Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Generalized inverses for monotonic operations. A survey

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Abstract

Generalized inverses are important tools for solving equations and inequalities in many applications of algebra and analysis. Our presentation is based on the semigroup theory with consideration of generalized inverses in regular, inverse and ordered semigroups. We describe properties of quasi-inverse and pseudo-inverse in semigroups and in particular cases of matrix product and composition of functions.

Keywords: Generalized inverse, quasi-inverse, pseudo-inverse, matrix inverse, function inverse, regular semigroup, inverse semigroup, ordered semigroup.

1 Introduction

Generalized inverses are important tools in soft computing and fuzzy mathematics. First consideration of this kind are connected with development of non-classical logic. Papers of Brouwer (1908) and Heyting (1930) concern conjunction inverse in the form of intuitionistic implication. Moore (1920) introduced and Penrose (1955) egzamined generalized inverse for matrix product. Von Neumann (1936) considered regular elements in semigroups of a ring. Another point of view was introduced by Ward (1937, 1940) as residuation in a lattice. Birkhoff

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(1940) summarized some points of view and introduced notions of Brouwerian lattice and relative pseudocomplement. Ore (1944) and Evarett (1944) considered generalized inverse for composition of functions and introduced name Galois connection.

We review some of these results in terminology of the semigroup theory with consideration of generalized inverses in regular, inverse and ordered semigroups. These results will be illustrated in applications for matrix product and for composition of functions.

At first, we discuss a general situation of regular semigroups and particular situations of idempotent semigroups and inverse semigroups, where the problem of natural order is presented (Sections 2, 3). Next, properties of matrix generalized inverses are summarized (Section 4). Then, we describe pseudo-inverses and Galois connection for functions between ordered sets (Section 5). Finally, the unique semigroup treatment of pseudo inverses is proposed (Section 6).

In our presentation important classed of semigroups are omitted (e.g. cancellation semigroups or positively and negatively ordered semigroups). All these considerations are based on the following monographs: [1], [2], [3], [5], [7], [8], [10], [15], [16], [17], [20].

2 Regular semigroups

The simplest case of inverses appears in regular semigroups (cf. [5], [10]).

Definition 1. Let $(S, *)$ be a semigroup.
• An element $a \in S$ is called regular, if there exist an element $b \in S$ such that $a * b * a = a$. Then b is called a generalized inverse of a. By $G(a)$ we denote the set of all generalized inverses of a.
• The semigroup S is called regular if al its elements are regular.
• The semigroup S is called idempotent if $a^2 = a * a = a$ for $a \in S$. The set of all idempotent elements of the semigroup S is denoted by $E = E_S$.
• Element $b \in G(a)$ is called a quasi-inverse of a, if additionally $b * a * b = b$.
• The regular semigroup S is called an inverse semigroup if any element $a \in S$ has exactly one quasi-inverse. Such inverse is denoted by a^-.
• A regular element $a \in S$ is called completely regular if $a * b = b * a$.
• Let $e \in S$ be the identity element of S. If $a, b \in S$, $a * b = e$, then a is called a left-hand inverse of b and b is called a right-hand inverse of a (or b is left invertible and a is right invertible).

Example 1. Any idempotent element a is regular and $a \in G(a)$ (own quasi-inverse). Thus, every idempotent semigroup is regular. In particular semigroups

$(L, \vee)$ and $(L, \wedge)$ in a lattice $(L, \vee, \wedge)$ are regular semigroups. Moreover, $G(a) = \{b \in L : b \leqslant a\}$ in $(L, \vee)$ and $G(a) = \{b \in L : b \geqslant a\}$ in $(L, \wedge)$ but only $b = a$ is a quasi inverse of a. Therefore, $a^- = a$ in both the above semigroups.

In this section we follow mainly after Chapter 1 in [5].

Lemma 1. *Let $(S, *)$ be a semigroup.*
- *If an element $a \in S$ is regular with a generalized inverse $x \in S$, then elements $e_l = a * x$, $e_r = x * a$ are idempotent and $e_r * a = a$, $a * e_l = a$.*
- *Every regular element has a quasi inverse.*
- *If an element is commuting with a quasi inverse, then there exist a subgroup of S in which this quasi inverse is the inverse.*
- *If an element $a \in S$ is left invertible or a is right invertible, the it is regular.*

Theorem 1. *A regular semigroup is an inverse semigroup if and only if all its idempotent elements are commuting.*

The above theorem generalizes the case of lattice semigroups considered in Example 1.

Theorem 2. *In an inverse semigroup S we have:*

$$(a^-)^- = a, \quad (a * b)^- = b^- * a^- \text{ for } a, b \in S.$$

3 Matrix generalized inverses

The above considerations can be applied to the semigroup of square matrices with fixed order. However, in the case of rectangular matrices the matrix product is a partial operation and we obtain a partial semigroup only. Thus we put here more attention. Presented results can be found in the booklet [20] (cf. also [1], [15]).

Theorem 3. *Any matrix has a generalized-inverse (i.e. every matrix is a regular element).*

Directly form Lemma 1 we obtain

Corollary 1. *Arbitrary matrix A has left and right identities: $E_l = AX$, $E_r = XA$, where $X \in g(A)$. The above identity matrices are idempotent and $E_l \in G(E_l)$, $E_r \in G(E_r)$ (own quasi-inverses).*

Theorem 4. *We consider rectangular matrices $A \in \mathbb{R}^{m \times n}$ and $r(A)$ denotes the matrix rank.*

• *If a matrix A has a left-hand (right-hand) inverse, then all its generalized inverses are also left-hand (right-hand) inverses.*
• *Matrix A has a left inverse if and only if $r(A) = n$.*
• *Matrix A has a right inverse if and only if $r(A) = m$.*
• *A matrix $X \in g(A)$ is a quasi-inverse of A if and only if $r(X) = r(A)$.*

Definition 2. Let $A \in \mathbb{R}^{m \times n}$.
• A generalized inverse X is called left (right) symmetric if $(E_l)^T = E_l$ $((E_r)^T = E_r)$.
• A quasi-inverse X is called Moore-Penrose inverse if it is left and right symmetric, i.e.

$$AXA = A, \quad XAX = X, \quad (AX)^T = AX, \quad (XA)^T = XA.$$

Theorem 5. *Let $X \in g(A)$.*
• *Matrix X is left symmetric if and only if $X^T X \in g(AA^T)$.*
• *Matrix X is right symmetric if and only if $XX^T \in g(A^T A)$.*
• *The Moore-Penrose inverse is unique (is given by a formula) and it has the least Euclidean norm in $g(A)$.*

4 Natural order in a semigroup

In this section we follow mainly after Chapter 11 in [3].

Definition 3. A semigroup with order relation $(S, *, \leqslant)$ is called ordered if its operation is order preserving, i.e.

$$a \leqslant b \Rightarrow (a * c \leqslant b * c,\ c * a \leqslant c * b) \text{ for } a, b, c \in S.$$

In general, there are many order relation in the set S, but only some of them are compatible with the algebraic structure. This leads us to the question if arbitrary semigroup has a compatible order relation. This question is solved under additional assumptions about the semigroup. The simpliest case if for idempotent semigroups, where order relation is modelled on lattice order.

Theorem 6. *Any idempotent semigroup is ordered by the relation:*

$$a \leqslant b \Leftrightarrow (a = a * b = b * a) \text{ for } a, b \in S.$$

Lemma 2. *In arbitrary inverse semigroup S we have:*

$$(a*a^- = a*b^-) \Leftrightarrow (a*a^- = b*a^-) \Leftrightarrow (a^-*a = a^-*b) \Leftrightarrow (a^-*a = b^-*a) \Leftrightarrow$$
$$(a * b^- * a = a) \Leftrightarrow (a^- * b * a^- = a^-)$$

for $a, b \in S$.

Theorem 7. *Any inverse semigroup is ordered by relations from Lemma 2. In particular,*

$$a \leqslant b \Leftrightarrow (a * a^{-} = a * b^{-}) \text{ for } a, b \in S.$$

Theorem 8. *Any regular semigroup has partial order of the form:*

$$a \leqslant b \Leftrightarrow \underset{e,f \in E}{\exists} (a = e * b = b * f) \text{ for } a, b \in S,$$

which need not be compatible with $*$.

Definition 4. The partial order from the above theorems is called natural one.

Theorem 9. *The natural order in the inverse semigroup can be characterized by conditions:*

$$a \leqslant b \Leftrightarrow \underset{e \in E}{\exists} (a = e * b) \Leftrightarrow \underset{f \in E}{\exists} (a = b * f)$$

for $a, b \in S$. *Moreover,*

$$a \leqslant b \Leftrightarrow a^{-} \leqslant b^{-}$$

for $a, b \in S$.

5 Pseudo-inverses of functions

Since composition of functions is associative, we can consider the semigroup S of all the functions $f : X \to X$. However, this operation need not be order preserving (it is left order preserving). Thus we reduce our considerations to monotone functions between two ordered sets. In this section we follow mainly after Chapter 4 in [17].

Definition 5. Let $(X, \leqslant)$ and $(Y, \leqslant)$ be two partially ordered sets.
• We say that an order-preserving function $f : X \to Y$ has a pseudo-inverse if there exists an order-preserving function $h : Y \to X$ fulfilling the inequalities

$$f \circ h \geqslant I_Y, \quad h \circ f \leqslant I_X.$$

• We say that an order-reversing function $f : X \to Y$ has a pseudo-inverse if there exists an order-reversing function $h : Y \to X$ fulfilling the inequalities

$$f \circ h \geqslant I_Y, \quad h \circ f \geqslant I_X.$$

Lemma 3. *There exists at most one function* h *from Definition 5. It is denoted by* $h = f^{+}$.

Theorem 10. *Let* $(X, \leqslant)$ *and* $(Y, \leqslant)$ *be two partially ordered sets.*
• *An order-preserving function* $f : X \to Y$ *has a pseudo-inverse if and only if every set*

$$H_y = \{x : f(x) \geqslant y\}$$

has the least element for $y \in f(X)$*. Then*

$$f^+(y) = \min H_y \textit{ for } y \in F(X).$$

• *An order-reversing function* $f : X \to Y$ *has a pseudo-inverse if and only if every set*

$$H_y = \{x : f(x) \geqslant y\}$$

has the greatest element for $y \in f(X)$*. Then*

$$f^+(y) = \max H_y \textit{ for } y \in F(X).$$

Example 2. Function images and pre-images by an arbitrary function $f : X \to Y$ are order-preserving between 2^X and 2^Y. Then $f^{-1} : 2^Y \to 2^X$ is a pseudo-inverse of $f : 2^X \to 2^Y$, because

$$A \subset f^{-1}(f(A)) \text{ for } A \subset X, \quad f(f^{-1}(B)) = B \text{ for } B \subset Y.$$

Lemma 4. *Pseudo-inverse is a particular case of quasi-inverse with respect to composition of functions, i.e.*

$$f * f^+ * f = f, \quad f^+ * f * f^+ = f^+.$$

Theorem 11. *If functions* $f : X \to Y$*,* $g : Y \to Z$ *have pseudo-inverses in semigroup* $(S, *, e, \leqslant)$*, then also* $f \circ g$ *has pseudo-inverse, and*

$$(a * b)^+ = b^+ * a^+.$$

Corollary 2. *The set of all function* $f : X \to X$ *with pseudo-inverses in an ordered set* X *forms a semigroup (similarly does the set of all pseudo-inverses).*

Theorem 12. *If* $f, g : X \to X$ *have pseudo-inverses, then*

$$f \leqslant g \Rightarrow g^+ \leqslant f^+, \quad ((f^+)^+)^+ = f^+.$$

According to Theorem 7 we see that the pseudo-inverse need not coincide with the quasi-inverse (cf. Lemma 4). The notion of pseudo-inverse is strictly connected with a concept of Galois connection (cf. [8], Chapter 0).

Definition 6. Let $(X, \leqslant)$ and $(Y, \leqslant)$ be two partially ordered sets.
• A Galois connection (f, h) between these sets consists of two order-preserving functions: $f : X \to Y$ and $h : Y \to X$, such that

$$f(x) \leqslant y \Leftrightarrow x \leqslant h(y) \text{ for } x \in X,\ y \in Y.$$

In this situation, f is called the lower adjoint of h and h is called the upper adjoint of f.
• Dually, a Galois connection (f, h) with two order-reversing functions is such that

$$y \leqslant f(x) \Leftrightarrow x \leqslant h(y) \text{ for } x \in X,\ y \in Y.$$

Then f and h are called polarities (because of symmetry).

It can be proved that adjoints or polarities in a Galois connection are quasi-inverses of each other, i.e. $f \circ h \circ f = f$ and $h \circ f \circ h = h$ and one uniquely determines the other (formulas similar to used in Theorem 10). However, they are not left-hand or right-hand inverses of each other. Their usefulness is connected with the following property:

Theorem 13. *Let $(X, \leqslant)$ and $(Y, \leqslant)$ be two complete lattices. Any upper adjoint preserves arbitrary infima, and any lower adjoint preserves arbitrary suprema, i.e.*

$$\underset{t\in T}{inf}\ f(x_t) = f(\underset{t\in T}{inf}\ x_t), \quad \underset{u\in U}{sup}\ h(y_u) = h(\underset{u\in U}{sup}\ y_u),$$

where $x_t \in X$ for $t \in T$ and $y_u \in Y$ for $u \in U$.

6 Appendix. Semigroup pseudo-inverse

We list here generalizations of inverse in diverse domains. Are there other ways of generalizations? We consider it from more general point of view.
Let $(S, *, e, \leqslant)$ be a partially ordered semigroup with identity e. We can generalize invertibility condition $a * b = e$, $b * a = e$ by replacing equalities by inequalities. Formally we have 9 possibilities from $\{\leqslant, =, \geqslant\} \times \{\leqslant, =, \geqslant\}$. Not all of them are useful.

Example 3. Let us consider and increasing operation $*$ and replace $(=, =)$ by $(\leqslant, \leqslant)$. If $a \leqslant e$, then $a * e = a \leqslant e$, $e * a = a \leqslant e$, i.e. a is a generalized inverse of e. Thus such generalization is not correct. Similarly, in the generalization $(\geqslant, \geqslant)$ every $a \geqslant e$ will be a generalized inverse of e, which discredit such generalization. In consequence, the mixed conditions $(\leqslant, \geqslant)$ and $(\geqslant, \leqslant)$ are most interesting.

The following considerations are modelled only on function semigroups from the above section. Thus the last section will be presented with proofs.

Definition 7. Let $(S, *, e, \leqslant)$ be an ordered semigroup with identity e. Element $b \in S)$ is called a pseudo-inverse of $a \in S$, if simultaneously

$$a * b \geqslant e, \quad b * a \leqslant e. \tag{1}$$

Lemma 5. *In an ordered semigroup with identity $(S, *, e, \leqslant)$ element a and b from (1) are regular, i.e. b is a quasi inverse of a.*

Proof. According to (1) we obtain

$$a * b * a \geqslant e * a = a, \quad a * b * a \leqslant a * e = a,$$

which proves that $a * b * a = a$, an similarly we get $b * a * b = b$. □

Lemma 6. *Any element of an ordered semigroup with identity $(S, *, e, \leqslant)$ has at most one pseudo-inverse.*

Proof. Suppose that an element $a \in S$ has two pseudo-inverses $b, c \in S$. Thus $a * c \geqslant e$ and $b * a \leqslant e$, what gives

$$b = b * e \leqslant b * a * c \leqslant e * c = c,$$

i.e. $b \leqslant c$. Similarly we get $c \leqslant b$ what leads uniqueness of pseudo-inverse. □

According to the above lemma we can introduce unique notation $b = a^+$ for pseudo-inverse (similarly as in the case of functions). Thus we have

$$a * a^+ \geqslant e, \quad a^+ * a \leqslant e.$$

Theorem 14. *If $a, b \in S$ have pseudo-inverses in $(S, *, e, \leqslant)$, then also $a * b$ and $b * a$ have pseudo-inverses, and*

$$(a * b)^+ = b^+ * a^+.$$

Proof. Because of uniqueness (Lemma 6), it suffice to check if $a * b$ and $b^+ * a^+$ fulfills conditions from (1). By associativity we get:

$$(a * b) * (b^+ * a^+) = a * (b * b^+) * a^+ \geqslant a * a^+ \geqslant e,$$

$$(b^+ * a^+) * (a * b) = b^+ * (a^+ * a) * b \leqslant b^+ * b \leqslant e,$$

what finishes the proof. □

By analogy to function semigroup we expect

Conjecture 1. If $a, b \in S$ have pseudo-inverses in $(S, *, e, \leqslant)$, then

$$a \leqslant b \Rightarrow b^+ \leqslant a^+, \quad ((a^+)^+)^+ = a^+.$$

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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