New Trends in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
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Semigroups and semirings of Atanassov's intuitionistic fuzzy relations

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Abstract

The paper deals with $\sup -*$ composition of intuitionistic fuzzy relations, but its considerable part concerns properties of arbitrary fuzzy relations. At first, auxiliary properties of binary operations $* : [0,1]^2 \to [0,1]$ are completed. Next, properties of composition and dual composition of fuzzy relations are discussed. Then, the existence results for $\sup -*$ composition of intuitionistic fuzzy relations are presented. Finally, the above results are used for construction of semigroups and semirings of intuitionistic fuzzy relations.

Keywords: Ordered semigroup, ordered semiring, fuzzy relation, relation composition, dual composition, intuitionistic fuzzy relation.

1 Introduction

After the introduction of intuitionistic fuzzy sets (cf. [1]) there was a period of over ten years without considerations about compositions of intuitionistic fuzzy relations. One of the early papers in this direction was discussion by Burillo and Bustince [4] about the possibility of composition of intuitionistic fuzzy relations with simultaneous application of triangular norms and conorms. As a result of

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this discussion these authors further examined $\sup -T$ and $\inf -S$ compositions (cf. [5]). A new understanding of such compositions was presented by Cornelis et al. [6], where triangular norms on IFS-lattice were applied. A new kind of compositions based on Bandler-Kohout operations was examined by Deschrijver and Kerre [8].

In this paper we apply properties of fuzzy relations in a description of similar properties of intuitionistic fuzzy relations. We pay here attention to algebraic properties of relation composition. In particular we describe fundamental algebraic structures of intuitionistic fuzzy relations. Thus we need auxiliary results from algebra (Section 2), real analysis (Section 3) and fuzzy relation theory (Sections 4 and 5). The longest Section 5 contains theorems and examples describing behaviour of fuzzy relation composition under diverse assumptions about the operation $*$. Then, intuitionistic fuzzy relations are described and properties of their compositions are examined (Sections 6-8).

2 Semigroups and semirings

We begin with algebraic structures with one binary operation. The notion of a semigroup is a simplification of that of a group.

Definition 1 ([14], Chapter X)**.** Let $S \neq \emptyset$ and $* : S \times S \to S$.
• An algebraic structure $(S, *)$ is called a semigroup if the operation $*$ is associative.
• A semigroup $(S, *, e)$ is called a monoid, if the operation $*$ has the neutral element e.
• A semigroup (monoid) is called commutative, if the operation $*$ is commutative.
• A semigroup (monoid) is called idempotent, if the operation $*$ is idempotent, i.e. $a * a = a$ for $a \in S$.
• A semigroup (monoid) has the zero element z if $a * z = z * a = z$ for $a \in S$ and we write $(S, *, z)$ or $(S, *, e, z)$.
• A semigroup (monoid) is called ordered, if $(S, \leqslant)$ is a partially ordered set and the operation $*$ is isotonic (increasing), i.e.

$$a \leqslant b \Rightarrow (c * a \leqslant c * b, \quad a * c \leqslant b * c) \text{ for } a, b, c \in S.$$

The next step is a consideration of algebraic structures with two binary operations. The notion of a semiring is a simplification of that of a ring.

Definition 2 ([13])**.** Let $S \neq \emptyset$ and $*, \circ : S \times S \to S$.
• An algebraic structure $(S, *, \circ, 0, e)$ is called a semiring if:

a) $(S, *, 0)$ is a commutative monoid;
b) $(S, \circ, e, 0)$ is a monoid with the zero element 0;
c) the operation $\circ$ is distributive with respect to the operation $*$, i.e.

$$c \circ (a * b) = (c \circ a) * (c \circ b), \quad (a * b) \circ c = (a \circ c) * (b \circ c) \text{ for } a, b, c \in S.$$

• A semiring is called commutative, if the operation $\circ$ is commutative.
• A semiring $(S, *, \circ, 0, e, \leqslant)$ is called ordered, if both of its semigroups are ordered and $0 < e$ (positive order).

Algebraic structures $(\mathbb{N}, +, \cdot, 0, 1, \leqslant)$ and $(2^X, \cup, \cap, \emptyset, X, \subset)$ are known examples of ordered semirings. We pay more attention to a semiring called bounded distributive lattice $(L, \vee, \wedge, 0, 1)$, where both operations $\vee, \wedge$ are associative, commutative, idempotent and mutually distributive. In particular, a bounded distributive lattice is an ordered semiring with respect to the partial order:

$$a \leqslant b \Leftrightarrow a \vee b = b \text{ for } a, b \in S.$$

With this order the lattice $(L, \vee, \wedge, 0, 1)$ has bounds $\inf L = 0,\ \sup L = 1$.
The lattice is called complete if it is a complete ordered set with the above order. Next, we consider an additional binary operation $*$ in a lattice L.

Definition 3 ([2], Chapter XIV)**.** Let $(L, \vee, \wedge, 0, 1)$ be a complete, distributive lattice, $T \neq \emptyset$ and $* : L \times L \to L$.
• The operation $*$ is called join-distributive if it is distributive with respect to $\vee$.
• The operation $*$ is called meet-distributive if it is distributive with respect to $\wedge$.
• The operation $*$ is called infinitely sup-distributive if

$$a * (\sup_{t \in T} b_t) = \sup_{t \in T}(a * b_t), \quad (\sup_{t \in T} b_t) * a = \sup_{t \in T}(b_t * a) \text{ for } a, b_t \in L. \tag{1}$$

• The operation $*$ is called infinitely inf-distributive if

$$a * (\inf_{t \in T} b_t) = \inf_{t \in T}(a * b_t), \quad (\inf_{t \in T} b_t) * a = \inf_{t \in T}(b_t * a) \text{ for } a, b_t \in L. \tag{2}$$

From infinite distributivity we obtain finite distributivity but the converse is possible in a finite lattice only.

3 Semigroups in $[0, 1]$

Binary operations in interval $[0, 1]$ play an important role in generalizing connectives ‘and’ and ‘or’ of fuzzy logic. We begin with the elementary properties of increasing operations.

Lemma 1 ([10], Lemma 1). *Let* $*: [0,1]^2 \to [0,1]$. *The following three conditions are equivalent:*

$$\underset{a,b,c\in[0,1]}{\forall} (b \leqslant c) \Rightarrow (a * b \leqslant a * c,\ b * a \leqslant b * c), \tag{3}$$

$$\underset{a,b,c\in[0,1]}{\forall} a*\max(b,c) = \max(a*b, a*c),\ \max(b,c)*a = \max(b*a, c*a), \tag{4}$$

$$\underset{a,b,c\in[0,1]}{\forall} a*\min(b,c) = \min(a*b, a*c),\ \min(b,c)*a = \min(b*a, c*a). \tag{5}$$

Lemma 2 (cf. [16], Section I.1). *Let* $f : [p,q] \to \mathbb{R}$ *be an increasing real function and* $x \in [p,q]$.

- *The function f is left-continuous if and only if* $f(x) = \sup\limits_{t<x} f(t)$ *for* $x > p$.
- *The function f is right-continuous if and only if* $f(x) = \inf\limits_{t>x} f(t)$ *for* $x < q$.

Lemma 3 (cf. [16], Introduction). *Let* $f : [p,q] \to \mathbb{R}$ *be an increasing real function and* $B \subset [p,q]$.

- *If the function f is left-continuous, then* $f(\sup B) = \sup f(B)$.
- *If the function f is right-continuous, then* $f(\inf B) = \inf f(B)$.

Proof. Let $x = \sup B$. By monotonicity we get

$$b \leqslant x \Rightarrow f(b) \leqslant f(x) \text{ for } b \in B,\ \sup f(B) = \sup_{b\in B} f(b) \leqslant f(x) = f(\sup B).$$

Moreover, there exists an increasing sequence (b_k) in B, such that $\lim\limits_{k\to\infty} b_k = x$ and by left-continuity we get $f(\sup B) = f(x) = \lim\limits_{k\to\infty} f(b_k) \leqslant \sup f(B)$. This finishes the proof of the first part of the lemma and the proof of the second one is similar. □

Theorem 1. *Let* $*: [0,1]^2 \to [0,1]$.

- *The operation $*$ is infinitely* sup*-distributive if and only if it is increasing and left-continuous with respect to both arguments.*
- *The operation $*$ is infinitely* inf*-distributive if and only if it is increasing and right-continuous with respect to both arguments.*

Proof. At first we assume that the operation $*$ is increasing and left-continuous. We prove the left condition in (1) and the proof of the right case is similar. Let $T \neq \emptyset$, $a, b_t \in [0,1]$, $t \in T$, $B = \{b_t : t \in T\}$, $x = \sup B$. Since the function f_a, $f_a(t) = a * t$, $t \in [0,1]$ is increasing and left-continuous, then by Lemma 3 we get (1):

$$a * (\sup_{t\in T} b_t) = a * x = f_a(x) = f_a(\sup B) = \sup f_a(B) = \sup_{t\in T}(a * b_t).$$

Conversely, let us assume that the operation $*$ is infinitely sup-distributive. Putting $T = \{1, 2\}$, $b_1 = b, b_2 = c$ from (1) we get the property (4) and the operation $*$ is increasing by Lemma 1 (cf. (3)).

Now let $a, x \in [0, 1]$, $x > 0$. Putting $T = [0, x)$, from (1) we get

$$\sup_{t<x} f_a(t) = \sup_{t<x}(a * t) = a * (\sup_{t<x} t) = a * x = f_a(x).$$

By Lemma 2 this proves the left-continuity of the function f_a, i.e. the left-continuity of the operation $*$ in the second argument. Similarly, we can prove the left-continuity of the operation $*$ in the first argument. Thus the operation $*$ is increasing and left-continuous.

The proof of the second part of the theorem may be obtained dually. □

Recently many papers have been devoted to important semigroups in the interval $[0, 1]$ such as triangular norms and conorms, uninorms, nullnorms and their diverse generalizations.

Definition 4 ([15])**.** Let operation $* : [0, 1]^2 \to [0, 1]$ be an increasing, associative and commutative. It is called:
• a uninorm if it has the neutral element $e \in [0, 1]$,
• a nullnorm if it has the zero element $z \in [0, 1]$ and $0 * x = x$ for $x \leqslant z$, $1 * x = x$ for $x \geqslant z$.
The uninorm $*$ is called
• conjunctive if $0 * 1 = 0$,
• disjunctive if $0 * 1 = 1$,
• triangular norm if $e = 1$,
• triangular conorm if $e = 0$.

Example 1 ([15], Examples 1.2, 1.14)**.** As a common representation of the above classes we put here the drastic triangular norm $*$, which is a conjuctive uninorm $(e = 1)$ and nullnorm with $z = 0$, and also the drastic triangular conorm $*^d$, which is a disjuctive uninorm $(e = 0)$ and nullnorm with $z = 1$. These operations are used in our considerations as some counter-examples, because the drastic triangular norm is not left-continuous and the drastic triangular conorm is not right-continuous:

$$x * y = \begin{cases} x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ 0, & \text{if } x, y < 1 \end{cases}, \quad x *^d y = \begin{cases} x, & \text{if } y = 0 \\ y, & \text{if } x = 0 \\ 1, & \text{if } x, y > 0 \end{cases}, \quad x, y \in [0, 1].$$

4 Fuzzy relations

A fuzzy relation in a set $X \neq \emptyset$ is an arbitrary mapping $R : X \times X \to [0,1]$ and we shall write $R \in FR(X)$. For $R, S \in FR(X)$ we use the induced order and the lattice operations:

$$R \leqslant S \quad \Leftrightarrow \quad (R(x,y) \leqslant S(x,y), \;\; x, y \in X),$$

$$(R \vee S)(x,y) = R(x,y) \vee S(x,y), \; (R \wedge S)(x,y) = R(x,y) \wedge S(x,y), \; x, y \in X.$$

Lemma 4 (cf. [12])**.** *$(FR(X), \vee, \wedge, 0_{X\times X}, 1_{X\times X})$ is a complete, distributive lattice (and a commutative, ordered semiring).*

The most important operations on fuzzy relations are their compositions. We shall restrict here to $\sup -*$ and $\inf -*$ compositions.

Definition 5 ([12])**.** Let $* : [0,1]^2 \to [0,1]$. By $\sup -*$ composition of fuzzy relations $R, S \in FR(X)$ we call the fuzzy relation $R \circ S$, where

$$(R \circ S)(x,z) = \sup_{y \in X} (R(x,y) * S(y,z)), \quad x, y \in X. \tag{6}$$

Similarly, $\inf -*$ composition (dual composition) is defined by

$$(R \bullet S)(x,z) = \inf_{y \in X} (R(x,y) * S(y,z)), \quad x, y \in X. \tag{7}$$

Example 2. The operation $*$ can be retrieved from compositions (6), (7) by the application of constant fuzzy relations $c_{X\times X}$, where

$$c_{X\times X}(x,y) = c \text{ for } c \in [0,1], \; x, y \in X.$$

Let $a, b \in [0,1]$, and $x, z \in X$. Directly from Definition 5 we obtain

$$(a_{X\times X} \circ b_{X\times X})(x,z) = \sup_{y \in X} (a_{X\times X}(x,y) * b_{X\times X}(y,z))$$

$$= \sup_{y \in X} (a * b) = a * b = (a * b)_{X\times X}(x,z),$$

$$(a_{X\times X} \bullet b_{X\times X})(x,z) = \inf_{y \in X} (a_{X\times X}(x,y) * b_{X\times X}(y,z))$$

$$= \inf_{y \in X} (a * b) = a * b = (a * b)_{X\times X}(x,z).$$

Thus, the above compositions of constant fuzzy relations are constant fuzzy relations and we have

$$a_{X\times X} \circ b_{X\times X} = (a * b)_{X\times X}, \; a_{X\times X} \bullet b_{X\times X} = (a * b)_{X\times X}. \tag{8}$$

Example 3. We will show that a constant result of the composition can be also obtained for non-constant arguments. Let us consider $X = [0,1]$ with binary operations from Example 1. Putting

$$R(x,y) = y, \quad S(x,y) = \begin{cases} 0, & \max(x,y) = 1 \\ y, & \max(x,y) < 1 \end{cases}, \quad x, y \in [0,1],$$

for the drastic triangular norm $*$ we get

$$(R \circ S)(x,z) = \max(y * 0, \sup_{y<1} y * y) = \max(0,0) = 0,$$

i.e. $R \circ S = 0_{[0,1]^2}$. Dually, using the drastic triangular conorm $*^d$ and

$$R^d(x,y) = 1-y, \quad S^d(x,y) = \begin{cases} 1, & \max(x,y) = 1 \\ 1-y, & \max(x,y) < 1 \end{cases}, \quad x, y \in [0,1]$$

we get

$$(R^d \bullet^d S^d)(x,z) = \inf_{y\in[0,1]} (1-y) *^d \begin{cases} 1, & \max(y,z) = 1 \\ 1-z, & \max(y,z) < 1 \end{cases},$$

$$= \begin{cases} \min(0 *^d 1, \inf_{y<1}(1-y) *^d (1-z)) = \min(1,1) = 1, & z < 1 \\ \inf_{y\in[0,1]}(1-y) *^d 1 = 1, & z = 1 \end{cases}$$

for $x, y \in [0,1]$, i.e. $R^d \bullet^d S^d = 1_{[0,1]^2}$.

5 Properties of compositions of fuzzy relations

Dependencies between properties of the operation $*$ and these of $\sup -*$ composition were examined in details in [9]. We summarize or reprove some of these results. The above dependencies are not clear and not obvious. It is commonly known that the compositions from Definition 5 need not be commutative (similarly as the matrix product). Simultaneously, by Example 2 we can see that the compositions preserve constancy of fuzzy relations, but by Example 3 the constant result does not need constant arguments. These examples will help us to describe more exactly dependencies between operations $*$, $\circ$ and $\bullet$.

Lemma 5. *Let* $* : [0,1]^2 \to [0,1]$.
• *The* $\sup -*$ $(\inf -*)$ *composition is increasing in* $FR(X)$ *if and only if the operation* $*$ *is increasing.*

• *If the operation* $*$ *is increasing, then the* $\sup -*$ *composition is join-distributive and meet-subdistributive, i.e.*

$$T \circ (R \vee S) = T \circ R \vee T \circ S, \ (R \vee S) \circ T = R \circ T \vee S \circ T, \ R, S, T \in FR(X), \quad (9)$$

$$T \circ (R \wedge S) \leqslant T \circ R \wedge T \circ S, \ (R \wedge S) \circ T \leqslant R \circ T \wedge S \circ T, \ R, S, T \in FR(X). \quad (10)$$

• *If the operation* $*$ *is increasing, then the* $\inf -*$ *composition is meet-distributive and join-superdistributive, i.e.*

$$T \bullet (R \wedge S) = T \bullet R \wedge T \bullet S, \ (R \wedge S) \bullet T = R \bullet T \wedge S \bullet T, \ R, S, T \in FR(X), \quad (11)$$

$$T \bullet (R \vee S) \geqslant T \bullet R \vee T \bullet S, \ (R \vee S) \bullet T \geqslant R \bullet T \vee S \bullet T, \ R, S, T \in FR(X). \quad (12)$$

Proof. Let $R, S, T \in FR(X)$. We consider only the case of $\sup -*$ composition because considerations of $\inf -*$ composition are dual to the first one. If the operation $*$ is increasing, then similarly as in [12], Proposition 3A we get monotonicity, join-distributivity (9) and meet-subdistributivity (10).

Conversely, if the operation $\circ$ is increasing, then it is increasing for constant relations and for $a, b, c \in [0, 1]$ we get (cf. (8))

$$a \leqslant b \Leftrightarrow a_{X\times X} \leqslant b_{X\times X} \Rightarrow a_{X\times X} \circ c_{X\times X} \leqslant b_{X\times X} \circ c_{X\times X}$$

$$\Leftrightarrow (a * c)_{X\times X} \leqslant (b * c)_{X\times X} \Leftrightarrow a * c \leqslant b * c.$$

Similarly we get $a \leqslant b \Rightarrow c * a \leqslant c * b$, which finishes the proof of equivalence. □

Lemma 6. *The* $\sup -*$ $(\inf -*)$ *composition has the zero element* Z *if and only if the operation* $*$ *has the zero element* $z \in [0, 1]$ *and* $Z = z_{X\times X}$.

Proof. If $z \in [0, 1]$ is the zero element of the operation $*$ and $Z = z_{X\times X}$, then for every $R \in FR(X)$ we get

$$(Z \circ R)(x, w) = (z_{X\times X} \circ R)(x, w) = \sup_{y\in X}(z * R(y, w)) = \sup_{y\in X} z = z$$

$$= z_{X\times X}(x, w) = Z(x, w),$$

i.e. $Z \circ R = Z$ and similarly we get $R \circ Z = Z$.

Conversely, if a fuzzy relation $Z \in FR(X)$ is the zero element of operation $\circ$, then for $c \in [0, 1]$ we get $Z \circ c_{X\times X} = c_{X\times X} \circ Z = Z$, i.e.

$$(Z \circ c_{X\times X})(x, w) = \sup_{y\in X}(Z(x, y) * c) = Z(x, w),$$

$$(c_{X\times X} \circ Z)(x,w) = \sup_{y\in X}(c * Z(y,w)) = Z(x,w).$$

The result in the first line is independent of w and that in the second line is independent of x, i.e. $Z(x,w) = Z(x,x) = Z(w,w) = \text{const.}$, because $x, w \in X$ are arbitrary. Thus the relation Z is constant. Denoting this constant by z we have $Z = z_{X\times X}$. Moreover, from the above lines we obtain $z * c = c * z = z$, i.e. z is the zero element of operation $*$.

The proof in the case of $\inf -*$ composition is similar. □

Lemma 7. *Let operation* $* : [0,1]^2 \to [0,1]$ *be increasing.*
• *The* $\sup -*$ *composition* $\sup -*$ *has the neutral element* $E \in FR(X)$ *if and only if the operation* $*$ *has the zero element* $z = 0$*, the neutral element* $e \in (0,1]$ *and*

$$E(x,y) = \begin{cases} e, & x = y \\ 0, & x \neq y, \end{cases}, \quad x, y \in X. \tag{13}$$

• *The* $\inf -*$ *composition has the neutral element* $E' \in FR(X)$ *if and only if the operation* $*$ *has the zero element* $z = 1$*, the neutral element* $e' \in [0,1)$ *and*

$$E'(x,y) = \begin{cases} e', & x = y \\ 1, & x \neq y, \end{cases}, \quad x, y \in X. \tag{14}$$

Proof. Let the increasing operation $*$ has the zero element $z = 0$, the neutral element $e > 0$ and the fuzzy relation E be defined by (13). For every $R \in FR(X)$, $x, w \in X$, we obtain

$$(E \circ R)(x,w) = \sup_{y\in X}(E(x,y) * R(y,w)) = \max(e * R(x,w), \sup_{y\neq x}(0 * R(y,w)))$$

$$= e * R(x,w) = R(x,w),$$

i.e. $E \circ R = R$. Similarly we get $R \circ E = R$ and thus the fuzzy relation (13) is the neutral element of the operation $\circ$.

Conversely, let us assume, that a fuzzy relation $E \in FR(X)$ is the neutral element of the operation $\circ$ and $c \in [0,1]$. We consider diagonal fuzzy relations of the form

$$D_c(x,y) = \begin{cases} c, & x = y \\ 0, & x \neq y, \end{cases}, \quad D^d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y, \end{cases}, \quad x, y \in X.$$

At first we get $E \circ D_c = D_c \circ E = D_c$. Thus for $c = 0$ we get

$$\sup_{y\in X}(E(x,y) * 0) = \sup_{y\in X}(0 * E(x,y)) = 0,$$

which gives

$$E(x,y) * 0 = 0 * E(x,y) = 0 \text{ for } x,y \in X. \tag{15}$$

Using this property we obtain

$$(E \circ D_c)(x,x) = \sup_{y\in X}(E(x,y) * c) = \max(E(x,x) * c, \sup_{y\neq x}(E(x,y) * 0))$$

$$= E(x,x) * c = D_c(x,x) = c.$$

Similarly, we get $c * E(x,x) = c$, i.e. $e = E(x,x)$ is the neutral element of the operation $*$. Thus $E(w,w) = e$ for $w \in X$ because the neutral element is unique.

Now by (15) we obtain

$$D_e(x,w) = (D_e \circ E)(x,w) = \max(e * E(x,w), \sup_{y\neq x}(0 * E(y,w))) = E(x,w),$$

i.e. $E = D_e$.

Finally, we have $E \circ D^d = D^d \circ E = D^d$ and for $x \in X$ we obtain

$$0 = D^d(x,x) = (E \circ D^d)(x,x) = \max(e * 0, \sup_{y\neq x}(0 * 1)) = 0 * 1.$$

Similarly, we get $1 * 0 = 0$, which by monotonicity proves that $z = 0$ is the zero element of the operation $*$ (e.g. we have $0 \leqslant x * 0 \leqslant 1 * 0 = 0$). $\square$

Lemma 8. *Let* $* : [0,1]^2 \to [0,1]$.
• *The* $\sup -*$ *composition is infinitely* $\sup$*-distributive if and only if the operation* $*$ *is increasing and left-continuous.*
• *The* $\inf -*$ *composition is infinitely* $\inf$*-distributive if and only if the operation* $*$ *is increasing and right-continuous.*

Proof. Let $T \neq \emptyset$, $R, S_t \in FR(X)$, $t \in T$ and $x, w \in X$. If the operation $*$ is increasing and left-continuous, then by Theorem 1 it is infinitely sup-distributive and we have

$$(R \circ \sup_{t\in T} S_t)(x,w) = \sup_{y\in X}(R(x,y) * \sup_{t\in T} S_t(y,w)) = \sup_{y\in X}\sup_{t\in T}(R(x,y) * S_t(y,w))$$

$$= \sup_{t\in T}\sup_{y\in X}(R(x,y) * S_t(y,w)) = \sup_{t\in T}(R \circ S_t)(x,w),$$

which proves that

$$R \circ (\sup_{t\in T} S_t) = \sup_{t\in T}(R \circ S_t).$$

Similarly, we get

$$(\sup_{t\in T} S_t)\circ R = \sup_{t\in T}(S_t \circ R),$$

i.e. the $\sup -*$ composition is infinitely sup-distributive.

Conversely, let us assume that the $\sup -*$ composition is infinitely sup-distributive. We apply this to the constant fuzzy relations.
At first, let $T \neq \emptyset$, $a, b_t \in [0,1]$, $t \in T$, $x, y \in X$. Formally we have

$$(\sup_{t\in T}(b_t)_{X\times X})(x,y) = \sup_{t\in T} b_t = (\sup_{t\in T} b_t)_{X\times X}(x,y).$$

Thus

$$\sup_{t\in T}(b_t)_{X\times X} = (\sup_{t\in T} b_t)_{X\times X}.$$

Now we get

$$(a * \sup_{t\in T} b_t)_{X\times X} = a_{X\times X}\circ (\sup_{t\in T} b_t)_{X\times X} = a_{X\times X}\circ (\sup_{t\in T} b_t)_{X\times X}$$

$$= \sup_{t\in T} a_{X\times X}\circ (b_t)_{X\times X} = \sup_{t\in T}(a * b_T)_{X\times X} = \sup_{t\in T}(a * b_t)_{X\times X}.$$

Thus

$$a * \sup_{t\in T} b_t = \sup_{t\in T}(a * b_t)$$

and similarly we get

$$\sup_{t\in T} b_t * a = \sup_{t\in T}(b_t * a),$$

i.e. the operation $*$ is infinitely sup-distributive. By Theorem 1 it is increasing and left-continuous.

The proof in the case of $\inf -*$ composition is similar. $\square$

Lemma 9. *Let* $* : [0,1]^2 \to [0,1]$.
• *If the* $\sup -*$ $(\inf -*)$ *composition is associative in* $FR(X)$*, then the operation* $*$ *is associative in* $[0,1]$.
• *If the operation* $*$ *is increasing, left-continuous and associative, then the* $\sup -*$ *composition is associative.*
• *If the operation* $*$ *is increasing, right-continuous and associative, then the* $\inf -*$ *composition is associative.*

Proof. Let $a, b, c \in [0,1]$. If the composition $\circ$ is associative in $FR(X)$, then we can apply the constant fuzzy relations and (8):

$$((a*b)*c)_{X\times X} = (a*b)_{X\times X}\circ c_{X\times X} = (a_{X\times X}\circ b_{X\times X})\circ c_{X\times X}$$

$$= a_{X\times X} \circ (b_{X\times X} \circ c_{X\times X}) = a_{X\times X} \circ (b * c)_{X\times X} = (a * (b * c))_{X\times X}.$$

Thus, $(a*b)*c = ((a*b)*c)_{X\times X}(x,y) = (a*(b*c))_{X\times X}(x,y) = a*(b*c)$, i.e. the operation $*$ is associative and the proof in the case of the $\inf -*$ composition is analogical.

Now, let $R, S, T \in FR(X)$, $x, w \in X$. If the operation $*$ is increasing, left-continuous and associative, then by Lemma 8 the $\sup -*$ composition is infinitely $\sup$-distributive. Thus we get

$$[(R \circ S) \circ T](x,w) = \sup_{z\in X}(R \circ S)(x,z) * T(z,w)$$

$$= \sup_{z\in X}(\sup_{y\in X} R(x,y) * S(y,z)) * T(z,w) = \sup_{z,y\in X}[(R(x,y) * S(y,z)) * T(z,w)]$$

$$= \sup_{y,z\in X}[R(x,y) * (S(y,z) * T(z,w)] = \sup_{y\in X} R(x,y) * (\sup_{z\in X} S(y,z) * T(z,w))$$

$$= \sup_{y\in X} R(x,y) * (S \circ T)(y,w) = [R \circ (S \circ T)](x,w),$$

which proves associativity of the $\sup -*$ composition.

The proof in the case of $\inf -*$ composition is similar. □

It is troublesome fact that we have no equivalence in the above lemma. However, if we delete additional assumptions about the operation $*$, then we can lose the positive result.

Example 4 (cf. [9], Example 6)**.** We show that the continuity assumption from the above lemma cannot be deleted in consideration of the composition associativity. From Examples 1, 3 we use the drastic triangular norm $*$ (which is not left-continuous), relations $R, S \in [0,1]^2$ and additionally relation $T = 1_{[0,1]^2}$. By Example 3 we have $R \circ S = 0_{[0,1]^2}$ and by (8) we get

$$(R \circ S) \circ T = 0_{[0,1]^2} \circ 1_{[0,1]^2} = (0 * 1)_{[0,1]^2} = 0_{[0,1]^2}.$$

For $x, z \in [0,1]$ we also have

$$(S \circ T)(x,z) = 0 * 1 \vee \sup_{y<1} y * 1 = 0 \vee 1 = 1.$$

Similarly, $R \circ 1_{[0,1]^2} = 1_{[0,1]^2}$, i.e. $R \circ (S \circ T) = 1_{[0,1]^2}$.

In the case of $\inf -*$ composition we also use Examples 1, 3 with the drastic triangular conorm $*^d$, $X = [0,1]$, relations R^d, S^d and additionally $T^d = 0_{[0,1]^2}$. We get

$$(R^d \bullet^d S^d) \bullet^d T^d = 1_{[0,1]^2} > R^d \bullet^d (S^d \bullet^d T^d) = 0_{[0,1]^2}.$$

Therefore, the $\sup -*$ $(\inf -*^d)$ composition is not associative, while the operation $*$, $(*^d)$ is associative and increasing.

Example 5. We show that the monotonicity assumption cannot be deleted from Lemma 9 in consideration of the composition associativity. We consider the binary operation $x * y = 2xy - x - y + 1$, $x, y \in [0,1]$, for which $z = 0.5$, $e = 1$ and $x * 0 = 0 * x = 1 - x$. It is associative and continuous but not increasing, because $0 < 1$ and $0 * 0 = 1 > 0 * 1 = 0$. Putting $X = [0,1]$, $R(x,y) = 0.5y$, $S(x,y) = 0.5x$, $x, y \in [0,1]$ and $T = 0_{[0,1]^2}$, for $\sup -*$ composition we obtain

$$(R \circ S)(x,z) = \sup_{y \in [0,1]} (0.5y^2 - y + 1) = 1, \ (S \circ T)(x,y) = \sup_{y \in [0,1]} (1 - 0.5x)$$

$$= 1 - 0.5x, \ x, z \in [0,1],$$

$$(R \circ S) \circ T = 0_{[0,1]^2} < R \circ (S \circ T) = 0.5_{[0,1]^2}.$$

Dually we put the operation $x *^d y = x + y - 2xy$, $x, y \in [0,1]$, and relations $R^d(x,y) = 1 - 0.5y$, $S^d(x,y) = 1 - 0.5x$, $T^d = 1_{[0,1]^2}$. We get

$$(R^d \bullet^d S^d) \bullet^d T^d = 1_{[0,1]^2} > R^d \bullet^d (S^d \bullet^d T^d) = 0.5_{[0,1]^2}.$$

Therefore, the $\sup -*$ ($\inf -*^d$) composition is not associative, while the operation $*$ ($*^d$) is associative and continuous.

Under suitable assumptions about the operation $*$ we can obtain a semigroup or a semiring of fuzzy relations. Directly from the above lemmas (Lemmas 5-9) we get

Theorem 2. *Let* $* : [0,1]^2 \to [0,1]$ *be an increasing, associative operation.*
- *If the operation* $*$ *is left-continuous, then* $(FR(X), \circ, \leqslant)$ *is an ordered semigroup of fuzzy relations.*
- *If the operation* $*$ *is left-continuous with the zero* $z = 0$ *and the neutral element* $e > 0$*, then* $(FR(X), \circ, E, 0_{X \times X}, \leqslant)$ *is an ordered monoid with the zero* $0_{X \times X}$*. Moreover,* $(FR(X), \vee, \circ, 0_{X \times X}, E, \leqslant)$ *is an ordered semiring of fuzzy relations, where* E *is given by* (13).
- *If the operation* $*$ *is right-continuous, then* $(FR(X), \bullet, \leqslant)$ *is an ordered semigroup of fuzzy relations.*
- *If the operation* $*$ *is right-continuous with the zero* $z = 1$ *and the neutral element* $e < 1$*, then* $(FR(X), \bullet, 1_{X \times X}, E', \leqslant)$ *is an ordered monoid with the zero* $1_{X \times X}$*. Moreover,* $(FR(X), \wedge, \bullet, 1_{X \times X}, E', \leqslant)$ *is an ordered semiring of fuzzy relations, where* E' *is given by* (14).
- *If the operation* $*$ *is continuous, then we get both ordered semigroups:* $(FR(X), \circ, \leqslant)$ *and* $(FR(X), \bullet, \leqslant)$.

Let us observe, that fixed operation $*$ cannot generate both monoids from the last result, because they have different zero elements. The above theorem can be simplified in the case of commonly known triangular norms, triangular conorms, uninorms and nullnorms:

Corollary 1. *Let* $* : [0,1]^2 \to [0,1]$.
• *If the operation* $*$ *is a left-continuous nullnorm, then* $(FR(X), \circ, \leqslant)$ *is an ordered semigroup of fuzzy relations.*
• *If the operation* $*$ *is a right-continuous nullnorm, then* $(FR(X), \bullet, \leqslant)$ *is an ordered semigroup of fuzzy relations.*
• *If the operation* $*$ *is a left-continuous, conjuctive uninorm with the neutral element* $e > 0$*, then* $(FR(X), \circ, E, 0_{X\times X}, \leqslant)$ *is an ordered monoid with the zero* $0_{X\times X}$ *and* $(FR(X), \vee, \circ, 0_{X\times X}, E, \leqslant)$ *is an ordered semiring of fuzzy relations, where* E *is given by* (13) *(for* $e = 1$ *it is the case of left-continuous triangular norms).*
• *If the operation* $*$ *is right-continuous, disjunctive uninorm with the neutral element* $e < 1$*, then* $(FR(X), \bullet, 1_{X\times X}, E', \leqslant)$ *is an ordered monoid with the zero* $1_{X\times X}$*, and* $(FR(X), \wedge, \bullet, 1_{X\times X}, E', \leqslant)$ *is an ordered semiring of fuzzy relations, where* E' *is given by* (14) *(for* $e = 0$ *it is the case of right-continuous triangular conorms).*

6 Intuitionistic fuzzy relations

Intuitionistic fuzzy relations are pairs of fuzzy relations.

Definition 6 ([3])**.** Let fuzzy relations $R, R^d \in FR(X)$ fulfil the condition

$$R(x,y) + R^d(x,y) \leqslant 1,\ x, y \in X.$$

A pair $\rho = (R, R^d)$ is called an (Atanassov's) intuitionistic fuzzy relation.
The family of all intuitionistic fuzzy relations in X is denoted by $IFR(X)$.

According to the condition from the above definition we consider the triangle

$$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leqslant 1\}$$

with order relation

$$(x_1, x_2) \leqslant_L (y_1, y_2) \Leftrightarrow (x_1 \leqslant y_1,\ x_2 \geqslant y_2).$$

Lemma 10 ([7]). $(L^*, \leqslant_L) = (L^*, \vee_L, \wedge_L)$ *is a complete, distributive lattice with bounds* $0_L = (0,1), 1_L = (1,0)$*, where*

$$(x_1, x_2)\vee_L(y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)),$$

$$(x_1, x_2)\wedge_L(y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2)).$$

Thus, intuitionistic fuzzy relations are functions with values in the lattice $(L^*, \vee_L, \wedge_L)$. Using this lemma we can introduce induced order and induced lattice operations in $IFR(X)$.

Definition 7 ([5]). Let $\rho, \sigma \in IFR(X)$, $\rho = (R, R^d), \sigma = (S, S^d)$. We put

$$\rho\leqslant_L\sigma \Leftrightarrow (R \leqslant S, R^d \geqslant S^d),$$

$$\rho\vee_L\sigma = (R \vee S, R^d \wedge S^d),\ \rho\wedge_L\sigma = (R \wedge S, R^d \vee S^d).$$

Directly from Lemma 4 we get

Corollary 2. $(IFR(X), \leqslant_L) = (IFR(X), \vee_L, \wedge_L)$ *is a complete, distributive lattice with bounds* $0_{IFR} = (0_{X\times X}, 1_{X\times X}), 1_{IFR} = (1_{X\times X}, 0_{X\times X})$.

7 Composition of intuitionistic fuzzy relations

After detailed discussion of diverse formulas of composition of intuitionistic fuzzy relations in [4] we can use a general version of $\sup -*$ composition:

Definition 8 ([5]). Let $*, *^d : [0,1]^2 \to [0,1]$, $* \leqslant (*^d)'$ and $\rho, \sigma \in IFR(X)$, $\rho = (R, R^d), \sigma = (S, S^d)$. By composition of intuitionistic fuzzy relations $\rho, \sigma \in IFR(X)$ we call $\rho\circ_L\sigma = (R \circ S, R^d \bullet^d S^d)$, i.e.

$$(\rho\circ_L\sigma)(x,z) = (\sup_{y\in X} R(x,y) * S(y,z),\ \inf_{y\in X} R^d(x,y) *^d S^d(y,z)),\ \ x, y \in X,$$

where $a *' b = 1 - (1-a) * (1-b)$, $a, b \in [0,1]$.

According to [4], we get

Lemma 11 ([4], Proposition 1). *Let* $*, *^d : [0,1]^2 \to [0,1]$*. If* $* \leqslant (*^d)'$ *and* $\rho, \sigma \in IFR(X)$*, then* $\rho\circ_L\sigma$ *is an intuitionistic fuzzy relation.*

This provides us consideration of semigroups and semirings of intuitionistic fuzzy relations.

8 Algebraic structures of intuitionistic fuzzy relations

Directly from Lemma 11 and Theorem 2 we obtain

Theorem 3. *Let* $*, *^d : [0,1]^2 \to [0,1]$ *be increasing, associative operations and* $* \leqslant (*^d)'$.
• *If the operation* $*$ *is left-continuous and the operation* $*^d$ *is right-continuous, then* $(IFR(X), \circ_L, \leqslant)$ *is an ordered semigroup of intuitionistic fuzzy relations.*
• *If additionally the operation* $*$ *has the zero* $z = 0$ *and the neutral element* $e > 0$, *the operation* $*^d$ *has the zero* $z^d = 1$ *and the neutral element* $e^d < 1$, *then we obtain an ordered monoid* $(IFR(X), \circ_L, E_L, 0_{IFR(X)}, \leqslant)$ *of intuitionistic fuzzy relations and* $(IFR(X), \vee_L, \circ_L, 0_{IFR(X)}, E_L, \leqslant)$ *is an ordered semiring of intuitionistic fuzzy relations where,* $E_L = (E, E')$.

Corollary 3. *Let* $*, *^d : [0,1]^2 \to [0,1]$, $* \leqslant (*^d)'$.
• *If the operation* $*$ *is a left-continuous nullnorm and the operation* $*^d$ *is a right-continuous nullnorm, then* $(IFR(X), \circ_L, \leqslant)$ *is an ordered semigroup of intuitionistic fuzzy relations.*
• *If the operation* $*$ *is a conjunctive, left-continuous uninorm with* $e > 0$, *the operation* $*^d$ *is a disjunctive, right-continuous uninorm with* $e^d < 1$, *then* $(IFR(X), \circ_L, E_L, 0_{IFR(X)}, \leqslant)$ *is an ordered monoid of intuitionistic fuzzy relations, and* $(IFR(X), \vee_L, \circ_L, 0_{IFR(X)}, E_L, \leqslant)$ *is an ordered semiring of intuitionistic fuzzy relations, where* $E_L = (E, E')$ *(for* $e = 1$ *and* $e^d = 0$ *it is the case of left-continuous triangular norms and right-continuous triangular conorms).*

Example 6. Using the drastic triangular operations and fuzzy relations from Example 4 we put $\rho = (R, R^d), \sigma = (S, S^d), \tau = (T, T^d)$. Using the composition $\circ_L$ we get $R \circ_L (S \circ_L T) = 1_{IFR} \neq (R \circ_L S) \circ_L T = 0_{IFR}$. Therefore, the composition $\circ_L$ is not associative, while operations $*$ and $*^d$ are associative and increasing.

Example 7. Using the operations and fuzzy relations from Example 5 we similarly put $\rho = (R, R^d), \sigma = (S, S^d), \tau = (T, T^d)$. Using the composition $\circ_L$ we get $R \circ_L (S \circ_L T) = 0_{IFR} \neq (R \circ_L S) \circ_L T = 0.5_{IFR}$. Therefore, the composition $\circ_L$ is not associative, while operations $*$ and $*^d$ are associative and continuous.

9 Conclusion

The goal of the paper is a presentation of some algebraic structures of intuitionistic fuzzy relations. Such results are important from two points of view. First of all, the composition of fuzzy relations is the main operation on fuzzy relations and appears in many applications. Thus, the composition properties can be applied in many domains. From the second point of view, algebraic structures are standard notions in mathematics and their properties are summarized in many monographs. Identification of such structures in relational calculus gives possibility of their consideration on higher level of abstraction with application of known theories. In such a way many particular cases may be replaced with one general statement (e.g. definition, theorem, proof or example).

Some generalizations of $\sup - \inf$ composition need additional assumptions connected with monotonicity and continuity. In practical computation on a finite domain the continuity assumptions are not necessary. However, a general theorem need precise assumptions. Examples of such results are presented in Section 5. Further considerations will concern properties of powers of intuitionistic fuzzy relations based on results of paper [11].

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eleventh International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2012) organized in Warsaw on October 12, 2012 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Eleventh International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2012) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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