Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Fuzzy logical connectives in approximate reasoning. Dual operations

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Abstract

This paper concerns applications of Duality Principle in fuzzy algebra, fuzzy logics and approximate reasoning. Usually dual definitions, theorems, proofs and examples are neglected, omitted or left as a simple exercise. They are treated as an image in mirror. However, such opinion can be quite false, because duality is an equivalence relation. Thus both dual objects are equally important or non important. It can be easy seen in the case of linear programming: one handbook describes mainly maximization of linear functionals, while another concentrates on minimization. We tray to fill up this gap in treating of dual objects by presentation of examples of important dual notions and properties from lattice theory to relation theory and from fuzzy logic to fuzzy relational equations. In particular we describe properties of dual connectives of multivalued logic and summarize properties of dual relation compositions. Finally, a similarity between fuzzy coimplication and poset difference is analyzed.

Keywords: Duality Principle, lattice duality, dual negation, dual conjunction, coimplication, dual relation composition, dual relation equation, difference.

1 Introduction

Approximate reasoning in fuzzy environment initiated by Zadeh [26] is a motivation of many algebraical and logical researches. In particular, algebra of fuzzy

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relations [15], fuzzy relation equations [23] or algebra of fuzzy numbers [14], provide answers for diverse questions of fuzzy reasoning. Similarly, axiomatizations of fuzzy connectives by Baldwin and Pilsworth [2] or Magrez and Smets [24] were based on needs of inference rules. Duality is a fundamental notion of lattice theory [3]. However, another versions of duality appears in linear algebra, Fourier analysis, linear programming, nonlinear optimization, mathematical linguistic, relation theory, category theory, graph theory, game theory or quantum field theory. Recently, approximate reasoning with interval-valued fuzzy sets [16] and intuitionistic fuzzy sets [4] needs attention to dual operations and properties in fuzzy algebra and fuzzy logic. This paper is concentrated around precise formulation of diverse consequences of Duality Principle. We pay special attention to applications of Duality Principle in fuzzy algebra, fuzzy logics and fuzzy relation algebra. At first, we describe basic consequences of Duality Principle in complete and infinitely distributive lattices (Sections 2, 3). Next, dual algebraic and functional properties of unary and binary connectives of fuzzy logic are examined (Sections 4 -7). Then, dual relation algebras based on dual relation compositions are presented. Finally, axioms of difference are compared with axioms of coimplication.

2 Duality in lattices

Let $(L, \leqslant)$ be a partially ordered set (poset). It is called a lattice if there exist extremal bounds

$$a \vee b = \sup\{a, b\},\ a \wedge b = \inf\{a, b\} \in L.$$

Thus (cf. [3], Chapter I), the lattice $(L, \leqslant)$ is an algebraic structure $(L, \vee, \wedge)$ with substructures: a join semilattice $(L, \vee)$ and a meet semilattice $(L, \wedge)$. Lattice operations $\vee$ and $\wedge$ are idempotent, commutative, associative and fulfil the absorption laws:

$$a \vee (a \wedge b) = a,\ a \wedge (a \vee b) = a \text{ for } a, b \in L.$$

The lattice is bounded if there exist $1 = \max L$, and $0 = \min L$, i.e.

$$a \vee 0 = 0 \vee a = a,\ a \wedge 1 = 1 \wedge a = a \text{ for } a \in L.$$

The lattice is complete if every nonempty subset $A \subset L$ has bounds $\sup A$, $\inf A \in L$. The lattice is called distributive if its binary operations are mutually distributive, i.e.

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ for } a, b, c \in L.$$

The lattice $(L, \vee, \wedge, ', 0, 1)$ with a unary operation $' : L \to L$ is called complemented if

$$(a')' = a,\ a \leqslant b \Rightarrow b' \leqslant a' \text{ for } a, b \in L.$$

At first, let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Because of symmetry in the algebraic lattice properties we get an isomorphism between semilattices $(L, \vee, 0, \leqslant)$ and $(L, \wedge, 1, \geqslant)$. Thus every property from one semilattice has its dual property in the other semilattice. For example

$$a \leqslant a \vee b \Leftrightarrow a \geqslant a \wedge b,\ a \vee 0 = a \Leftrightarrow a \wedge 1 = a,$$

$$(a \leqslant c,\ b \leqslant c \Rightarrow a \vee b \leqslant c) \Leftrightarrow (a \geqslant c,\ b \geqslant c \Rightarrow a \wedge b \geqslant c)$$

for $a, b, c \in L$. Now, if $(L, \vee, \wedge, ', 0, 1)$ is a complemented lattice (or Boolean algebra), then we also have

$$(a \vee b)' = (a' \wedge b') \Leftrightarrow (a \wedge b)' = (a' \vee b'),$$

$$a \vee a' = 1 \Leftrightarrow a \wedge a' = 0,\ a \to b = a' \vee b \Leftrightarrow a \leftarrow b = a' \wedge b$$

for $a, b \in L$. The above properties can be summarized in a general rule called 'Duality Principle'.

Theorem 1 (Duality Principle, cf. [5], p. 76)**.** *Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Replacing $\leqslant$ by $\geqslant$, $\vee$ by $\wedge$, 0 by 1, and conversely in a lattice statement, we obtain the equivalent one.*

This principle is a very useful tool in mathematical considerations because many proofs can be omitted as dual. In extremal thrift whole chapters can be omitted as dual results (an exercise for the reader). Simultaneously, in applications we need precise definitions, theorems and examples, and many people try to deduce omitted statements.

Example 1. Besides primary dual pairs $(\leqslant, \geqslant)$, $(\vee, \wedge)$, $(0, 1)$ from Duality Principle, we have many secondary dual pairs such as: (above, below), (left, right), (maximum, minimum), (full, empty), (positive, negative), (odd, even), (supremum, infimum), (maximal element, minimal element), (upper bound, lower bound), (the greatest element, the least element), (convex, concave), (contraction, expansion), (restriction, extension), (limes superior, limes inferior), (upper semicontinuity, lower semicontinuity), (left continuity, right continuity), (minimization, maximization), (closure, interior), (closed, open), (ideal, filter), (subdistributivity, superdistributivity), (real, imaginary), (abscissa, ordinate), (perpendicular, parallel) etc.

3 Duality in infinitely distributive lattices

Now we consider more general case: a lattice with additional binary operation $* : L^2 \to L$.

Definition 1. Let $(L, \vee, \wedge, *, 0, 1)$ be a complete lattice with an additional binary operation $* : L^2 \to L$.
• The operation $*$ is infinitely distributive with respect to supremum (infinitely $\sup -$distributive) if it fulfils

$$\underset{a,b_t \in L}{\forall} \; a * (\sup_{t\in T} b_t) = \sup_{t\in T}(a * b_t), \qquad \underset{a,b_t \in L}{\forall} \; (\sup_{t\in T} b_t) * a = \sup_{t\in T}(b_t * a). \tag{1}$$

for arbitrary index set $T \neq \emptyset$.
• The operation $*$ is infinitely distributive with respect to infimum (infinitely $\inf -$distributive) if it fulfils

$$\underset{a,b_t \in L}{\forall} \; a * (\inf_{t\in T} b_t) = \inf_{t\in T}(a * b_t), \qquad \underset{a,b_t \in L}{\forall} \; (\inf_{t\in T} b_t) * a = \inf_{t\in T}(b_t * a). \tag{2}$$

for arbitrary index set $T \neq \emptyset$.
• The complete lattice L is called infinitely $\sup -$distributive if the operation $* = \wedge$ is infinitely $\sup -$distributive and it is infinitely $\inf -$distributive if the operation $* = \vee$ is infinitely $\inf -$distributive.
• The complete lattice L is called infinitely distributive if it is infinitely $\sup -$ and $\inf -$distributive.

Corollary 1. *Let L be a complete lattice with an additional binary operation $*$.*
• *If the operation $*$ is infinitely $\sup -$distributive, then it is distributive with respect to $\vee$.*
• *If the operation $*$ is infinitely $\inf -$distributive, then it is distributive with respect to $\wedge$.*
• *In both the above cases the operation $*$ is increasing.*

Proof. Let $a, b, c \in L$. Putting $T = \{1, 2\}$, $b_1 = b$, $b_2 = c$ in (1) we get

$$a * (b \vee c) = (a * b) \vee (a * c), \; (b \vee c) * a = (b * a) \vee (c * a). \tag{3}$$

Thus the operation $*$ is distributive with respect to $\vee$. Dually, using (2) we get distributivity with respect to $\wedge$:

$$a * (b \wedge c) = (a * b) \wedge (a * c), \; (b \wedge c) * a = (b * a) \wedge (c * a). \tag{4}$$

Now, if $b \leqslant c$, then $b \vee c = c$. Using (3) we have

$$(a * b) \vee (a * c) = (a * c),\ (b * a) \vee (c * a) = (c * a)$$

which proves that $(a * b) \leqslant (a * c)$ and $(b * a) \leqslant (c * a)$, i.e. the operation $*$ is increasing. Similarly, it is increasing in the case of (4). $\square$

According to [19] (Proposition 1.22), in the case $L = [0, 1]$ we have a characterization of infinite distributivity.

Lemma 1. *An operation* $* : [0, 1]^2 \to [0, 1]$ *is infinitely* $\sup -$*distributive if and only if it is increasing and left-continuous. Dually, it is infinitely* $\inf -$*distributive if and only if it is increasing and right-continuous.*

4 Duality in fuzzy logics

Diversity of fuzzy logic connectives was described in details in monographs [11] and [17]. Connectives of fuzzy logics are unary and binary functions: $U : [0, 1] \to [0, 1]$, $B : [0, 1]^2 \to [0, 1]$. Unary operations can be continuous, monotonic, convex, idempotent (involutive) or has fixed points. ¿From the algebraic point of view, binary operations can be commutative, associative, cancellable, invertible, idempotent, with zero element (null), with zero divisors or with neutral element (identity). ¿From the point of view of mathematical analysis, binary operations can be continuous, monotonic, Lipschitz or Archimedean. In particular we deal with simple Lipschitz condition (B is $C-$Lipschitz)

$$\underset{C>0}{\exists}\ |B(x, y) - B(u, v)| \leqslant C(|x - u| + |y - v|) \text{ for } x, y, u, v \in [0, 1] \qquad (5)$$

and Archimedean conditions for continuous operations (cf. [17], Proposition 5.1.12)

$$\underset{x\in(0,1)}{\forall}\ B(x, x) < x \text{ or } \underset{x\in(0,1)}{\forall}\ B(x, x) > x. \qquad (6)$$

The standard concept of duality in fuzzy logics is connected with the Łukasiewicz negation

$$x' = 1 - x,\ x \in [0, 1]. \qquad (7)$$

The dual operations have the form:

$$U'(x) = 1 - U(1 - x),\ B'(x, y) = 1 - B(1 - x, 1 - y),\ x, y \in [0, 1]. \qquad (8)$$

Since $([0, 1], \max, \min, ', 0, 1)$ is a complemented lattice, then we can use Duality Principle for describing properties of dual operations. By direct verification of equations and inequalities for unary operations in real domain we get

Lemma 2. *Let $U : [0,1] \to [0,1]$, $A \subset [0,1]$, $1 - A = \{1 - a : a \in A\}$.*
- *U is increasing, if and only if U' is increasing.*
- *U is decreasing, if and only if U' is decreasing.*
- *U is continuous, if and only if U' is continuous.*
- *U is upper (lower) continuous, if and only if U' is lower (upper) continuous.*
- *U is left- (right-) continuous, if and only if U' is right- (left-) continuous.*
- *U is convex (concave), if and only if U' is concave (convex).*
- *$U(x) \leqslant x$, $x \in A \Leftrightarrow U'(x) \geqslant x$, $x \in 1 - A$.*
- *$U(x) \geqslant x$, $x \in A \Leftrightarrow U'(x) \leqslant x$, $x \in 1 - A$.*
- *U has a fixed point s, if and only if U' has a fixed point $1 - s$.*
- *$U(x) > 0$, $x \in A \Leftrightarrow U'(x) < 1$, $x \in 1 - A$.*
- *$U(x) < 1$, $x \in A \Leftrightarrow U'(x) > 0$, $x \in 1 - A$.*
- *$U(U(x)) \leqslant x$, $x \in A \Leftrightarrow U'(U'(x)) \geqslant x$, $x \in 1 - A$.*
- *$U(U(x)) \geqslant x$, $x \in A \Leftrightarrow U'(U'(x)) \leqslant x$, $x \in 1 - A$.*
- *U is an involution, if and only if U' is an involution.*
- *Boundary values:*

x	*0*	*1*
$U'(x)$	$1 - U(1)$	$1 - U(0)$

Remark 1. Let us observe the difference between properties of monotonicity and convexity. For example let us assume that operation U is increasing, i.e. $x \leqslant y \Rightarrow U(x) \leqslant U(y)$. By Duality Principle we obtain the condition $x \geqslant y \Rightarrow U'(x) \geqslant U'(y)$, which describes the same kind of monotonicity. Now let U be a convex function, i.e. $U(tx + (1-t)y) \leqslant tU(x) + (1-t)U(y)$. By Duality Principle we obtain $U'(tx + (1-t)y) \geqslant tU'(x) + (1-t)U'(y)$, which describes the dual kind of convexity.

In the case of binary operations the function (7) is an example of isomorphism between $([0,1], B)$ and $([0,1], B')$. Thus it saves typical algebraic properties of binary operations (cf. [13]) and we obtain

Lemma 3. *Let $B : [0,1]^2 \to [0,1]$.*
- *B is associative, commutative or idempotent, if and only if B' has the same properties.*
- *B has neutral element e, if and only if B' has neutral element $e' = 1 - e$.*
- *B has zero element z, if and only if B' has zero element $z' = 1 - z$.*
- *B is cancellable, if and only if B' is cancellable.*
- *B is invertible, if and only if B' is invertible.*
- *B has divisors of zero z , if and only if B' has divisors of zero $z' = 1 - z$.*

Lemma 4. *Let* $B : [0,1]^2 \to [0,1]$.
• B' *has the same monotonicity as* B *from* $(\uparrow,\uparrow)$, $(\uparrow,\downarrow)$, $(\downarrow,\uparrow)$ *or* $(\downarrow,\downarrow)$.
• B *is left (right) continuous with respect to one variable, if and only if* B' *is right (left) continuous with respect to the same variable.*
• B *is continuous and Archimedean, if and only if* B' *is continuous and Archimedean.*
• B *is* $C-$*Lipschitz, if and only if* B' *has the same property.*
• *Boundary values:*

$B'(x,y)$	*0*	*1*
0	$1-B(1,1)$	$1-B(1,0)$
1	$1-B(0,1)$	$1-B(0,0)$

Proof. Results for monotonicity are consequences of composition of monotonic functions. Similarly, continuity is a consequence of composition of continuous functions If $B(x,x) < x$ for $x \in (0,1)$, then $B'(x,x) = 1-B(x',x') > 1-x'$= x. Thus $B'(x,x) > x$ for $x \in (0,1)$ and we obtain the dual Archimedean condition from (6). Assuming that B is $C-$Lipschitz we obtain $|B'(x,y)-B'(u,v)| = |1-B(x',y')-1+B(u',v')| = |B(x',y')-B(u',v')| \leqslant C(|x'-u'|+|y'-v'|) = C(|x-u|+|y-v|)$ for $x,y,u,v \in [0,1]$. So B' is also $C-$Lipschitz. Finally, the boundary values are direct consequence of (8). □

5 Duality of fuzzy negations

Definition 2 (cf. [7]). A decreasing function $N\colon [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$, $N(1) = 0$. A fuzzy negation N is called
• a strict negation if it is a bijection;
• a strong negation if it is an involution;
• a non-vanishing negation if $N(x) > 0 \Leftrightarrow x < 1$;
• a non-filling negation if $N(x) < 1 \Leftrightarrow x > 0$;
• a contracting negation if

$$x \leqslant N(N(x)) \leqslant N(x) \text{ or } N(x) \leqslant N(N(x)) \leqslant x,\ x \in [0,1]; \tag{9}$$

• an expanding negation if

$$N(N(x)) \leqslant x \leqslant N(x) \text{ or } N(x) \leqslant x \leqslant N(N(x)),\ x \in [0,1]. \tag{10}$$

Corollary 2 ([1], pp. 14-15)**.** • *Every strong negation is strict, contracting and expanding.*
• *Every strict negation is non-vanishing and non-filling.*

Example 2 ([1], p. 15)**.** Sugeno negations $N_\lambda, \lambda > -1$ is a parametric family of strong negations, where

$$N_\lambda(x) = \frac{1-x}{1+\lambda x},\ x \in [0,1],\ N'_\lambda = N_{\lambda'},\ \text{with } \lambda' = \frac{-\lambda}{1+\lambda}.$$

As a particular case for $\lambda = 0$ we get the Łukasiewicz negation (7). Formulas $N^p(x) = 1 - x^p$, $N'(x) = (1-x)^p$, $p > 0$, $x \in [0,1]$ gives families of strict negations. They are contracting for $p \leqslant 1$ and expanding for $p \geqslant 1$. Now let us consider a three valued negation

$$N(x) = \begin{cases} 1, \text{ if } x < 0.5 \\ 0.5, \text{ if } x = 0.5 \\ 0, \text{ if } x > 0.5 \end{cases},\ N^2(x) = \begin{cases} 0, \text{ if } x < 0.5 \\ 0.5, \text{ if } x = 0.5 \\ 1, \text{ if } x > 0.5 \end{cases},\ x \in [0,1],\ N' = N.$$

It is neither strict nor non-filling or non-vanishing negation. However, it is an expanding negation.

Theorem 2. • *If N is a negation, strict negation or strong negation, then the same is N'.*
• *If N has a fixed point s, then N' has a fixed point $s' = 1 - s$.*
• *If N is a non-vanishing (non-filling) negation, then N' is a non-filling (non-vanishing) negation.*
• *If N is a contracting (expanding) negation, then N' is a contracting (expanding) negation.*

Proof. We consider only the case of contracting (expanding) negations, and another cases are direct consequence of Lemma 2. Let

$$A = \{x \in [0,1] : x \leqslant N(N(x)) \leqslant N(x)\}.$$

If $1-x \in A$, then $1-x \leqslant N(N(1-x)) \leqslant N(1-x)$, i.e. $x \geqslant 1-N(N(1-x)) \geqslant 1 - N(1-x)$. Thus $N'(x) \leqslant N'(N'(x)) \leqslant x$ for $x \in 1 - A$, which is the dual part of condition (9). Another conditions from (9), (10) can be check in a similar way. □

6 Duality between fuzzy conjunctions and disjunctions

Definition 3 (cf. [7])**.** An operation $C\colon [0,1]^2 \to [0,1]$ is called a fuzzy conjunction if it is increasing with respect to each argument and

$$C(1,1) = 1,\ C(0,0) = C(0,1) = C(1,0) = 0.$$

A fuzzy disjunction is an operation $D\colon [0,1]^2 \to [0,1]$ which is increasing with respect to each argument and

$$D(0,0) = 0,\ D(0,1) = D(1,0) = D(1,1) = 1.$$

A fuzzy conjunction C is called
• a weak triangular norm if $C(x,1) \leqslant x$, $C(1,y) = y$;
• a semicopula (triangular seminorm) if it has the neutral element $e = 1$;
• a pseudo triangular norm if it is associative with $e = 1$;
• a triangular norm if it is associative, commutative with $e = 1$;
• strict if it is continuous and strictly increasing in $(0,1]^2$.
• a quasi-copula if it is 1-Lipschitz with $e = 1$;
• a conjunctive uninorm if it is associative, commutative with $e \in (0,1]$.
A fuzzy disjunction D is called
• a weak triangular conorm if $D(x,1) \geqslant x$, $D(0,y) = y$;
• a triangular semiconorm if it has the neutral element $e = 0$;
• a pseudo triangular conorm if it is associative with $e = 0$;
• a triangular conorm if it is associative, commutative with $e = 0$;
• strict if it is continuous and strictly increasing in $[0,1)^2$.
• a disjunctive uninorm if it is associative, commutative with $e \in [0,1)$.

Example 3. The most important dual pairs of fuzzy conjunctions and disjunctions are known as triangular norms T and triangular conorms S, where $T' = S$:

$$T_M(x,y) = \min(x,y), T_P(x,y) = xy, T_L(x,y) = \max(x+y-1,0),$$

$$S_M(x,y) = \max(x,y), S_P(x,y) = x+y-xy, S_L(x,y) = \min(x+y,1),$$

$$T_D(x,y) = \begin{cases} x, \text{ if } y = 1 \\ y, \text{ if } x = 1 \\ 0, \text{ otherwise} \end{cases},\ S_D(x,y) = \begin{cases} x, \text{ if } y = 0 \\ y, \text{ if } x = 0 \\ 1, \text{ otherwise} \end{cases},$$

$$T_{FD}(x,y) = \begin{cases} \min(x,y), x+y > 1 \\ 0, \text{ otherwise} \end{cases},\ S_{FD}(x,y) = \begin{cases} \max(x,y), x+y < 1 \\ 1, \text{ otherwise} \end{cases},$$

for $x, y \in [0,1]$, where T_P, S_P are strict and T_L, S_L, T_D, S_D are nilpotent.

A good example of fuzzy conjunction and disjunction gives geometric mean

$$G(x,y) = \sqrt{xy},\ G'(x,y) = 1 - \sqrt{(1-x)(1-y)},\ x, y \in [0,1].$$

Useful examples of triangular seminorms and semiconorms can be obtained by convex combination of triangular norms (conorms)

$$C(x,y) = \lambda xy + (1-\lambda)\min(x,y),\ C'(x,y) = \lambda(x+y-xy) + (1-\lambda)\max(x,y),$$

where $\lambda, x, y \in [0,1]$. As examples of conjunctive and disjunctive uninorms for fixed $e \in (0,1)$ we consider the least uninorm $\underline{U_e}$ and the least idempotent uninorm ${U_e}^{\min}$ (cf. [12]) with their duals (the greatest uninorm and the greatest idempotent uninorm with neutral element $1-e$), where

$$\underline{U_e} = \begin{cases} 0 & \text{in } [0,e)^2 \\ \max & \text{in } [e,1]^2 \\ \min & \text{otherwise} \end{cases}, \quad \underline{U_e}' = \overline{U_{1-e}} = \begin{cases} 1 & \text{in } (1-e,1]^2 \\ \min & \text{in } [0,1-e]^2 \\ \max & \text{otherwise} \end{cases},$$

$${U_e}^{\min} = \begin{cases} \max & \text{in } [e,1]^2 \\ \min & \text{otherwise} \end{cases}, \quad ({U_e}^{\min})' = U_{1-e}^{\max} = \begin{cases} \min & \text{in } [0,1-e]^2 \\ \max & \text{otherwise} \end{cases}.$$

For the boundary case $e = 1$ we obtain $\underline{U_1} = T_D$, $\overline{U_0} = S_D$ and ${U_1}^{\min} = T_M$, ${U_0}^{\max} = S_M$.

Duality properties of fuzzy conjunctions and disjunctions are commonly known (de Morgan Triples). Directly from Lemmas 3, 4 we get

Theorem 3. *The dual operation of a fuzzy conjunction, weak triangular norm, triangular seminorm, pseudo triangular norm, triangular norm, strict triangular norm or conjunctive uninorm is a fuzzy disjunction, weak triangular conorm, triangular semiconorm, pseudo triangular conorm, triangular conorm, strict triangular conorm or disjunctive uninorm, respectively.*
Moreover, a fuzzy conjunction is continuous, strict, Archimedean or 1-Lipschitz, if and only if its dual operation is continuous, strict, Archimedean or 1-Lipschitz fuzzy disjunction.

7 Duality between fuzzy implications and coimplications

Definitions and examples of fuzzy implications are based mainly on the recent monograph by Baczyński and Jayaram [1].

Definition 4 (cf. [1])**.** Let functions $I, I^*\colon [0,1]^2 \to [0,1]$ be decreasing with respect to the first variable and increasing with respect to the second one.

• I is called fuzzy implication if $I(0,0) = I(0,1) = I(1,1) = 1,\ I(1,0) = 0$;
• I^* is called fuzzy coimplication if
$I^*(0,0) = I^*(1,0) = I^*(1,1) = 0,\ I^*(0,1) = 1$.
A fuzzy implication I is said to satisfy:
• (NP), the left neutral property, if $I(1,y) = y,\ y \in [0,1]$,
• (EP), the exchange principle, if $I(x, I(y,z)) = I(y, I(x,z)),\ x,y,z \in [0,1]$,
• (IP), the identity principle, if $I(x,x) = 1,\ x \in [0,1]$,
• (OP), the ordering property, if $I(x,y) = 1 \Leftrightarrow x \leqslant y,\ x,y \in [0,1]$,
• (CP), the law of contraposition, if $I(x,y) = I(1-y, 1-x),\ x,y \in [0,1]$.

Example 4. List of important fuzzy implications (cf. [1]) with their coimplications (dual operation in this case is denoted by I^*, because I' is usually used for reciprocal fuzzy implications as in [1], Definition 1.6.1):

$$I_{LK}(x,y) = \min(1-x+y, 1), \qquad I^*_{LK}(x,y) = \max(0, y-x)$$

$$I_{GD}(x,y) = \begin{cases} 1, \text{ if } x \leq y \\ y, \text{ if } x > y \end{cases}, \qquad I^*_{GD}(x,y) = \begin{cases} 0, \text{ if } x \geq y \\ y, \text{ if } x < y \end{cases},$$

$$I_{RC}(x,y) = 1 - x + xy, \qquad I^*_{RC}(x,y) = (1-x)y,$$

$$I_{KD}(x,y) = \max(1-x, y), \qquad I^*_{KD}(x,y) = \min(1-x, y),$$

$$I_{GG}(x,y) = \begin{cases} 1, \text{ if } x \leq y \\ \frac{y}{x}, \text{ if } x > y \end{cases}, \qquad I^*_{GG}(x,y) = \begin{cases} 0, \text{if } x \geq y \\ \frac{y-x}{1-x}, \text{ if } x < y \end{cases},$$

$$I_{RS}(x,y) = \begin{cases} 1, \text{ if } x \leq y \\ 0, \text{ if } x > y \end{cases}, \qquad I^*_{RS}(x,y) = \begin{cases} 0, \text{ if } x \geq y \\ 1, \text{ if } x < y \end{cases},$$

$$I_{WB}(x,y) = \begin{cases} 1, \text{ if } x < 1 \\ y, \text{ if } x = 1 \end{cases}, \qquad I^*_{WB}(x,y) = \begin{cases} 0, \text{ if } x > 0 \\ y, \text{ if } x = 0 \end{cases},$$

$$I_{FD}(x,y) = \begin{cases} 1, x \leq y \\ \max(1-x, y), x > y \end{cases}, \qquad I^*_{FD}(x,y) = \begin{cases} 1, x \geq y \\ \max(1-x, y), x < y \end{cases}$$

for $x, y \in [0,1]$.

Theorem 4. *Let* $I\colon [0,1]^2 \to [0,1]$.
• *The dual of a fuzzy implication is a coimplication and vice versa.*
• *Fuzzy implication fulfils (EP) or (CP) if and only if its dual has the same property.*
• *If fuzzy implication fulfils (NP), then* $I^*(0,y) = y$, $y \in [0,1]$.
• *If fuzzy implication fulfils (IP), then* $I^*(x,x) = 0$, $y \in [0,1]$.
• *If fuzzy implication fulfils (OP), then* $I^*(x,y) = 0 \Leftrightarrow x \geqslant y,\ x,y \in [0,1]$.

Proof. Using monotonicity and boundary values from Lemma 4 it is evident that I^* fulfils the definition of coimplication. Let us observe, that $I^*(x,y) = 1 - I(x',y')$. Thus, if I fulfils (EP), then

$$I^*(x, I^*(y,z)) = 1 - I(x', I(y',z')) = 1 - I(y', I(x',z')) = I^*(y, I^*(x,z)),$$

for $x, y, z \in [0,1]$, i.e. I^* also fulfils (EP). Similarly, if I fulfils (CP), then

$$I^*(y',x') = 1 - I(y,x) = 1 - I(x',y') = I(x,y),\ x,y \in [0,1],$$

which proves, that I^* fulfils (CP). In the case of other properties from Definition 4 we obtain new conditions:
if I fulfils (NP), then $I^*(0,y) = 1 - I(1,y') = 1 - y' = y$,
if I fulfils (IP), then $I^*(x,x) = 1 - I(x',x') = 1 - 1 = 0$,
if I fulfils (OP), then $I^*(x,y) = 0 \Leftrightarrow 1 - I(x,y) = 0 \Leftrightarrow I(x,y) = 1 \Leftrightarrow x \leqslant y$ for $x, y \in [0,1]$, what finishes the proof. □

8 Dual relation compositions

Fuzzy relations generalize characteristic functions of binary relations. Let $X, Y \neq \emptyset$ and $L = (L, \vee, \wedge, 0, 1)$ be a complete lattice. An $L-$fuzzy relation between sets X and Y is an arbitrary mapping $R : X \times Y \to L$ (a fuzzy relation for $L = [0,1]$). In the case $X = Y$ we say about $L-$fuzzy relation on a set X. The family of all $L-$fuzzy relations on X is denoted by $LR(X)$ ($FR(X)$ for fuzzy relations). For $R, S \in LR(X)$ we use the induced order and the lattice operations:

$$R \leqslant S \ \Leftrightarrow \ \underset{x,y\in X}{\forall} \ (R(x,y) \leqslant S(x,y)), \tag{11}$$

$$(R \vee S)(x,y) = R(x,y) \vee S(x,y),\ (R \wedge S)(x,y) = R(x,y) \wedge S(x,y),\ x,y \in X. \tag{12}$$

Usually these operations are considered as the simplest version of inclusion, sum and intersection of fuzzy relations, respectively (cf. [25]). However, the most important operation on fuzzy relations is their composition.

Definition 5 ([15])**.** Let L be a complete lattice with a binary operation $* : L^2 \to L$. By $\sup -*$ composition of $L-$fuzzy relations R, S we call the $L-$fuzzy relation $R \circ S$, where

$$(R \circ S)(x,z) = \sup_{y\in X}(R(x,y) * S(y,z)), \quad x,y \in X. \tag{13}$$

Similarly, $\inf -*$ composition (dual composition) is defined by

$$(R \circ' S)(x,z) = \inf_{y\in X}(R(x,y) * S(y,z)), \quad x,y \in X. \tag{14}$$

Properties of composition $\circ$ depends on properties of operation $*$, what was examined in details in the paper [8] in the case $L = [0, 1]$. We recall here some of these results.

Theorem 5 ([8]). *Let* $* : [0, 1]^2 \to [0, 1]$.
• *Monotonicity of the operation $*$ (it is increasing or decreasing with respect to the first or to the second argument) is equivalent to suitable property of the composition $\circ$.*
• *The operation $*$ has (left, right) zero element $z \in [0, 1]$ if and only if the composition $\circ$ has suitable zero element $Z(x, y) = z,\ x, y \in X$.*
• *If the operation $*$ is increasing, then the composition $\circ$ is distributive over $\vee$ and subdistributive over $\wedge$, i.e.*

$$T \circ (R \vee S) = T \circ R \vee T \circ S,\ (R \vee S) \circ T = R \circ T \vee S \circ T,\ R, S, T \in FR(X).$$

$$T \circ (R \wedge S) \leqslant T \circ R \wedge T \circ S,\ (R \wedge S) \circ T \leqslant R \circ T \wedge S \circ T,\ R, S, T \in FR(X).$$

• *The operation $*$ is infinitely* $\sup -$*distributive if and only if the composition $\circ$ is infinitely* $\sup -$*distributive, i.e.*

$$R \circ (\sup_{t \in T} S_t) = \sup_{t \in T}(R \circ S_t), \quad (\sup_{t \in T} S_t) \circ R = \sup_{t \in T}(S_t \circ R),$$

where $R, S_t \in FR(X), t \in T$ for arbitrary index set $T \neq \emptyset$.
• *Let the operation $*$ be infinitely* $\sup -$*distributive. The operation $*$ is associative in $[0, 1]$ if and only if the composition $\circ$ is associative in $FR(X)$.*
• *Let $z = 0$ be the zero element of the operation $*$. The operation $*$ has (left, right) neutral element $e \in (0, 1]$ if and only if the composition $\circ$ has suitable neutral element $E \in FR(X)$, where*

$$E(x, y) = \begin{cases} e, & x = y \\ 0, & x \neq y, \end{cases}, \quad x, y \in X. \tag{15}$$

Directly from Theorem 5 and Lemma 1 we get

Corollary 3. *Let an operation $* : [0, 1]^2 \to [0, 1]$ be increasing.*
• *If the operation $*$ is left-continuous in $[0, 1]$, then the composition $\circ$ is infinitely* $\sup -$*distributive in $FR(X)$.*
• *If the operation $*$ is left-continuous and associative in $[0, 1]$, then the composition $\circ$ is associative in $FR(X)$.*

Definition 6 ([18]). Let the composition $\circ$ be associative. Powers of fuzzy relation R are defined by the recurrence:

$$R^1 = R,\ R^{m+1} = R^m \circ R,\ m = 1, 2, \ldots$$

Additionally we consider the closure $R^{\vee}$ and kernel $R^{\wedge}$ of R:

$$R^{\vee} = \sup_{k\in\mathbb{N}} R^k,\ R^{\wedge} = \inf_{k\in\mathbb{N}} R^k.$$

Dual powers, closure and kernel will be denoted by $R^{\bullet n}$, $R^{\bullet\vee}$ and $R^{\bullet\wedge}$, respectively.

Theorem 6 ([9]). *Let $R, S \in FR(X)$. If the operation $*$ is left-continuous in $[0,1]$, then*

$$(R\vee S)^n \geqslant R^n \vee S^n,\ (R\wedge S)^n \leqslant R^n \wedge S^n,$$

$$(R\vee S)^{\vee} \geqslant R^{\vee}\vee S^{\vee},\ (R\vee S)^{\wedge} \geqslant R^{\wedge}\vee S^{\wedge},\ (R\wedge S)^{\vee} \leqslant R^{\vee}\wedge S^{\vee},$$

$$(R\wedge S)^{\wedge} \leqslant R^{\wedge}\wedge S^{\wedge},\ R^n\circ R^{\vee} = R^{\vee}\circ R^n = (R^{\vee})^{n+1},$$

$$(R^{\vee})^n = \bigvee_{k=n}^{\infty} R^k \geqslant (R^n)^{\vee}, (R^{\wedge})^{n+1} \leqslant \left\{ \begin{array}{c} R^n\circ R^{\wedge} \\ R^{\wedge}\circ R^n \end{array} \right. \leqslant \bigwedge_{k=n+1}^{\infty} R^k \leqslant (R^{n+1})^{\wedge}.$$

$$(R^{\vee})^{\vee} = R^{\vee}, (R^{\wedge})^{\wedge} \leqslant R^{\wedge}, (R^{\wedge})^{\vee} \leqslant (R^{\vee})^{\wedge},\ n = 1, 2, \ldots$$

Using Duality Principle, the above results can be reformulated also for dual composition $\circ'$ in $FR(X)$. By direct verification we get

Theorem 7 (cf. [10], Theorem 2). *Compositions* $\sup -*$ *and* $\inf -*'$ *are connected by the formula*

$$R\circ' S = 1-(1-R)\circ(1-S)\, \textit{for}\ R, S \in FR(X).$$

Moreover, if the operation $*$ is increasing, left-continuous and associative, then

$$R^{\bullet n} = 1-(1-R)^n,\ R^{\bullet\wedge} = 1-(1-R)^{\vee},\ R^{\bullet\vee} = 1-(1-R)^{\wedge}\, \textit{for}\ R\in FR(X).$$

Theorem 8. *Let $* : [0,1]^2 \to [0,1]$.*
• *Monotonicity of the operation $*$ is equivalent to suitable property of the composition $\circ'$.*
• *The operation $*$ has (left, right) zero element $z \in [0,1]$ if and only if the composition $\circ'$ has suitable zero element $Z'(x,y) = 1-z,\ x, y\in X$.*
• *If the operation $*$ is increasing, then the composition $\circ'$ is superdistributive over $\vee$ and distributive over $\wedge$.*

• *The operation $*$ is infinitely* $\inf -$*distributive if and only if the composition $\circ'$ is infinitely* $\inf -$*distributive.*
• *Let the operation $*$ be infinitely* $\inf -$*distributive. The operation $*$ is associative in $[0,1]$ if and only if the composition $\circ'$ is associative in $FR(X)$.*
• *Let $z = 1$ be the zero element of the operation $*$. The operation $*$ has (left, right) neutral element $e \in (0,1]$ if and only if the composition $\circ'$ has suitable neutral element $E' \in FR(X)$, where*

$$E'(x,y) = \begin{cases} 1-e, & x = y \\ 1, & x \neq y, \end{cases}, \quad x, y \in X.$$

Corollary 4. *Let an operation $* : [0,1]^2 \to [0,1]$ be increasing.*
• *If the operation $*$ is right-continuous in $[0,1]$, then the composition $\circ'$ is infinitely* $\inf -$*distributive in $FR(X)$.*
• *If the operation $*$ is right-continuous and associative in $[0,1]$, then the composition $\circ'$ is associative in $FR(X)$.*

Theorem 9. *Let $R, S \in FR(X)$ If the operation $*'$ is right-continuous in $[0,1]$, then*

$$(R \vee S)^{\bullet n} \geqslant R^{\bullet n} \vee S^{\bullet n},\ (R \wedge S)^{\bullet n} \leqslant R^{\bullet n} \wedge S^{\bullet n},$$

$$(R \vee S)^{\bullet\vee} \geqslant R^{\bullet\vee} \vee S^{\bullet\vee},\ (R \vee S)^{\bullet\wedge} \geqslant R^{\bullet\wedge} \vee S^{\bullet\wedge},\ (R \wedge S)^{\bullet\vee} \leqslant R^{\bullet\vee} \wedge S^{\bullet\vee},$$

$$(R \wedge S)^{\bullet\wedge} \leqslant R^{\bullet\wedge} \wedge S^{\bullet\wedge},\ R^{\bullet n} \circ' R^{\bullet\wedge} = R^{\bullet\wedge} \circ' R^{\bullet n} = (R^{\wedge})^{\bullet(n+1)},$$

$$(R^{\bullet\wedge})^{\bullet n} = \bigwedge_{k=n}^{\infty} R^{\bullet k} \leqslant (R^{\bullet n})^{\bullet\wedge},$$

$$(R^{\bullet\vee})^{\bullet(n+1)} \geqslant \left\{ \begin{array}{l} R^{\bullet n} \circ' R^{\bullet\vee} \\ R^{\bullet\vee} \circ' R^{\bullet n} \end{array} \right. \geqslant \bigvee_{k=n+1}^{\infty} R^{\bullet k} \geqslant (R^{\bullet(n+1)})^{\bullet\vee}.$$

$$(R^{\bullet\vee})^{\bullet\vee} \geqslant R^{\bullet\vee}, (R^{\bullet\wedge})^{\bullet\wedge} = R^{\bullet\wedge}, (R^{\bullet\wedge})^{\bullet\vee} \leqslant (R^{\bullet\vee})^{\bullet\wedge}, n = 1, 2, \ldots$$

9 Duality in fuzzy relation equations

Let $A \in [0,1]^{m\times n}$ and $b \in [0,1]^m$. Dual fuzzy systems of equations $A \circ x = b$ and $A \circ' x = b$ has dual properties.

Theorem 10. *Let $* : [0,1]^2 \to [0,1]$, $* \leqslant \min$ and $x *' y = 1 - (1-x) * (1-y)$ for $x, y \in [0,1]$.*

• *If the system $A \circ x = b$ is solvable, uniquely solvable or unsolvable, then the dual system has the same properties.*
• *If the system $A \circ x = b$ has the greatest solution u, then the dual system has the least solution u' and these solutions has dual formulas using fuzzy implication and coimplication:*

$$u_j = \bigwedge_{i=1}^{m} (a_{ij} \overset{*}{\rightarrow} b_i), \ u'_j = \bigvee_{i=1}^{m} (a_{ij} \overset{*'}{\leftarrow} b_i), \ j = 1, \ldots, n. \tag{16}$$

• *If the system $A \circ x = b$ has minimal solutions, then the dual system has the same number of maximal solutions.*
• *If the system $A \circ x = b$ has the least solution, then the dual system has the greatest solution.*

For example we have

Theorem 11 (cf. [6], Theorems 4.1, 4.2)**.** *If an operation $*$ with neutral element $e = 1$ is increasing and left-continuous, then the greatest solution u of the inequality $A \circ x \leqslant b$ is given by* (16).

Dually we obtain

Theorem 12. *If an operation $*$ with neutral element $e = 0$ is increasing and right-continuous, then the least solution u' of the inequality $A \circ' x \geqslant b$ is given by* (16).

10 Coimplication and difference axioms

In set theory we have two equivalent formulas for set difference:

$$B \setminus A = A' \cap B = \inf\{C | B \subset A \cup C\}.$$

These formulas from Boolean algebra can be used in the complemented lattice $([0, 1], \max, \min, ')$ in order to define two differences in $[0, 1]$

$$y \ominus_1 x = \min(1 - x, y), \ x, y \in [0, 1],$$

$$y \ominus_2 x = \inf\{t \in [0, 1] | y \leqslant \max(x, t)\} = \begin{cases} y, x < y \\ 0, \text{ otherwise} \end{cases}, \ x, y \in [0, 1].$$

We also have the bounded difference $y \ominus x = \max(0, y - x)$, $x, y \in [0, 1]$. After change of variables it is easy to check, that we obtain examples of coimplications from Example 4: $I^*_{KD}(x, y) = y \ominus_1 x$, $I^*_{GD}(x, y) = y \ominus_2 x$, $I^*_{LK} = y \ominus x$. This leads to consideration of axiomatic systems for difference. For example we have

Definition 7 ([20])**.** By difference poset (D-poset) we call a structure $(P,\leqslant,\ominus,0,1)$, where $(P,\leqslant,0,1)$ is a bounded partially ordered set and the operation $\ominus : P^2 \to P$ fulfils conditions:
D1) $y \ominus x \leqslant y$,
D2) $x \leqslant y \Rightarrow z \ominus y \leqslant z \ominus x$,
D3) $(z \ominus y) \ominus x = (z \ominus x) \ominus y$,
D4) $x \leqslant y \Leftrightarrow y \ominus (y \ominus x) = x$,
for $x, y, z \in [0,1]$.

Example 5. Let φ be an increasing bijection in [0,1]. Then

$$y \ominus x = \varphi^{-1}(\max(0, \varphi(y) - \varphi(x))) = \max(0, \varphi^{-1}(\varphi(y) - \varphi(x))),\ x, y \in [0,1]$$

defines a difference in $[0,1]$, i.e. $([0,1],\leqslant,\ominus,0,1)$ is a D-poset, because $\ominus$ fulfils axioms D1) - D4).

Another axioms of difference were presented in [21].

Definition 8. A function $F : [0,1]^2 \to [0,1]$ is a difference in $[0,1]$, if it fulfills conditions:
F1) $F(x,0) = x$,
F2) F is increasing with respect to the first argument,
F3) F is decreasing with respect to the second argument,
F4) $F(0,x) = 0$,
where $x \in [0,1]$.

By comparison with Definition 4 we have

Corollary 5. *If F is a difference from Definition 8, then function $I(x,y) = 1 - F(x,y)$ is a fuzzy implication and $I^*(x,y) = F(1-x, 1-y)$ is a fuzzy coimplication dual to I.*

11 Concluding remarks

We listed here diverse consequences of Duality Principle which have applications in fuzzy logic, fuzzy algebra, soft computing, discrete mathematics, artificial intelligence, approximate reasoning and decision making. Such survey of duality properties can be useful during considerations of the above domains. Simultaneously, the presented properties and notions can be a subject of further examination, specification or generalization. For example, paper [22] describes interval extensions of fuzzy coimplications generated from aggregation functions. In particular, this paper contains a list of 19 properties of fuzzy implications and coimplications with their interval generalizations.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) organized in Warsaw on September 30, 2011 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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