scenario may be simulated (see e.g. [7, 8, 21]).

By $C(j,t)$ we denote claims caused by losses for cell j on the time interval $[0,t]$ and by $C(t)$ the aggregated demands for all cells on the time interval $[0,t]$, i.e. $C(t) = \sum_{i=1}^{m} C(j,t)$. These claims depend on losses $L(j,t)$. We apply the proportional contract with lower bound given by formula

$$C(j,t) = q_j L(j,t) I\left(L(j,t) > k_0\right) , \tag{8}$$

where q_j is the proportion parameter, $I(.)$ is the characteristic function and k_0 is the lower bound. By $L(t)$ we denote the aggregated losses, i.e. $L(t) = \sum_{i=1}^{m} L(j,t)$, by $\Pi(j,t)$ – the insurance premiums for the cell j on the time interval $[0,t]$, and by $\Pi(t)$ – the aggregated premiums for all cells, i.e. $\Pi(t) = \sum_{i=1}^{m} \Pi(j,t)$. Then the classical insurance model (see [2]) describing the evolution of the profit process for the insurer has the form

$$R(t) = R(0) + \Pi(t) - C(t) , \tag{9}$$

where $R(0)$ is the initial capital. In (9) we assume that the risk-free yield is equal to zero.

The main goal for insurer is the maximization of the profit given by random variable

$$\Theta = R(0) + \Pi(T) - C(T) \tag{10}$$

under the "fairness" constraints, i.e. we limit the probability of insurer bankruptcy by value p_1

$$\mathrm{P}\left(R(T) \leq 0\right) \leq p_1 \tag{11}$$

and the the probability of overpayment for insureds by p_2 (see [19, 26])

$$\mathrm{P}\left(c(j,T) \leq \Pi(j,T)\right) \leq p_2 \,. \tag{12}$$

Because for natural catastrophes standard insurance mechanism may not be adequate, we add some extra layers in order to create the whole portfolio of various insurance and financial instruments. The second layer in our portfolio is the catastrophe bond issued by government, insurer or other enterprise. If insurer issues such bond, then at the maturity time of the bond T the process (10) has the form

$$\Theta = R(0) + \mathrm{PV}\left(\Pi(T)\right) - \mathrm{PV}\left(C(T)\right) + \Pi^{cb} - \mathrm{PV}\left[f^{cb}\left(C(T)\right)\right] \,, \tag{13}$$

where PV denotes the present value of cash flow, Π^{cb} – the aggregated premiums from cat bond issuing, $f^{cb}(.)$ – the payment function for the considered kind of cat bond. In our case the triggering point is connected with surpassing the limit k_1 by aggregated losses $L(T)$.

The next layers are connected with some form of external help. We assume that the probability space $(\Omega', \mathcal{F}', P')$ describes this help.

If the aggregated losses are above some level k_2, i.e. $L(T) > k_2$, then there is possibility that special governmental fund may be used and this probability is modelled as independent random event A_2. We may assume that the value of such fund is modelled by independent random variable X_2 or this value is proportional to the aggregated losses, i.e. $\varphi_2 L(T)$, where φ_2 is the proportion parameter. The next layer is the foreign help. Such help may be used if the aggregated losses is above level k_3, i.e. $L(T) > k_3$ and the probability of using this fund may be modelled as independent random event A_3. As previously, the value of such fund may be denoted by independent random variable X_3 or it may be proportional to the aggregated losses, i.e. $\varphi_3 L(T)$.

Taking into account all the layers of the portfolio, the process (10) has now the form

$$\begin{aligned}\Theta = R(0) + \mathrm{PV}\left(\Pi(T)\right) - \mathrm{PV}\left(C(T)\right) + \Pi^{cb} - \mathrm{PV}\left[f^{cb}\left(C(T)\right)\right] + \\ + \mathrm{PV}\left(X_2\right) I(A_2) I\left(L(T) > k_2\right) + \mathrm{PV}\left(X_3\right) I(A_3) I\left(L(T) > k_3\right)\end{aligned} \tag{14}$$

or

$$\begin{aligned}\Theta = R(0) + \mathrm{PV}\left(\Pi(T)\right) - \mathrm{PV}\left(C(T)\right) + \Pi^{cb} - \mathrm{PV}\left[f^{cb}\left(C(T)\right)\right] + \\ + \mathrm{PV}\left(\varphi_2 L(T)\right) I(A_2) I\left(L(T) > k_2\right) + \mathrm{PV}\left(\varphi_3 L(T)\right) I(A_3) I\left(L(T) > k_3\right)\end{aligned} \tag{15}$$

depending on the way we model values of the external help, under mentioned previously constraints (11) and (12).

For the formula (14) or (15) we need some additional parameters, which may be also calculated, like price of the considered type of catastrophe bond (see Section 3.3). Because problems similar to the above equations are extremely sensitive for the constraints, special maximization procedure may be used (see e.g. [6]).

3.3 Cat bond pricing

Let $(W_t)_{t\in[0,T']}$ be a Brownian motion. Let $\tau : \Omega \to [0, T']$ be a stopping time defined by the formula

$$\tau(\omega) = \inf_{t\in[0,T']} \{C(t)(\omega) > 0\},$$

where $C(t)$ denotes aggregated demands introduced in previous section.

We assume that $(W_t)_{t\in[0,T']}$ and τ are independent. We denote by $(F_t)_{t\in[0,T']}$ the filtration defined by formulas

$$F_t = \sigma\left(F_t^0 \cup F_t^1 \cup N\right), F_t^0 = \sigma(W_s, s \le t),$$
$$F_t^0 = \sigma(\{\tau \le u\}, u \le t), t \in \left[0, T'\right],$$

where

$$N = \{A \in F : P(A) = 0\}.$$

We assume that $F = \sigma\left(\bigcup_{t\in[0,T']} F_t\right)$. Then the filtered probability space $\left(\Omega, F, (F_t)_{t\in[0,T']}, P\right)$ satisfies the standard assumptions, i.e. σ-algebra F is P-complete, the filtration $(F_t)_{t\in[0,T']}$ is right continuous, what means that for each $t \in [0, T')$

$$F_{t+} = \bigcap_{s>t} F_s = F_t$$

and F_0 contains all the sets in F of P-probability zero.

We denote by $(B_t)_{t\in[0,T']}$ banking account satisfying the following equation:

$$dB_t = r(t)B_t dt\,, B_0 = 1,$$

where r is a risk-free spot interest rate. The solution of the above equation has the form:

$$B_t \exp\left(\int_0^t r(u)du\right), t \in \left[0, T'\right].$$

We denote by $B(t, T)$ the price at the time t zero-coupon bond with maturity date $T \le T'$ and the face value equal to 1.

Definition 2. *We denote by* $IB(\tau, T, \mathrm{Fv}, w)$ *a catastrophe bond satisfying the following assumptions:*

a) *If the catastrophe does not occur in the period* $[0, T]$*, i.e.* $\tau > T$*, the bondholder is paid the face value* Fv;

b) *If* $\tau \leq T$*, the bondholder receives the face value minus a write-down coefficient in percentage* w*.*

c) *A cash payments are done at date of maturity* T*.*

Definition 3. $B(t, T)$*,* $t \leq T \leq T'$ *is called the arbitrage-free family of zero-coupon bond prices with respect to* r*, if the following conditions are satisfied:*

a) $B(T, T) = 1$ *for each* $T \in [0, T']$ *.*

b) *There exists a probability* Q*, equivalent to* P*, such that for each* $T \in [0, T']$ *the process of discounted zero-coupon bond price*

$$B(t, T) / B_t \,, \, t \in [0, T] \,,$$

is a martingale with respect to Q*. Then we have the following pricing formula*

$$B(t, T) = E^Q \left(e^{-\int_t^T r(u) du} | F_t^Q \right) \,, t \in [0, T] \,.$$

Let $\lambda_u = -\lambda$ denote the risk premium for risk-free bonds. The following Radon-Nikodym derivative defines a probability measure Q, equivalent to P:

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda_u dW_u - \frac{1}{2} \int_0^T \lambda_u^2 du \right) P\text{-a.s.}, \tag{16}$$

such that $B(t, T) / B_t$, $t \in [0, T]$, is a martingale with respect to Q.

We assume the Vasicek model of the risk-free spot interest rate r. The interest rate satisfies the following equation

$$dr(t) = a(b - r(t)) dt + \sigma dW_t \tag{17}$$

for positive constants a, b and σ.

We assume that financial market is independent of the catastrophe risk and investors are neutral toward nature jump risk.

We apply the methodology from [27] to price the catastrophe bond.

Proposition 1. *Let $IB\,(0)$ be the price of a $IB\,(\tau, T, \mathrm{Fv}, w)$ at time 0. Let Φ denote cumulative distribution function of τ. Then*

$$IB\,(0) = \mathrm{Fv}\, e^{-TR(T,r(0))} \left\{1 - w\Phi\,(T)\right\}, \tag{18}$$

where

$$R\,(\theta, r) = R_{\infty} - \frac{1}{a\theta}\left\{(R_{\infty} - r)\left(1 - e^{-a\theta}\right) - \frac{\sigma^2}{4a^2}\left(1 - e^{-a\theta}\right)^2\right\}$$

and

$$R_{\infty} = b - \frac{\lambda\sigma}{a} - \frac{\sigma^2}{2a^2}.$$

From now we assume that the parameters of the spot interest rate are not precisely known. To model this uncertainty we introduce fuzzy numbers $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$ and $\tilde{r}_0$ in place of a, b, σ and $r\,(0)$. We also treat the market price of risk as a small fuzzy number. Therefore we replace the parameter λ by its negative fuzzy counterpart $\tilde{\lambda}$.

Let $\mathcal{F}\,(R)$ the set of all fuzzy numbers. It is proved in [28] that each function $f : R \to R$, for which the inverse image of any value is compact, induces a fuzzy-valued function $\tilde{f} : \mathcal{F}\,(R) \to \mathcal{F}\,(R)$ via the extension principle and the α-level set of $\tilde{f}(\tilde{\Lambda})$ is $\tilde{f}(\tilde{\Lambda})_{\alpha} = \{f(x) : x \in \tilde{\Lambda}_{\alpha}\}$.

From this fact we obtain the following fuzzy version of the pricing formula.

Proposition 2.

$$\tilde{IB}\,(0) = \mathrm{Fv} \otimes e^{-T\otimes\tilde{R}(T)} \otimes \left\{1 - w\Phi\,(T)\right\}, \tag{19}$$

where

$$\begin{aligned}\tilde{R}\,(T) = \tilde{R}_{\infty} \ominus \Big\{\left(\tilde{R}_{\infty} \ominus \tilde{r}_0\right) \otimes \left(1 \ominus e^{-\tilde{a}\otimes T}\right) \ominus \tilde{\sigma}\otimes\tilde{\sigma} \otimes \left(1 \ominus e^{-\tilde{a}\otimes T}\right) \\ \otimes \left(1 \ominus e^{-\tilde{a}\otimes T}\right) \oslash (4 \otimes \tilde{a}\otimes\tilde{a})\Big\} \oslash (\tilde{a} \otimes T)\end{aligned} \tag{20}$$

and

$$\tilde{R}_{\infty} = \tilde{b} \ominus \tilde{\lambda}\otimes\tilde{\sigma} \oslash \tilde{a} \ominus \tilde{\sigma}\otimes\tilde{\sigma} \oslash (2\otimes\tilde{a}\otimes\tilde{a})\,. \tag{21}$$

To calculate the α-level sets of $\tilde{IB}\,(0)$ we use formulas similar to (19), (20) and (21), replacing the operators $\oplus$, $\ominus$, $\otimes$, $\oslash$ by $\oplus_{int}$, $\ominus_{int}$, $\otimes_{int}$, $\oslash_{int}$.

3.4 Simulations

In order to analyse the features of the portfolio proposed in Section 3.2, the appropriate simulations were conducted. We assume that quantity of losses is modelled by Poisson process with expected value $\mu = 0.05$ and the value of each loss is given by random variable from Gamma distribution with scale parameter $\alpha = 10$ and shape parameter $\beta = 10$. Therefore the generated losses have catastrophic nature, i.e. they are rare, but with high value. Other types of distributions for modelling the value of losses are also possible, e.g. Weibull distribution.

The trading horizon is set on 5 years and constant continuous risk-free yield r is equal to 0.05. For each portfolio we generate $n = 100000$ simulations.

Only insurance contract is taken into account in the *Portfolio I*. It is assumed that insurance premium is equal to 0.02, it is paid by 100 insureds and the insurance contract is proportional with lower bound. For simplicity the lower bond is set for the whole portfolio of insurance contacts to $k_0 = 5$ and proportion parameter is set to $q = 0.95$. For initial capital $R(0) = 45$ the probability of insurer bankruptcy is equal to 2.318%.

Then the price of cat bond is calculated. According to Section 3.3, the Vasicek model with fuzzy parameters is assumed. We use α-sets which may be derived e.g. from triangular fuzzy numbers or L-R numbers (see Section 2).

In the first discussed case, the α-level sets for model parameters are given by

$$a_\alpha = [0.025, 0.03]\ ,\ b_\alpha = [0.05, 0.075]\ ,\ \lambda_\alpha = [-0.02, -0.01]\ ,\\ \sigma_\alpha = [0.01, 0.02]\ ,\ r_\alpha(0) = [0.05, 0.06]\ , \quad (22)$$

the face value of the bond is set to 1 and triggering point is connected with surpassing the limit $k_1 = 40$ by aggregated losses. If the triggering point occurs, then the bond holder receives only 50% of face value, i.e. $w = 0.5$. The price of this cat bond is calculated according to formula (19) with operators replaced by $\oplus_{int}$, $\ominus_{int}$, $\otimes_{int}$, $\oslash_{int}$ via Monte Carlo simulations (see e.g. [20] for similar approach). We obtain $\widetilde{IB}_\alpha = [0.734072, 0.777102]$.

All the instruments discussed in Section 3.2 are taken into account in *Portfolio II*. It is assumed that $R(0) = 45$, the features of cat bond are the same as mentioned above and the external help is proportional to the losses with parameters $k_2 = 40, k_3 = 60, \varphi_2 = 0.01, \varphi_3 = 0.01$. The occurrence of governmental help is independent event with probability 0.05 and the occurrence of foreign help is independent event with probability 0.02. Because the cat bond price is given as α-level set, we may calculate intervals of measures for our portfolio based on simulations (see Table 1). As we could see, for this portfolio the probability of ruin is substantially lower with relatively low average value of external help. However, even the small difference in cat bond price causes rather huge differences in average value of portfolio, its median and maximum loss for probability 5%. But the quantiles and the probability of ruin have rather stable values. For higher cat bond prices the mean of money flows connected with cat bonds is more positive and portfolio mean has higher value. This last value becomes similar to initial capital.

Table 1: Numerical features of Portfolio II

	Value
Average value of portfolio	[41.7529, 45.9854]
Median of value of portfolio	[42.5271 , 46.8301]
Standard deviation of value of portfolio	[6.23276, 6.53629]
5% quantile of value of portfolio	[42.5271, 46.8301]
95% quantile of value of portfolio	[42.5271 , 46.8301]
1% quantile of value of portfolio	[8.1041 , 9.87955]
99% quantile of value of portfolio	[42.5271 , 46.8301]
Max. loss for portfolio with prob. 5%	[2.47288 , -1.8301]
Max. loss for portfolio with prob. 1%	[36.8959 , 35.1204]
Probability of ruin (%)	[0.68 , 0.63]
Average value of flows for cat bond	[-3.60062 , 0.725716]
Average value of flows for external help	[0.000842729 , 0.00131172]

For the *Portfolio III*, the cat bond has the greater impact, because we set $k_3 = 100$. Therefore, the foreign help is more limited. The appropriate intervals of measures for such portfolio are given in Table 2. As we could see, the values are similar to the previous case, but the probability of ruin is slightly higher.

Then the α-level sets of cat bond parameters for lower α are used which give the wider intervals

$$a_\alpha = [0.02, 0.035] \, , \, b_\alpha = [0.04, 0.08] \, , \, \lambda_\alpha = [-0.03, -0.005] \, , \\ \sigma_\alpha = [0.005, 0.03] \, , \, r_\alpha(0) = [0.05, 0.06] \, . \quad (23)$$

In this case we have $\tilde{IB}_\alpha = [0.723755, 0.77929]$. Then the intervals of measures for both Portfolio II (see Table 3) and Portfolio III (see Table 4) are recalculated.

Table 2: Numerical features of Portfolio III

	Value
Average value of portfolio	[41.7081, 46.0065]
Median of value of portfolio	[42.5271 , 46.8301]
Standard deviation of value of portfolio	[6.40444, 6.77022]
5% quantile of value of portfolio	[42.5271, 46.8301]
95% quantile of value of portfolio	[42.5271 , 46.8301]
1% quantile of value of portfolio	[7.39254 , 11.307]
99% quantile of value of portfolio	[42.5271 , 46.8301]
Max. loss for portfolio with prob. 5%	[2.47288 , -1.8301]
Max. loss for portfolio with prob. 1%	[37.6075 , 33.693]
Probability of ruin (%)	[0.79 , 0.6]
Average value of flows for cat bond	[-3.56947 , 0.682882]
Average value of flows for external help	[0.000901882 , 0.00128055]

Table 3: Numerical features of Portfolio II A

	Value
Average value of portfolio	[40.5361, 46.1237]
Median of value of portfolio	[41.4954 , 47.0489]
Standard deviation of value of portfolio	[7.29131, 6.94636]
5% quantile of value of portfolio	[41.4954, 47.0489]
95% quantile of value of portfolio	[41.4954 , 47.0489]
1% quantile of value of portfolio	[0.469452 , 7.79863]
99% quantile of value of portfolio	[41.4954 , 47.0489]
Max. loss for portfolio with prob. 5%	[3.50461 , -2.04888]
Max. loss for portfolio with prob. 1%	[44.5305 , 37.2014]
Probability of ruin (%)	[0.99 , 0.68]
Average value of flows for cat bond	[-4.535 , 1.04185]
Average value of flows for external help	[0.00109913 , 0.00118999]

Table 4: Numerical features of Portfolio III A

	Value
Average value of portfolio	[40.596, 46.2061]
Median of value of portfolio	[41.4954 , 47.0489]
Standard deviation of value of portfolio	[6.93468, 6.3416]
5% quantile of value of portfolio	[41.4954, 47.0489]
95% quantile of value of portfolio	[41.4954 , 47.0489]
1% quantile of value of portfolio	[2.5014 , 8.94574]
99% quantile of value of portfolio	[41.4954 , 47.0489]
Max. loss for portfolio with prob. 5%	[3.50461 , -2.04888]
Max. loss for portfolio with prob. 1%	[42.4986 , 36.0543]
Probability of ruin (%)	[0.91 , 0.59]
Average value of flows for cat bond	[-4.49606 , 0.987338]
Average value of flows for external help	[0.0014125 , 0.00144937]

4 Conclusions

In the paper we construct the portfolio which consists of the standard insurance contract, the catastrophe bond, the governmental and foreign help. In order to evaluate this portfolio, we price the catastrophe bond based on Vasicek model used for zero-coupon bound under assumption of independence between catastrophe occurrence and behaviour of financial market. Because of possible errors and uncertainties which arise from estimation of rare events with serious, catastrophic consequences and lack of precise, historical data, the fuzzy set approach is applied. Then appropriate simulations for the obtained fuzzy pricing formula are generated. Based on the simulations, we compare some properties for the presented portfolios.

References

[1] Bardossy A., Duckstein L. (eds.), Fuzzy Rule-Based Modeling with Applications to Geophysical, Biological and Engineering Systems (Systems Engineering), CRC-Press, 1995

[2] Borch K., The Mathematical Theory of Insurance, Lexington Books, Lexington, 1974

[3] Buckley J. J., Fuzzy Statistics, Springer, 2004

[4] Cummins J.D., Doherty N., Lo A., Can insurers pay for the ”big one”? Measuring the capacity of insurance market to respond to catastrophic losses. Journal of Banking and Finance 26, 2002

[5] Dubois D., Prade H., Fuzzy Sets and Systems – Theory and Applications. Academic Press, New York, 1980

[6] Ermoliev Yu. M., Ermolyeva T. Yu., McDonald G., Norkin V. I., Problems on Insurance of Catastrophic Risks. Cybernetics and Systems Analysis, 37 (2), 2001

[7] Ermolieva, T., Ermoliev, Y., Catastrophic Risk Management: Flood and Seismic Risks Case Studies. In: Applications of Stochastic Programming, Wallace, S., W., Ziemba, W., T. (eds.), MPS-SIAM Series on Optimization, Philadelphia, PA, USA, 2005

[8] Ermolieva, T., Romaniuk, M., Fischer, G., Makowski, M., Integrated model-based decision support for management of weather-related agricultural losses. In: Enviromental informatics and systems research. Vol. 1: Plenary and session papers - EnviroInfo 2007, Hryniewicz, O., Studziński, J., Romaniuk, M. (eds.), Shaker Verlag, IBS PAN, 2007

[9] Freeman, P, K., Kunreuther, H., Managing Environmental Risk Through Insurance. Boston, Kluwer Academic Press, 1997

[10] Froot K. A., The market for catastrophe risk: A clinical examination. Journal of Financial Economics, 60 (2), 2001

[11] George J. B., Alternative reinsurance: Using catastrophe bonds and insurance derivatives as a mechanism for increasing capacity in the insurance markets. CPCU Journal, 1999

[12] Harrington S. E., Niehaus G., Capital, corporate income taxes, and catastrophe insurance. Journal of Financial Intermediation, 12 (4), 2003

[13] MacKellar L., Freeman P., Ermolieva T., Estimating Natural Catastrophic Risk Exposure and the Benefits of Risk Transfer in Developing Countries, IIASA

[14] Malamud B. D., Turcotte D. L., The applicability of power-law frequency statistics to floods. Journal of Hydrology, 322, 2006

[15] McGhee C., Market Update: The Catastrophe Bond Market at Year-End 2003. Guy Carpenter & Company, Inc. and MMC Security Corporation, 2004

[16] Muermann A., Market Price of Insurance Risk Implied by Catastrophe Derivatives. North American Actuarial Journal, 12 (3), pp. 221 – 227, 2008

[17] Niedzielski J., USAA places catastrophe bonds, National Underwriter, Jun 16, 1997

[18] Nowak P., Romaniuk M., Ermolieva T., Integrated management of weather - related agricultural losses - computational approach. In: Information Systems Architecture and Technology, Wilimowska, E., Borzemski, L., Grzech, A., witek, J. (eds.), Wrocaw 2008

[19] Nowak P., Romaniuk M., Portfolio of financial and insurance instruments for losses caused by natural catastrophes. In: Information Systems Architecture and Technology. IT Technologies in Knowledge Oriented Management Process (Wilimowska Z., Borzemski L., Grzech A., witek J., Eds.), Wrocaw 2009

[20] Nowak P., Romaniuk M., Computing option price for Levy process with fuzzy parameters, European Journal of Operational Research, 201 (1), 2010

[21] Pinelli J.-P., Gurley K.R. et all, Validation of probabilistic model for hurricane insurance loss projections in Florida, Reliability Enigeeniring and System Safety, 93, 2008

[22] Rao A. R., Hamed K. H., Flood Frequency Analysis. CRC Press, 2000

[23] Romaniuk M., Ermolieva, T., Application EDGE software and simulations for integrated catastrophe management. International Journal of Knowledge and Systems Sciences, 2(2), pp. 1 - 9, 2005

[24] Walker G., Current Developments in Catastrophe Modelling in Britton N. R., Olliver J. (Eds.), Financial Risks Management for Natural Catastrophes, Brisbane, Griffith University, Australia, 1997

[25] Skees J., Varangis P., Larson D., Siegel P., Can Financial Markets be Tapped to Help Poor People Cope with Weather Risks?, Policy Research Working Paper, 2812, The World Bank Development Research Group, Rural Development, March 2002

[26] Stone, J., M., A Theory of Capacity and the Insurance of Catastrophe Risks. The Journal of Risk and Insurance, 40, 1973

[27] Vaugirard V. E., Pricing catastrophe bonds by an arbitrage approach. The Quarterly Review of Economics and Finance, 43, pp. 119 - 132, 2003

[28] Wu H.-Ch. (2004). Pricing European options based on the fuzzy pattern of Black-Scholes formula. Computers & Operations Research, 31, 1069 – 1081.

[29] Zadeh L. A., Fuzzy sets. Information and Control, 8, 338 – 353, 1965

The papers presented in this Volume 2 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

http://www.ibspan.waw.pl/ifs2009

The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

ISBN-13 9788389475305
ISBN 838947530-8
9 788389 475305