Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Developments in Fuzzy Sets,
Intuitionistic Fuzzy Sets,
Generalized Nets and Related Topics
Volume I: Foundations

Editors

Krassimir T. Atanassov
Michał Baczyński
Józef Drewniak
Janusz Kacprzyk
Maciej Krawczak
Eulalia Szmidt
Maciej Wygralak
Sławomir Zadrożny

Editors
Krassimir T. Atanassov
Michał Baczyński
Józef Drewniak
Janusz Kacprzyk
Maciej Krawczak
Eulalia Szmidt
Maciej Wygralak
Sławomir Zadrożny

SRI PAS IBS PAN

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics Volume I: Foundations

Systems Research Institute
Polish Academy of Sciences

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
Volume I: Foundations

Editors

Krassimir T. Atanassov
Michał Baczyński
Józef Drewniak
Janusz Kacprzyk
Maciej Krawczak
Eulalia Szmidt
Maciej Wygralak
Sławomir Zadrożny

IBS PAN SRI PAS

© **Copyright by Systems Research Institute**
Polish Academy of Sciences
Warsaw 2010

All rights reserved. No part of this publication may be reproduced, stored in retrieval system or transmitted in any form, or by any means, electronic, mechanical, photocopying, recording or otherwise, without permission in writing from publisher.

Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
www.ibspan.waw.pl

ISBN 9788389475299

Estimation of the relations: preference, equivalence and tolerance on the basis of pairwise comparisons

Leszek Klukowski
Systems Research Institute Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
Leszek.Klukowski@ibspan.waw.pl

Abstract

The paper presents unified approach to estimation of three relations: preference, equivalence and tolerance on the basis of multiple pairwise comparisons with random errors. The estimates are obtained on the basis of discrete optimization tasks, which minimize inconsistencies between relation form and comparisons. The properties of the estimators are obtained on the basis of well-known probabilistic inequalities and properties of quantiles. The approach allows relaxation of some assumptions about comparison errors applied in other methods based on pairwise comparisons.

Keywords: estimation of preference, equivalence and tolerance relation, pairwise comparisons, binary comparisons, multiple-valued comparisons.

1 Introduction

The paper presents estimators of three relations: preference, equivalence and tolerance, in finite set, based on pairwise comparisons with random errors, developed by the author (see Klukowski 1994, 2006, 2007, 2008a, b, c). They exploit a common idea: to determine an estimate (or estimates), which minimizes the sum of inconsistencies between relation form and pairwise comparisons treated as sample; each pair of elements is compared independently N times. Two types of estimators are examined. The first one is based on total sum of inconsistencies, i.e. with whole set of comparisons. The second type uses medians instead of individual comparisons of each pair. Two forms of comparisons

Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations (*K.T. Atanassow, M. Baczyński, J. Drewniak, J. Kacprzyk, M.Krawczak, E.Szmidt, M.Wygralak, S.Zadrożny****, Eds.), IBS PAN - SRI PAS, Warsaw, 2009.***

are discussed: binary and multiple-valued. Binary comparisons express qualitative features of compared elements, e.g. direction of preference in the case of preference relation. Multiple-valued comparisons express quantitative features, e.g. distance between elements in the case of preference relation. The assumptions about distributions of comparison errors are weaker than those commonly used in other pairwise methods (see David 1998); in particular expected values of errors can differ from zero and the form of distributions may be unknown. It is possible to verify some assumptions about distributions of comparison errors. The properties of estimators are based on well-known probabilistic inequalities and properties of quantiles (medians).

The relations considered are models of real phenomena. The equivalence and tolerance relations can be applied to classification problems – respectively with non-overlapping or overlapping subsets (a decision about relation type – equivalence or tolerance can be made with the use of statistical test – see Klukowski 2006, 2008c). The preference relation can be applied for selecting the best form of regression model in the set examined; comparisons of pairs of models can be obtained on the basis of appropriate statistical tests.

The paper consists of four sections. The second section presents problem formulation and basic definitions and assumptions. The form of estimators and their properties are discussed in the third section. Last section summarizes the results.

2 Problem formulation

The estimation problem can be stated as follows.

Given a finite set of elements $\mathbf{X} = \{x_1, ..., x_m\}$ ($3 \leq m < \infty$). There exists in the set $\mathbf{X}$ the equivalence relation $\mathbf{R}^{(e)}$ (reflexive, transitive, symmetric) or the tolerance relation $\mathbf{R}^{(\tau)}$ (reflexive, symmetric) or the preference relation $\mathbf{R}^{(p)}$ (alternative of the equivalence relation and strict preference relation). Each relation considered generates some family of subsets $\chi_1^{(\ell)*}, ..., \chi_n^{(\ell)*}$ ($\ell \in \{p, e, \tau\}$; $n \geq 2$).

The equivalence relation generates the family $\chi_1^{(e)*}, ..., \chi_n^{(e)*}$ with the following properties:

$$\bigcup_{q=1}^{n} \chi_q^{(e)*} = \mathbf{X}, \tag{1}$$

$$\chi_r^{(e)*} \cap \chi_s^{(e)*} = \{\mathbf{0}\}, \tag{2}$$

where: $\mathbf{0}$ – the empty set,

$x_i, x_j \in \chi_r^{(e)*} \equiv x_i, x_j$ – equivalent elements, (3)

$(x_i \in \chi_r^{(e)*}) \cap (x_j \in \chi_s^{(e)*}) \equiv x_i, x_j$ – non-equivalent elements for $i \neq j, r \neq s$. (4)

The tolerance relation generates the family $\chi_1^{(\tau)*}, ..., \chi_n^{(\tau)*}$ with the property (1) and the properties:

$\exists r, s\ (r \neq s)\ such\ that\ \chi_r^{(\tau)*} \cap \chi_s^{(\tau)*} \neq \{\mathbf{0}\}$,

$x_i, x_j \in \chi_r^{(\tau)*} \equiv x_i, x_j$ – equivalent elements, (5)

$(x_i \in \chi_r^{(\tau)*}) \cap (x_j \in \chi_s^{(\tau)*}) \equiv x_i, x_j$ – non-equivalent elements for $i \neq j$ and $(x_i, x_j) \notin \chi_r^{(\tau)*} \cap \chi_s^{(\tau)*}$, (6)

each subset $\chi_r^{(\tau)*}\ (1 \leq r \leq n)$ includes an element x_i such that $x_i \notin \chi_s^{(\tau)*}\ (s \neq r)$. (7)

The preference relation generates the family $\chi_1^{(p)*}, ..., \chi_n^{(p)*}$ with the properties (1), (2) and the property:

$(x_i \in \chi_r^{(p)*}) \cap (x_j \in \chi_s^{(p)*}) \equiv x_i$ is preferred to x_j for $r<s$. (8)

The relations defined by the conditions (1) - (8) can be described by some functions $T_\upsilon^{(\ell)}(x_i, x_j)$ $((x_i, x_j) \in \mathbf{X} \times \mathbf{X};\ \ell \in \{p, e, \tau\}, \upsilon \in \{b, \mu\})$, where symbols b, μ denote – respectively: binary and multiple-valued set of values. The functions can be defined as follows:

$$T_b^{(e)}(x_i, x_j) = \begin{cases} 0\ if\ exists\ r\ such\ that\ (x_i, x_j) \in \chi_r^{(e)*}, \\ 1\ otherwise; \end{cases} \tag{9}$$

- the function $T_b^{(e)}(x_i, x_j)$ expresses the fact if a pair (x_i, x_j) belong to a common subset or not;

$$T_b^{(\tau)}(x_i, x_j) = \begin{cases} 0\ if\ exists\ r, s\ (r = s\ not\ excluded)\ such\ that \\ \quad (x_i, x_j) \in \chi_r^{(\tau)*} \cap \chi_s^{(\tau)*}, \\ 1\ otherwise; \end{cases} \tag{10}$$

- the function $T_b^{(\tau)}(x_i, x_j)$ expresses the fact if a pair (x_i, x_j) belong to any conjunction of subsets (also to the same subset) or not; the condition (7) guarantee uniqueness of the description;

$$T_\mu^{(\tau)}(x_i, x_j) = \#(\Omega_i^* \cap \Omega_j^*), \tag{11}$$

where:

Ω_l^* - the set of the form $\Omega_l^* = \{s \mid x_l \in \chi_s^{(\tau)*}\}$,

$\#(\Xi)$ - number of elements of the set Ξ;

- the function $T_\mu^{(\tau)}(x_i, x_j)$ expresses the number of subsets of conjunction including both elements; the condition (7) guarantee uniqueness of the description;

$$T_b^{(p)}(x_i, x_j) = \begin{cases} 0 \textit{ if there exists } r \textit{ such that } (x_i, x_j) \in \chi_r^{(p)*}, \\ \mp 1 \textit{ if } x_i \in \chi_r^{(p)*}, x_j \in \chi_s^{(p)*} \textit{ and } r < s \ (r > s); \end{cases} \tag{12}$$

(more precisely – the value -1 reflects the case $r<s$, 1 – the case $r>s$);

- the function $T_b^{(p)}(x_i, x_j)$ expresses direction of preference in a pair or equivalency of its elements;

$$T_\mu^{(p)}(x_i, x_j) = d_{ij} \Leftrightarrow x_i \in \chi_r^{(p)*}, x_j \in \chi_s^{(p)*}, d_{ij} = r - s; \tag{13}$$

- the function $T_\mu^{(p)}(x_i, x_j)$ expresses difference of ranks of elements x_i and x_j.

The estimation problem

The relation form has to be estimated on the basis of N ($N \geq 1$) comparisons of each pair $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$, under the following assumptions.

A1. The relation type, i.e.: equivalence or tolerance or preference, is known.

A2. Each pair of elements $(x_i, x_j) \in \mathbf{X} \times \mathbf{X}$ is compared N times; any comparison $g_{\upsilon k}^{(\ell)}(x_i, x_j)$ ($\ell \in \{e, \tau, p\}$; $\upsilon \in \{b, \mu\}$; $k = 1, ..., N$) evaluates the value of the function $T_\upsilon^{(\ell)}(x_i, x_j)$ and can be disturbed by random error. The probabilities of errors $T_\upsilon^{(\ell)}(x_i, x_j) - g_{\upsilon k}^{(\ell)}(x_i, x_j)$ have to satisfy the following conditions:

$$P(T_b^{(\ell)}(x_i,x_j) - g_{bk}^{(\ell)}(x_i,x_j) = 0) \geq 1-\delta \quad (\delta \in (0, {}^1\!/_2)), \tag{14}$$

$$\sum_{l\leq 0} P(T_\mu^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) = l) > {}^1\!/_2 , \tag{15}$$

$$\sum_{l\geq 0} P(T_\mu^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) = l) > {}^1\!/_2 , \tag{16}$$

$$P(T_\mu^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) = l) \geq P(T_\mu^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) = l+1) \quad (l > 0), \tag{17}$$

$$P(T_\mu^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) = l) \geq P(T_\mu^{(\ell)}(x_i,x_j) - g_{\mu k}^{(\ell)}(x_i,x_j) = l-1) \quad (l < 0), \tag{18}$$

$$\begin{aligned} &P((g_{\upsilon k}^{(\ell)}(x_i,x_j) = T_\upsilon^{(\ell)}(x_i,x_j)) \cap (g_{\upsilon l}^{(\ell)}(x_r,x_s) = T_\upsilon^{(\ell)}(x_r,x_s))) = \\ &P(g_{\upsilon k}^{(\ell)}(x_i,x_j) = T_\upsilon^{(\ell)}(x_i,x_j)) P(g_{\upsilon l}^{(\ell)}(x_r,x_s) = T_\upsilon^{(\ell)}(x_r,x_s)) \quad (k \neq l). \end{aligned} \tag{19}$$

The inequalities (14) – (19) reflect the following properties of distributions of comparison errors:

- the probability of correct comparison is greater than incorrect one, in the case of binary comparisons (the inequality (14));
- zero is the median of each distribution of comparison error (the inequalities (14) – (16)),
- zero is the mode of each distribution of comparison error (the inequalities (14) – (18));
- *k*-th and *l*-th ($k \neq l$) comparisons $g_{\upsilon k}^{(\ell)}(x_i,x_j)$, $g_{\upsilon l}^{(\ell)}(x_r,x_s)$ are independent (the relationship (19).

It is clear that:

- expected value of *any* comparison error can differ from zero,
- *comparisons* $g_{\upsilon k}^{(\ell)}(x_i,x_j)$, $g_{\upsilon k}^{(\ell)}(x_r,x_s)$ (indexes: *i*, *j* not the same as: *r*, *s*) are not assumed independent;
- number *n* of *subsets* of any family $\chi_1^{(\ell)*},..., \chi_n^{(\ell)*}$ is not assumed known; therefore it is also estimated parameter;
- the *application* of estimators proposed below is possible in the case of unknown distributions of comparison errors; it is sufficient that the assumptions about the distributions are satisfied.

The assumptions about estimation problem are weaker than those commonly used in literature. Determining properties of estimates require the form of distributions of comparison errors, which can be unknown. In the case of appropriate value of N, at least several, the distributions can be estimated. In opposite case the properties can be roughly evaluated with the use of simulation approach (see Klukowski 2008b, Section 5).

3 The form of estimators

Two types of estimators are presented in the paper. The first one is based on total sum of inconsistencies between relation form (expressed by the function $T_{\upsilon}^{(\ell)}(x_i, x_j)$) and comparisons $g_{\upsilon k}^{(\ell)}(x_i, x_j)$, the second – based on inconsistencies between relation form and medians from comparisons of each pair. The properties of both estimators are determined under assumption that distributions of comparisons $g_{\upsilon,1}^{(\ell)}(x_i, x_j), ..., g_{\upsilon N}^{(\ell)}(x_i, x_j)$ $((x_i, x_j) \in \mathbf{X} \times \mathbf{X})$ are the same. The assumption simplifies formulas and can be relaxed (see Klukowski 1994, 2007, 2008b).

3.1 Estimator based on total sum of inconsistencies

The form of the estimator based on the total sum of inconsistencies results from the properties of the random variable:

$$\sum_{<i,j> \in R_m} \sum_{k=1}^{N} \left| T_{\upsilon}^{(\ell)}(x_i, x_j) - g_{\upsilon k}^{(\ell)}(x_i, x_j) \right|, \tag{20}$$

where: R_m - the set of indices $R_m = \{<i, j> \mid 1 \le i, j \le m; \ j > i\}$, corresponding to the actual relation $\chi_1^{(\ell)*}, ..., \chi_n^{(\ell)*}$ and the variable:

$$\sum_{<i,j> \in R_m} \sum_{k=1}^{N} \left| \tilde{t}_{\upsilon}^{(\ell)}(x_i, x_j) - g_{\upsilon k}^{(\ell)}(x_i, x_j) \right|, \tag{21}$$

corresponding to some relation $\tilde{\chi}_1^{(\ell)}, ..., \tilde{\chi}_{\tilde{n}}^{(\ell)}$, different than actual, described by the function $\tilde{t}_{\upsilon}^{(\ell)}(x_i, x_j)$ - defined similarly as $T_{\upsilon}^{(\ell)}(x_i, x_j)$.

The variables (20) and (21) satisfy the following inequalities:

$$E\Big(\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|T_{\upsilon}^{(\ell)}(x_i,x_j)-g_{\upsilon k}^{(\ell)}(x_i,x_j)\Big|\Big)<E\Big(\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|\tilde{t}_{\upsilon}^{(\ell)}(x_i,x_j)-g_{\upsilon k}^{(\ell)}(x_i,x_j)\Big|\Big)$$
$$(\ell\in\{e,\tau,p\},\upsilon\in\{b,\mu\}), \qquad (22)$$

$$P\Big(\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|T_{b}^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)\Big|<\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|\tilde{t}_{b}^{(\ell)}(x_i,x_j)-g_{bk}^{(\ell)}(x_i,x_j)\Big|\Big)\geq$$
$$1-\exp\{-2N(\tfrac{1}{2}-\delta)^2\} \qquad (\ell\in\{e,\tau,p\}),$$
$$(23)$$

$$P\Big(\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|T_{\mu}^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)\Big|<\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|\tilde{t}_{\mu}^{(\ell)}(x_i,x_j)-g_{\mu k}^{(\ell)}(x_i,x_j)\Big|\Big)\geq$$
$$1-$$
$$\exp\{-2N\frac{\Big(\sum_{T_{\mu}^{(\ell)}(\cdot)\neq\tilde{t}_{\mu}^{(\ell)}(\cdot)}E\Big(\Big|T_{\mu}^{(\ell)}(x_i,x_j)-g_{\mu,1}^{(\ell)}(x_i,x_j)\Big|-\Big|\tilde{t}_{\mu}^{(\ell)}(x_i,x_j)-g_{\mu,1}^{(\ell)}(x_i,x_j)\Big|\Big)\Big)^2}{(2\vartheta(m-1))^2}\}$$
$$(\ell\in\{e,\tau,p\}), \qquad (24)$$

where:

$T_{\mu}^{(\ell)}(\cdot)\neq\tilde{t}_{\mu}^{(\ell)}(\cdot)$ denotes the set

$\{<i,j>\ \big|\ T_{\mu}^{(\ell)}(x_i,x_j)\neq\tilde{t}_{\mu}^{(\ell)}(x_i,x_j);(x_i,x_j)\in\mathbf{X}\times\mathbf{X};\ j>i\}$,

ϑ - the number of elements of the set $T_{\mu}^{(\ell)}(\cdot)\neq\tilde{t}_{\mu}^{(\ell)}(\cdot)$.

The inequality (22) shows that the expected value of sum of inconsistencies $\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|T_{\upsilon}^{(\ell)}(x_i,x_j)-g_{\upsilon k}^{(\ell)}(x_i,x_j)\Big|$, corresponding to the actual relation $\chi_1^{(\ell)*},...,\chi_n^{(\ell)*}$ (the errorless estimate), is lower than the expected value of sum $\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|\tilde{t}_{\upsilon}^{(\ell)}(x_i,x_j)-g_{\upsilon k}^{(\ell)}(x_i,x_j)\Big|$, corresponding to any other relation $\tilde{\chi}_1^{(\ell)},...,\tilde{\chi}_{\tilde{n}}^{(\ell)}$. Moreover, the probability of the event

$$\{\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|T_{\upsilon}^{(\ell)}(x_i,x_j)-g_{\upsilon k}^{(\ell)}(x_i,x_j)\Big|<\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\Big|\tilde{t}_{\upsilon}^{(\ell)}(x_i,x_j)-g_{\upsilon k}^{(\ell)}(x_i,x_j)\Big|\}$$

satisfies the inequality (23), in the case $\upsilon=b$, and the inequality (24), in the case $\upsilon=\mu$. The right-hand sides of both inequalities include the terms converging exponentially to one for $N\to\infty$.

The inequalities (22) – (24) indicate the form of the estimators based on total sum of inconsistencies: to determine the relation $\hat{\chi}_1^{(\ell)},...,\hat{\chi}_{\hat{n}}^{(\ell)}$, which minimizes the sum of inconsistencies $\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|t_\upsilon^{(\ell\kappa)}(x_i,x_j)-g_{\upsilon k}^{(\ell)}(x_i,x_j)\right|$, where: $t_\upsilon^{(\ell\kappa)}(x_i,x_j)$ the function describing κ-th relation form (their number is finite) of ℓ-th type in the set **X**. Thus, the minimization task assumes the form:

$$\min_{\chi_1^{(\ell\kappa)},...,\chi_{r(\kappa)}^{(\ell\kappa)}\in F_X^{(\ell)}}\left\{\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|t_\upsilon^{(\ell\kappa)}(x_i,x_j)-g_{\upsilon k}^{(\ell)}(x_i,x_j)\right|\right\}, \tag{25}$$

where:
$F_X^{(\ell)}$ ($\ell\in\{e,\tau,p\}$) - the feasible set (the family of all relations of ℓ–th type),
$\chi_1^{(\ell\kappa)},...,\chi_{r(\kappa)}^{(\ell\kappa)}$ - κ-th element of the feasible set.

The solution of the task (25) may be not unique, because of its discrete form; the unique solution can be selected in random way or with the use of additional criterion.

Determination of the right-hand side of the inequality (23) requires only the value of δ. Determination of the right-hand side of the inequality (24) is more cumbersome. It requires the form of distributions of comparison errors - actual or estimated; estimation is efficient in the case of appropriate N – at least several. In the case of unknown distributions (actual or estimated) it is possible to obtain some rough evaluation of the right-hand, with the use of simulation approach (see Klukowski 2008b).

3.2 Estimator based on inconsistencies with medians from comparisons

The estimator based on medians from comparisons of each pair is applicable in the case of uneven N, i.e. N=2ω+1 (ω=0, 1, …,). The median $g_\upsilon^{(\ell,me)}(x_i,x_j)$ ($(x_i,x_j)\in\mathbf{X}\times\mathbf{X}$) is the value $g_{\upsilon((N+1)/2)}^{(\ell,me)}(x_i,x_j)$, where symbols $g_{\upsilon(1)}^{(\ell,me)}(x_i,x_j)$, …, $g_{\upsilon(N)}^{(\ell,me)}(x_i,x_j)$ denote comparisons $g_{\upsilon,1}^{(\ell)}(x_i,x_j),...,g_{\upsilon N}^{(\ell)}(x_i,x_j)$ ordered in non-decreasing manner. Replacing N comparisons of each pair by their median leads to the problem similar to those discussed in point 3.1, but with single comparison of each pair. However, the distribution of each median $g_\upsilon^{(\ell,me)}(x_i,x_j)$ and comparison error $T_\upsilon^{(\ell)}(x_i,x_j)-g_\upsilon^{(\ell,me)}(x_i,x_j)$ is not the same as the distribution of k-th comparison $g_{\upsilon k}^{(\ell)}(x_i, x_j)$ and error $T_{\upsilon}^{(\ell)}(x_i, x_j) - g_{\upsilon k}^{(\ell)}(x_i, x_j)$ (1≤k≤N; N≥3).

In the case of binary comparisons $g_b^{(\ell)}(x_i, x_j)$ $(\ell \in \{e, \tau, p\})$ the probabilities $P(T_b^{(\ell)}(x_i, x_j) - g_b^{(\ell,me)}(x_i, x_j) = 0)$ $((x_i, x_j) \in \mathbf{X} \times \mathbf{X})$ can be determined directly on the basis of binomial distribution or its evaluations (see Klukowski (1994), Section 5). They converge to one for $N \to \infty$ (because $\delta \in (0, \frac{1}{2})$).

In the case of multiple-valued comparisons the probabilities $P(T_{\mu}^{(\ell)}(x_i, x_j) - g_{\mu}^{(\ell,me)}(x_i, x_j) = 0)$ $((x_i, x_j) \in \mathbf{X} \times \mathbf{X})$ assume the form (see David 1970, Klukowski 2007, 2008b):

$$P(T_{\mu}^{(\ell)}(x_i, x_j) - g_{\mu}^{(\ell,me)}(x_i, x_j) = l) = \frac{N!}{(((N-1)/2)!)^2} \int_{G(l-1)}^{G(l)} t^{(N-1)/2} (1-t)^{(N-1)/2} dt, \tag{26}$$

where: $G(l) = P(T_{\mu}^{(\ell)}(x_i, x_j) - g_{\mu k}^{(\ell)}(x_i, x_j) \leq l)$.

The probability (26) is determined on the basis of beta distribution $B(p, q)$ with parameters $p = q = (N+1)/2$ (see David 1970). The expected value and variance of the random variable with such distribution assume the form, respectively: $\frac{1}{2}$ and $\frac{1}{4(N+2)}$. The assumptions about distributions of comparison errors $T_{\mu}^{(\ell)}(x_i, x_j) - g_{\mu k}^{(\ell)}(x_i, x_j)$ (see (14) – (19)) guarantee convergence of the probabilities $P(T_{\mu}^{(\ell)}(x_i, x_j) - g_{\mu}^{(\ell,me)}(x_i, x_j) = 0)$ $((x_i, x_j) \in \mathbf{X} \times \mathbf{X})$ to one for $N \to \infty$. The convergence of distributions of comparison errors $P(T_{\upsilon}^{(\ell)}(x_i, x_j) - g_{\upsilon}^{(\ell,me)}(x_i, x_j) = 0)$ $(\upsilon \in \{b, \mu\})$, i.e. binary and multiple-valued, indicates the inequalities:

$$E\Big(\sum_{<i,j> \in R_m} \left| T_{\upsilon}^{(\ell)}(x_i, x_j) - g_{\upsilon}^{(\ell,me)}(x_i, x_j) \right|\Big) < E\Big(\sum_{<i,j> \in R_m} \left| \tilde{t}_{\upsilon}^{(\ell)}(x_i, x_j) - g_{\upsilon}^{(\ell,me)}(x_i, x_j) \right|\Big), \tag{27}$$

$$\lim_{N \to \infty} E\Big(\sum_{<i,j> \in R_m} \left| T_{\upsilon}^{(\ell)}(x_i, x_j) - g_{\upsilon}^{(\ell,me)}(x_i, x_j) \right|\Big) < \lim_{N \to \infty} E\Big(\sum_{<i,j> \in R_m} \left| \tilde{t}_{\upsilon}^{(\ell)}(x_i, x_j) - g_{\upsilon}^{(\ell,me)}(x_i, x_j) \right|\Big) \quad (\ell \in \{e, \tau, p\};\ \upsilon \in \{b, \mu\}). \tag{28}$$

These results indicate the estimator form in the case of medians from comparisons:

$$\min_{\chi_1^{(\ell\kappa)},\ldots,\chi_{r(\kappa)}^{(\ell\kappa)}\in F_X^{(\ell)}} \{ \sum_{<i,j>\in R_m} \left| t_\upsilon^{(\ell\kappa)}(x_i,x_j) - g_\upsilon^{(\ell,me)}(x_i,x_j) \right| \} \qquad (\ell \in \{e,\tau,p\}), \quad (29)$$

The task (29) is, for N>1, simpler than the task (25), because includes $\frac{m(m-1)}{2}$ decision variables, while the task (25) includes $N\frac{m(m-1)}{2}$ variables. The solution of the task (29) can be also not unique.

In the case of binary comparisons it can be determined the evaluation of probability of the event (see Klukowski 1994, Section 5): $\{ \sum_{<i,j>\in R_m} \left| T_b^{(\ell)}(x_i,x_j) - g_b^{(\ell,me)}(x_i,x_j) \right| < \sum_{<i,j>\in R_m} \left| \tilde{t}_b^{(\ell)}(x_i,x_j) - g_b^{(\ell,me)}(x_i,x_j) \right|) \}$, which converges to one, for $N\to\infty$.

It assumes the form:

$$P(\sum_{<i,j>\in R_m} \left| T_b^{(\ell)}(x_i,x_j) - g_b^{(\ell,me)}(x_i,x_j) \right| < \sum_{<i,j>\in R_m} \left| \tilde{t}_b^{(\ell)}(x_i,x_j) - g_b^{(\ell,me)}(x_i,x_j) \right|) \geq$$
$$1 - 2\exp\{-2N(\tfrac{1}{2}-\delta)^2\} \qquad (\ell \in \{e,\tau,p\}). \quad (30)$$

The comparison of efficiency of estimators (25) and (29), in analytic way, seems difficult and can be performed with the use of simulation approach.

4 Conclusions

The paper presents the unified approach to estimation of the equivalence relation, tolerance relation and preference relation on the basis of multiple pairwise comparisons with random errors. They are based on the idea of minimization of inconsistencies between relation form and comparisons. Two types of estimators are considered: the first one based on total sum of inconsistencies and the second – based on medians from each pair comparisons. Two kinds of comparisons can be used – binary and multiple-valued. The important feature of the estimators proposed are weak assumptions about distributions of comparison errors. The estimators based on total sum of inconsistencies have good statistical properties, in particular the convergence of the probabilities (23), (24) to one, for $N\to\infty$. The estimators based on medians are simpler than those based on individual comparisons and also posses asymptotic properties - expressed by inequalities (26), (28). Some properties of the estimators are difficult to determine in analytic way; they have to be determined with the use of simulation approach.

Acknowledgment

The work has been supported by the grant No N N111434937 of the Polish Ministry of Science and Higher Education

References

[1] David H. A. (1970) Order Statistics. J. Wiley, New York

[2] David H. A. (1988) The Method of Paired Comparisons, 2nd ed. Ch. Griffin, London.

[3] Klukowski L. (1994) Some probabilistic properties of the nearest adjoining order method and its extensions. *Annals of Operational Research*, **51**, 241 – 261.

[4] Klukowski L. (2006) Tests for relation type - equivalence or tolerance - in finite set of elements. *Control and Cybernetics*, **35**, 369 - 384.

[5] Klukowski L. (2007) Estimation of tolerance relation the basis of multiple pairwise comparisons with random errors. *Control and Cybernetics*, **36**, 443-466.

[6] Klukowski L. (2008a) Determination of tolerance relation – alternative approach to intuitionistic and fuzzy sets. Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets an Related Topics. Vol. II. K. Atanassov (et al), eds. Ac. Publ. House EXIT, Warsaw.

[7] Klukowski L. (2008b) Estimation of the preference relation the basis of multiple pairwise comparisons in the form of differences of ranks. *Control and Cybernetics*, **37**, 711-729.

[8] Klukowski L. (2008c) Tests for Relation Type - Equivalence or Tolerance – Based on Averaging of Multiple Binary Comparisons. In: Development In Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Foundations. Vol. I. K. Atanassov (et al), eds. Ac. Publ. House EXIT, Warsaw.

[9] Slater P. (1961) Inconsistencies in a schedule of paired comparisons. *Biometrika*, **48**, 303 - 312.

The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) organized in Warsaw on October 16, 2009 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Centre for Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT – Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bistrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

http://www.ibspan.waw.pl/ifs2009

The Eighth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2009) has been meant to commence a new series of scientific events primarily focused on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Moreover, other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems are discussed.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

ISBN-13 9788389475299
ISBN 838947529-4
9 788389 475299