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# Methodology and applications of decision support systems

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Polish-Finnish Symposium
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Warszawa 1989

Secretary of the Conference
dr. Andrzej Stachurski

Wykonano z gotowych oryginałów tekstowych
dostarczonych przez autorów



ISBN 83-00-02543-X

# A NEW APPROACH FOR PAIRWISE COMPARISONS UTILIZATION IN MCDM

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**ABSTRACT**

A new approach for an interactive support of multicriteria decision-making is presented. It is based on pairwise comparisons of alternatives and accepts any number of comparisons.

**Keywords:** multicriteria decision-making, pairwise comparisons, impicit utility function

## 1. INTRODUCTION.

The basic problem in the area of interactive decision support for multicriteria problems is the extraction and utilization of the decision maker's (DM's) preferences. A rather large number of approaches have been developed during the last decade. This paper reports some of the basic ideas of a new one which seems to be promising because of its conceptual and methodological simplicity. The presented approach is based on the pairwise comparisons of alternatives and linear approximations of the DM's utility function. Since the approach is at an early stage of development and several of its aspects still have to be investigated, just a few basic ideas and motivations will be presented.

The basic feature of the approach is that the DM is not forced to compare pairs of alternatives which are presented to him but he chooses himself a subset of alternatives to be evaluated. An underlying quasiconvex DM's utility function is assumed.

## 2. MOTIVATIONS.

Let us consider the following multicriteria decision making problem. A decision maker (a person or an institution) wants to buy a new car and has some difficulties in choosing it from the variety of models available on the market. He is not an expert in cars and he knows just a few models: his old car and those possessed by his friends and relatives. So, all he is able to say about his preferences is a number of statements concerning cars he knows, like for example:

VW Golf is preferred to Opel Kadett ,

Fiat Uno is preferred to Peugeot 205 , etc.

He refuses to compare cars he does not know or to supply any other kinds of information about them. The reference point approach might be adopted in this case, but what if the DM would not be satisfied with the result?

The task can be formulated as follows. A relatively small number of pairwise comparisons of alternatives is available. What can be said about the DM's preferences on the basis of this small amount of information and what can be said about the quality of that information ? Notice that a statement: "a cheap good car is preferred to an expensive bad car" is rather low quality information since once price and performance have been established as criteria, this is an obvious statement. The DM should be informed about the quality of the evaluation of the alternatives which he has made. Also his inconsistencies should be discovered and disclosed.

## 3. BASIC IDEAS.

Let $F$ be the space of $m$ criteria, $\Lambda = R^m_+$ be the domination cone and let $Q \subset F$ be the set of feasible alternatives. We will assume that behind the DM's preferences there exists an underlying implicit quasiconvex utility function $U: F \rightarrow R$ . The DM need not recognize its existence. However, we will assume that whenever he decides that alternative $b \in Q$ is preferred to alternative $a \in Q$ , it is equivalent to $U(b) > U(a)$ .

The DM's utility function $U$ is in general a nonlinear function of criteria. Identification (even indirect) of such a function usually requires a large amount of data and significant computational effort. Therefore, keeping nonlinearity of $U$ in mind, we shall restrict ourselves to a set of linear approximations of $U$ only.

Suppose that $k$ pairs of alternatives were compared by the DM:

$$b_i \text{ is preferred to } a_i \quad , \quad a_i, b_i \in Q \; , \; i=1,\ldots,k$$

This set of data may be considered as a set $W$ of $k$ vectors in the criteria space $F$, pointing from a less preferred alternative to a more preferred $b_i$.

$$W = \{ w_i : w_i=[a_i,b_i] \; , \; a_i b_i \in Q \; , \; i=1,\ldots,k \}$$

Let us also consider the set $V$ of normalized vectors $w \in W$ :

$$V = \{ v_i : v_i=w_i/\|w_i\| \; , \; w_i \in W \}$$

Each of the vectors $v_i$ represents a direction of the utility function $U(f)$ improvement in the space of criteria. Hence, the cone $C$ spanned by vectors $v \in V$ is the cone of $U(f)$ improvement

$$C = \{ \sum_i \alpha_i v_i : \alpha_i \in R_+ \; , \; v_i \in V \subset Q \; , \; i=1,\ldots,k \} .$$

The cone $C^*$ is the corresponding polar cone

$$C^* = \{ y \in F : \langle y_i v_i \rangle \; , \; v_i \in V \; , \; i=1,\ldots,k \} .$$

Both $C$ and $C^*$ cones can be expressed by their generators. The set of cone generators is the minimal subset of vectors belonging to that cone that still span the cone. The cone $C$ and $C^*$ generators will be denoted by $c^o \in C^o$ and $c^{o*} \in C^{o*}$, respectively.

$$C = \{ \sum_i \alpha_i c_i^o \; , \; \alpha_i \in R_+ \; , \; c_i \in C^o \} \; ,$$

$$C^* = \{ \sum_i \beta_i c_i^{o*} , \; \beta_i \in R_+ \; , \; c_i^{o*} \in C^{o*} \} .$$

Let us return to the pairwise comparisons. Since we shall consider the linear approximations of the utility function, for the sake of presentation simplicity let us assume that $U$ is linear. If the DM has decided that alternative $b \in Q$ is preferred to alternative $a \in Q$, then

$U(b)>U(a)$ . It is clear that $\langle v,u\rangle\geq 0$ , where $v=[a,b]$ , and $u$ is a vector normal to hyperplanes $U(f)=const$ . Hence, clearly vector $u$ is contained in cone $C^*$.

From the above analysis it follows that the accurate determination of vector $u$ normal to the hyperplanes of $U$ will be possible only in the case when the cone $C^*$ is spanned by a single vector (namely $u$) i.e. the cone $C$ is a halfspace of $F$. In this case the DM's utility function (or rather actually its linear approximation only) has been obtained and we can easily calculate the DM's most preferred solution by minimizing $U$ over the set $Q$.

In general, for obvious reasons, the cone $C$ will be smaller than a halfspace and its polar cone $C^*$ will have a nonempty interior. In such a case each of the vectors contained in $C^*$ may appear to be the vector $u$ . Fortunately, we can restrict ourselves to the cone $C^*$ generators $c^*$ only. Considering each $c_i^*$ to be the vector $u$ (minimizing linear function based on $c_i^*$) one can obtain a set of $q_i\in Q$ being the DM's utility function linear approximations minimizers. These elements $q_i$ determine a subset $S\subset Q$ of nondominated elements of $Q$ in which the DM's most preferred alternative (minimizer of $U$) is contained.

As it can be seen now, our approach does not claim to determine the DM's most preferred solution exactly. It will rather tend to find a domain in which it is contained. The more information about the DM's preferences is contained in the supplied pairwise comparisons of alternatives, the smaller this domain will be. Besides, a good candidate for the most preferred solution may be also presented to the DM. It can be determined as the cone $C^*$ "middle-most" vector: a sum of $c_i^*$ , a sum of $v_i$ , a gravity center of $v_i$ , etc. The author's favorite method for candidate selection is the calculation of the minimal (euclidean) norm element from the convex hull spanned by the cone $C^*$ generators $c^{o*}$. This technique based on the method of P. Wolfe [1] appeared to be very useful in our approach, serving also for some other purposes. Let us denote the minimal norm element from the convex hull spanned by the set $V$ of vectors $v$ as

$$z = MNECH(V) \ .$$

## 4. SOME DETAILS.

In this chapter we shall discuss the basic cases that can occur for different sets of pairwise comparisons of alternatives supplied by the DM. Let $H$ be a halfspace of $F$.

**Case A.** $C = H$ , $\|z\| = 0$ - full information case.

As it has already been mentioned, in this case the linear approximation of the DM's utility function is defined by the vector $u$ normal to the halfspace spanned by $C$. The DM's most preferred solution may be found by the optimization of the linear function based on $u$.

**Case B.** $C \supset H$ , $\|z\|=0$ - inconsistent information case.

Since the DM's utility function is assumed to be quasiconvex, the set $V$ of pairwise comparisons supplied by the DM is inconsistent. Conflicting elements should be selected from the set $V$ and presented to the DM. They are those elements which span a convex hull containing zero and hence cause $\|z\|=0$. Their selection is automatic during calculation of $z$ . This case may also appear when in the process of alternatives evaluation the DM actually uses at least one more criterion besides the specified $m$ .

**Case C.** $\Lambda \subset C \subset H$ , $\|z\|>0$ , - usual case - discriminative type of information supplied.

This is the basic case. After the sequence of $m$ optimizations of linear functions based on vectors $c^{o*}$, a subset of nondominated elements of set $Q$ will be obtained. This subset, spanned by the set of the utility function linear approximations optimizers, contains the DM's most preferred alternative . A candidate for the DM's most preferred solution will also be found by the linear function based on vector $z$ optimization over the set $Q$. Notice that if the number of supplied pairwise comparisons is to small (to span a non-degenerate cone $C$, for example), then generators of the domination cone A can be added to the set $V$.

**Case D.** $C \subset \Lambda \subset H$ , $\|z\|>0$ – non-discriminative information case

This is the case when the information contained in pairwise comparisons of alternatives supplied by the DM is of low quality (Statements like: "a good cheap car is preferred to an expensive bad car"). The DM should be informed about this fact and perhaps he will be able to give some more restrictive statements. If he refuses for some reasons, we cannot proceed along the Case C lines. However, instead of of considering the supplied information as being of a discriminative type we can treat it as an instructive type information. Each of the vectors $v \in V$ can be treated now as an approximation of the DM's improvement direction or his utility gradient approximation. Hence we can proceed just like in Case C, taking cone $C$ instead of $C^*$ into consideration. Of course the DM should be aware of the new interpretation of the information he has supplied.

Notice that the last two cases can be distinguished by checking whether $\Lambda \subset C$ or $C \subset \Lambda$ , respectively. The selection of a proper case is easy and it is done during the calculation of $z$ as proposed by P. Wolfe (1975).

## 5. CONCLUDING REMARKS.

The presented approach can be used for both arbitrary small and large amounts of available pairwise comparisons. If the DM is able to supply a large amount of evaluations for arbitrarily chosen pairs of alternatives, then a technique similar to one presented in by Koksalan,... (1984) may be used in order to eliminate dominated alternatives from further considerations. If it is not the case, the DISCRET (see Majchrzak (1988)) package methodology should be applied. Actually, the presented approach is planned to be included into the DISCRET framework.

Several aspects of the presented approach are still not quite clear at the moment. In general, most of them concerns the selection of some additional techniques that should be combined together with the presented one to form an operational decision support system. An interesting question is, for example, how to select a small sample of those alternatives whose evaluation by the DM may result in significant improvement of the DM's preferences approximation.

**REFERENCES:**

**Koksalan, M., M.H. Karwan, S. Zionts,** (1984), "An Improved Method for Solving Multiple Criteria Problems Involving Discrete Alternatives", *IEEE Transactions on Systems, Man and Cybernetics, Vol. SMC-14, No.1, 24-34.*

**Majchrzak, J.,** (1988), "Methodological Guide to the Decision Support System "DISCRET" for Discrete Alternatives Problems", *in A. Lewandowski, A.P. Wierzbicki (eds.), "Theory, Software and Testing Examples in Decision Support Systems", International Institute for Applied Systems Analysis, WP-88-o71, July 1988.*

**Wolfe, P.,** (1975), "Finding the Nearest Point in a Polytope", *Mathematical Programming , Vol.11, 128-149.*