Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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A Jensen type inequality for Sugeno integrals with respect to intuitionistic fuzzy measures

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Abstract

We prove a Jensen inequality for fuzzy integral of Sugeno kind of intuitionistic fuzzy-valued functions with respect to intuitionistic fuzzy-valued fuzzy measures.

Keywords: Sugeno integral, intuitionistic fuzzy measure, integral inequality.

1 Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [12] and intensively studied. The monographs [9] and [13] are dedicated to this topic. Recently, several classical inequalities were generalized to fuzzy integrals. In [11] a Jensen type inequality for fuzzy integral was provided.

The theory of intuitionistic fuzzy measures was developed in the last years (see [4]-[7]). In this contribution we use the theorem of decomposition of the Sugeno integral with respect to intuitionistic fuzzy-valued fuzzy measures ([4]) and the result in [11] to obtain a Jensen type inequality for intuitionistic fuzzy-valued fuzzy integrals of measurable intuitionistic fuzzy-valued functions with respect to intuitionistic fuzzy-valued fuzzy measures. At the end of the paper we give some illustrative examples.

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2 Preliminaries

We recall some basic definitions and previous results which will be used in the sequel.

Throughout this paper, X is a nonempty set, $\mathcal{A}$ is a σ-algebra of subsets of $X, \overline{\mathbb{R}_+} = [0, +\infty]$ and all considered subsets belong to $\mathcal{A}$.

Definition 1 *([10]) A set function $\mu : \mathcal{A} \to \overline{\mathbb{R}_+}$ is called a fuzzy measure if the following properties are satisfied:*

(i) $\mu(\varnothing) = 0$;

(ii) $A \subseteq B$ *implies* $\mu(A) \leq \mu(B)$;

(iii) $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$ *imply*

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n);$$

(iv) $A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ...$ *and* $\mu(A_1) < +\infty$ *imply*

$$\mu\left(\cap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

If $\mu(X) = 1$ then μ is called a normalized fuzzy measure.

When μ is a fuzzy measure, the triple $(X, \mathcal{A}, \mu)$ is called a fuzzy measure space and we denote by $\mathcal{A}_+^{\mu}(X)$ the set of all nonnegative μ-measurable functions with respect to $\mathcal{A}$.

Definition 2 *([9], [13]) Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space and $A \in \mathcal{A}$. The Sugeno integral of $f \in \mathcal{A}_+^{\mu}(X)$ on A, with respect to the fuzzy measure μ, is defined by*

$$(S)\int_A f d\mu = \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu(A \cap F_\alpha)\right),$$

where $F_\alpha = \{x \in X : f(x) \geq \alpha\}, \alpha \geq 0.$

The set of intuitionistic fuzzy values

$$\mathcal{L} = \{(x, y) : x, y \in [0, 1], x + y \leq 1\}$$

is very important in intuitionistic fuzzy set theory ([1], [2]). The set $\mathcal{L}$ is a complete lattice ([8]),

$$(x_1, y_1) \leq_{\mathcal{L}} (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_2 \leq y_1,$$
$$(x_1, y_1) <_{\mathcal{L}} (x_2, y_2) \text{ if and only if } (x_1, y_1) \leq_{\mathcal{L}} (x_2, y_2)$$
$$\text{and } x_1 < x_2 \text{ or } y_2 < y_1$$

$$\begin{aligned} sup_{\mathcal{L}}W &= (\sup\{x\in[0,1]\,|\exists y\in[0,1]:(x,y)\in W\}, \\ &\quad \inf\{y\in[0,1]\,|\exists x\in[0,1]:(x,y)\in W\}), \\ inf_{\mathcal{L}}W &= (\inf\{x\in[0,1]\,|\exists y\in[0,1]:(x,y)\in W\}, \\ &\quad \sup\{y\in[0,1]\,|\exists x\in[0,1]:(x,y)\in W\}), \end{aligned}$$

for each $W \subseteq \mathcal{L}$. If $(a_n)_{n\in\mathbb{N}} \subset \mathcal{L}, a_n = (x_n, y_n)$ is increasing, that is $a_n \leq_{\mathcal{L}} a_{n+1}$, for every $n \in \mathbb{N}$, or decreasing, that is $a_{n+1} \leq_{\mathcal{L}} a_n$, for every $n \in \mathbb{N}$, then it is convergent and (see [6], p. 168)

$$\lim_{n\to\infty}{}_{\mathcal{L}} a_n = \left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right).$$

Definition 3 *(see [6]) A mapping $v : \mathcal{A} \to \mathcal{L}$ is called an intuitionistic fuzzy measure if the following properties are satisfied:*

(i) $v(\varnothing) = (0,1)$ *and* $v(X) = (1,0)$;

(ii) $A \subseteq B$ *implies* $v(A) \leq_{\mathcal{L}} v(B)$;

(iii) $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$ *imply*

$$v\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty}{}_{\mathcal{L}} v(A_n);$$

(iv) $A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ...$ *imply*

$$v\left(\cap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty}{}_{\mathcal{L}} v(A_n).$$

The triple $(X, \mathcal{A}, v)$ is called an intuitionistic fuzzy measure space.

Theorem 1 *If $v : \mathcal{A} \to \mathcal{L}, v = (v_1, v_2)$ is an intuitionistic fuzzy measure then $v_1, v_2^c : \mathcal{A} \to [0,1], v_2^c(A) = 1 - v_2(A)$, for every $A \in \mathcal{A}$, are normalized fuzzy measures.*

Proof. See [6], Definition 3.3, p. 174 and Theorem 3.6, p. 177. ∎

Definition 4 *([6], p. 206, see also [4]) Let $(X, \mathcal{A}, v)$ be an intuitionistic fuzzy measure space. A function $f : X \to \mathcal{L}$ is called v-measurable with respect to $\mathcal{A}$ if*

$$\{x \in X : f(x) \leq_{\mathcal{L}} \alpha\} \in \mathcal{A}$$

and

$$\{x \in X : f(x) \geq_{\mathcal{L}} \alpha\} \in \mathcal{A},$$

for every $\alpha \in \mathcal{L}$.

We denote by $\mathcal{A}^v(X)$ the set of all v-measurable functions with respect to $\mathcal{A}$.

As a matter of fact, the measurability of an intuitionistic fuzzy-valued function reduces to the measurability of its components.

Theorem 2 *([6], p. 206 or [4]) Let $f : X \to \mathcal{L}, f(x) = (g(x), h(x))$. Then $f \in \mathcal{A}^v(X)$ if and only if $g \in \mathcal{A}_+^{v_1}(X)$ and $h \in \mathcal{A}_+^{v_2^c}(X)$.*

The Sugeno integral of an intuitionistic fuzzy-valued mapping, on a crisp set, with respect to an intuitionistic fuzzy measure is defined as follows.

Definition 5 *([6], p. 206 or [7]) Let $(X, \mathcal{A}, v)$ be an intuitionistic fuzzy measure space and $f \in \mathcal{A}^v(X)$. The Sugeno integral type of f on $A \in \mathcal{A}$ with respect to v, denoted by $(S)\int_X^{\mathcal{L}} f dv$, is defined by*

$$(S)\int_X^{\mathcal{L}} f dv = \underset{a\in\mathcal{L}}{sup_{\mathcal{L}}} inf_{\mathcal{L}}(a, v(A \cap F_a)),$$

where $F_a = \{x \in X : f(x) \geq_{\mathcal{L}} a\}$, for every $a \in \mathcal{L}$.

3 Jensen type inequality

The following results help us to reach a Jensen type inequality for Sugeno integrals with respect to intuitionistic fuzzy measures.

Theorem 3 *([6], p. 215, see also [4]) Let $v : \mathcal{A} \to \mathcal{L}, v = (v_1, v_2)$ be an intuitionistic fuzzy measure and $f \in \mathcal{A}^v(X), f(x) = (g(x), h(x))$. Then*

$$(S)\int_X^{\mathcal{L}} f dv = \left((S)\int_X g dv_1, 1 - (S)\int_X h^c dv_2^c\right),$$

where $h^c(x) = 1 - h(x), x \in X, v_2^c(A) = 1 - v_2(A), A \in \mathcal{A}$.

In the paper [11] the following Jensen type inequality for fuzzy integrals was proved.

Theorem 4 *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space and let $\Phi : [0, +\infty) \to [0, +\infty)$ be a strictly increasing function such that $\Phi(x) \leq x$, for every $x \in [0, \mu(X)]$. Then*

$$\Phi\left((S)\int_X f d\mu\right) \leq (S)\int_X \Phi(f) d\mu,$$

for all $f \in \mathcal{A}_+^{\mu}(X)$.

In the intuitionistic fuzzy case we obtain the following result, the main result of the paper.

Theorem 5 *Let* $(X, \mathcal{A}, v)$ *be an intuitionistic fuzzy measure space and* $\varphi_i : [0,1] \to [0,1]\,, i \in \{1,2\}$ *be strictly increasing functions such that* $\varphi_i(x) \leq x$*, for every* $x \in [0,1]$ *and* $\varphi_1(x) \leq \varphi_2(y)\,, \forall x, y \in [0,1]\,, x \leq y$*. Then*

$$\Phi\left((S)\int_X^{\mathcal{L}} f dv\right) \leq_{\mathcal{L}} (S)\int_X^{\mathcal{L}} \Phi(f)\, dv,$$

for all $f \in \mathcal{A}^v(X)$*, where* $\Phi : \mathcal{L} \to \mathcal{L}$ *is defined by*

$$\Phi(x, y) = (\varphi_1(x)\,, 1 - \varphi_2(1 - y))\,.$$

Proof. If $v = (v_1, v_2)\,, f = (g, h)$ then, taking into account Theorem 3, we obtain

$$\begin{aligned}
&\Phi\left((S)\int_X^{\mathcal{L}} f dv\right) \\
&= \Phi\left((S)\int_X g dv_1, 1 - (S)\int_X h^c dv_2^c\right) \\
&= \left(\varphi_1\left((S)\int_X g dv_1\right), 1 - \varphi_2\left((S)\int_X h^c dv_2^c\right)\right).
\end{aligned}$$

Because $\varphi_i, i \in \{1,2\}$ satisfy the conditions in Theorem 4, $g \in \mathcal{A}_+^{v_1}(X)$ and $h \in \mathcal{A}_+^{v_2^c}(X)$ (see Theorem 2) we get

$$\varphi_1\left((S)\int_X g dv_1\right) \leq (S)\int_X \varphi_1(g)\, dv_1$$

and

$$\varphi_2\left((S)\int_X h^c dv_2^c\right) \leq (S)\int_X \varphi_2(h^c)\, dv_2^c$$

that is (taking into account again Theorem 3)

$$\begin{aligned}
&\Phi\left((S)\int_X^{\mathcal{L}} f dv\right) \\
&\leq_{\mathcal{L}} \left((S)\int_X \varphi_1(g)\, dv_1, 1 - (S)\int_X \varphi_2(h^c)\, dv_2^c\right) \\
&= (S)\int_X^{\mathcal{L}} (\varphi_1(g)\,, 1 - \varphi_2(h^c))\, dv \\
&= (S)\int_X^{\mathcal{L}} \Phi(f)\, dv.
\end{aligned}$$

■

Remark 1 *The condition* $\varphi_1(x) \leq \varphi_2(y), \forall x, y \in [0,1]$ *such that* $x \leq y$ *assures that the mapping* Φ *is well-defined. Indeed, if* $(x,y) \in \mathcal{L}$ *then*

$$\varphi_1(x) + 1 - \varphi_2(1-y) \leq \varphi_2(1-y) + 1 - \varphi_2(1-y) = 1.$$

Example 1 *Let* $(X, \mathcal{A}, v)$ *be an intuitionistic fuzzy measure space and* $f \in \mathcal{A}^v(X)$. *Taking*

$$\varphi_1(x) = \alpha x$$

and

$$\varphi_2(x) = \beta x,$$

with $0 < \alpha \leq \beta \leq 1$, *we get*

$$\Phi(x,y) = (\alpha x, 1 - \beta + \beta y), \forall (x,y) \in \mathcal{L}$$

and the conditions in Theorem 5 are satisfied. If we consider the mapping (which generates an universal operator on intuitionistic fuzzy sets, see [2], p. 103)

$$X_{a,b,c,d,e,f} : \mathcal{L} \to \mathcal{L}$$

defined by

$$X_{a,b,c,d,e,f}(x,y) = (ax + b(1 - x - cy), dy + e(1 - fx - y)),$$

where $a, b, c, d, e, f \in [0,1]$ *and*

$$\begin{aligned} a + e - ef &\leq 1, \\ b + d - bc &\leq 1 \end{aligned}$$

then

$$\Phi = X_{\alpha,0,0,1,1-\beta,0},$$

therefore

$$X_{\alpha,0,0,1,1-\beta,0}\left((S)\int_X^{\mathcal{L}} f dv\right) \leq_{\mathcal{L}} (S)\int_X^{\mathcal{L}} X_{\alpha,0,0,1,1-\beta,0}(f)\, dv,$$

for all $f \in \mathcal{A}^v(X)$.

In the particular case $\alpha = \beta = \frac{1}{2}$ *we get*

$$\Phi(x,y) = \left(\frac{1}{2}x, \frac{1}{2} + \frac{1}{2}y\right), \forall (x,y) \in \mathcal{L}$$

and $\Phi = \boxplus$, where $\boxplus$ generates an operator on intuitionistic fuzzy sets (see [2], p. 121). Then

$$\boxplus\left((S)\int_X^{\mathcal{L}} f dv\right) \leq_{\mathcal{L}} (S)\int_X^{\mathcal{L}} \boxplus(f)\, dv,$$

for all $f \in \mathcal{A}^v(X)$.

If $\varphi_1(x) = \varphi_2(x) = x, \forall x, y \in [0,1]$, that is $\Phi(x,y) = (x,y), \forall (x,y) \in \mathcal{L}$ then inequality in Theorem 5 becomes equality. In the following we prove that the strict inequality is also possible.

Example 2 *Let $X = [0,1]$, $f = (g,h)$, $g(x) = x^2$ and $h(x) = 1-x$. Let the intuitionistic fuzzy measure $v = (v_1, v_2)$ be defined as $v_1(A) = m^2(A), v_2(A) = 1 - m(A), A \in \mathcal{A}$, where m is the Lebesgue measure, $\varphi_1(x) = x^2$ and $\varphi_2(x) = x$, that is $\Phi(x,y) = \left(x^2, y\right)$. Then*

$$(S)\int_{[0,1]} g dv_1 = \bigvee_{\alpha\in[0,1]} \left(\alpha \wedge \left(1-\sqrt{\alpha}\right)^2\right) = \frac{1}{4},$$

and

$$(S)\int_{[0,1]} h^c dv_2^c = \bigvee_{\alpha\in[0,1]} (\alpha \wedge (1-\alpha)) = \frac{1}{2}$$

that is

$$\Phi\left((S)\int_X^{\mathcal{L}} f dv\right) = \Phi\left(\frac{1}{4}, \frac{1}{2}\right) = \left(\frac{1}{16}, \frac{1}{2}\right).$$

Because

$$\Phi(f) = (G, H),$$

where $G(x) = x^4$ and $H(x) = 1 - x$, we obtain (Theorem 3)

$$\begin{aligned}
&(S)\int_X^{\mathcal{L}} \Phi(f)\, dv \\
&= \left((S)\int_{[0,1]} x^4 dv_1, 1 - \int_{[0,1]} x dv_2^c\right) \\
&= \left(\bigvee_{\alpha\in[0,1]} \left(\alpha \wedge \left(1 - \sqrt[4]{\alpha}\right)^2\right), 1 - \bigvee_{\alpha\in[0,1]} (\alpha \wedge (1-\alpha))\right) \\
&= \left(\frac{7-3\sqrt{5}}{2}, \frac{1}{2}\right)
\end{aligned}$$

and

$$\Phi\left((S)\int_X^{\mathcal{L}} f dv\right) <_{\mathcal{L}} (S)\int_X^{\mathcal{L}} \Phi(f)\, dv.$$

4 Conclusions

The present paper continues the study of integral inequalities in the intuitionistic fuzzy case, which began with Chebyshev inequality [3]. We have introduced a version of Jensen inequality for the Sugeno integral of intuitionistic fuzzy-valued functions with respect to intuitionistic fuzzy-valued fuzzy measures. The proof of the main result is based on a result of componentwise decomposition of intuitionistic fuzzy-valued fuzzy integrals. Some examples are presented at the end of the paper.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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