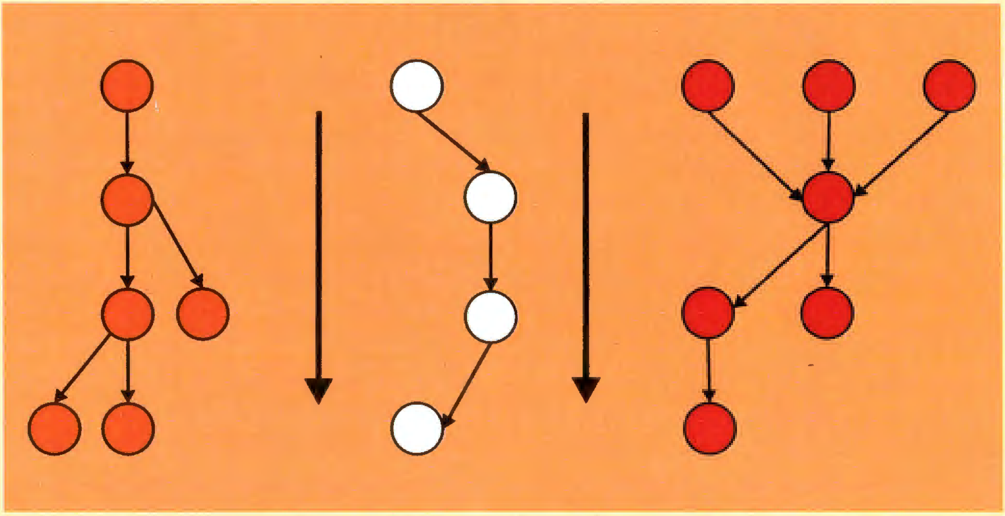
SYSTEMS RESEARCH INSTITUTE
POLISH ACADEMY OF SCIENCES

# MULTICRITERIA ORDERING AND RANKING:
## PARTIAL ORDERS, AMBIGUITIES AND APPLIED ISSUES



**Jan W. Owsiński and Rainer Brüggemann**
**Editors**

Warsaw 2008

SYSTEMS RESEARCH INSTITUTE
POLISH ACADEMY OF SCIENCES

# MULTICRITERIA ORDERING AND RANKING:
## PARTIAL ORDERS, AMBIGUITIES AND APPLIED ISSUES

Jan W.Owsiński and Rainer Brüggemann
Editors

Warsaw 2008

This book is the outcome of the international workshop held in Warsaw in October 2008 within the premises of the Systems Research Institute. All papers were refereed and underwent appropriate modification in order to appear in the volume. The views contained in the papers are, however, not necessarily those officially held by the respective institutions involved, especially the Systems Research Institute of the Polish Academy of Sciences.



ISBN 83-894-7521-9
EAN 9788389475213

Technical editing and typesetting:
Jan W.Owsiński, Anna Gostyńska, Aneta M.Pielak

# Applications and Comparisons

*Multicriteria Ordering and Ranking: Partial Orders, Ambiguities and Applied Issues*
*Jan W. Owsiński and Rainer Brüggemann, Editors*

# Analysing the Structure of Poverty by Fuzzy Partial Order

**Paola Annoni*, Marco Fattore** and Rainer Brüggemann*****

* Dept. of Economics, Business and Statistics, University of Milan, via Conservatorio 7, Milan, Italy (paola.annoni@unimi.it)

** Dept. of Statistics, University of Milano-Bicocca, via Bicocca degli Arcimboldi 8, Milan, Italy (marco.fattore@unimib.it)

*** Leibniz-Institute of Fresh Water Ecology and Inland Fisheries, Müggelseedamm 310, 12587 Berlin, Germany (brg_home@web.de)

We propose to employ Fuzzy Multi-Criteria Analysis, FMCA, which originated in the decision theory context, to describe the structure of poverty. In the last years, poverty has been increasingly recognized as a multidimensional, fuzzy and complex phenomenon, which cannot be duly represented by mono-dimensional monetary indicators. Our purpose is not to meausure poverty but to provide its structural representation in terms of the pattern of implications existing among different poverty descriptors in the context of a specific scenario, that is a selected geographical region. FMCA is adapted to address the case of the most southern Italian region, Sicily. The case study of Sicilian households is based on the European Statistics about Income and Living Conditions (EU-SILC) referring to the year 2004. The analysis shows deep-rooted poverty conditions with many relational implications across poverty descriptors, some of them describing high levels of deprivation and social exclusion.

Keywords: partial order, fuzzy logic, poverty

## 1. Introduction

In the last decades, monitoring and reduction of poverty and social exclusion are emerging as primary goals for many national governments and international organizations. At European level, after the definition of the Lisbon Agenda (March 2000) and the subsequent European Council in Nice (December, 2000), the European Commission has focused on these goals too. A subset of the Laeken indicators (Laeken indicators are a set of common European statistical indicators on poverty and social exclusion, established at the European Council of December 2001 in the Brussels suburb of Laeken, Belgium) have been explicitly devoted to these issues (Atkinson et al., 2004) and annual surveys have been launched to gather data on income, employment, health, material deprivation and other social-economical aspects, in different European countries. It is now generally agreed that poverty and social exclusion are complex multidimensional phenomena that cannot be represented faithfully by currently-in-use monetary poverty measures, usually based on income (Lemmi and Betti, 2006). Multi-dimensionality calls for new statistical methodologies and many new proposals have appeared in the last twenty years to address such an issue. Most of the research has been devoted to define multivariate poverty indicators and to extend classical poverty measures in a fuzzy direction (Cerioli and Zani, 1990; Lemmi and Betti, 2006), overcoming the rigid distinction of people into 'poor' and 'not poor'. Nevertheless, it is our opinion that complexity of the poverty phenomenon requires some kind of structural description, going beyond and complementing the collection of numerical indicators. With this aim, in this paper we investigate the relational pattern among different poverty facets, describing the network of logical implications among attributes which describe poverty within a given scenario, here an Italian region. Multidimensional poverty related to this scenario is thus characterized by means of a 'structure' and not by means of a simple, albeit necessary, set of point measures. In order to analyze poverty in terms of its multi-dimensionality we aim at establishing a set of implications among attributes which describe poverty. We consider this relational set of implications as a *structure*. Correspondingly the basis of our analysis is the theory of fuzzy relations and partially ordered sets, namely Fuzzy Multi-Criteria Analysis (FMCA), which provides powerful tools to describe complex relational structures which may be also appropriate for ordinal variables. We also present a first application of these techniques to poverty data for Sicily, the most southern Italian region, extracted from the European panel about Income and Living Conditions (EU-SILC) and referring to year 2004.

The paper is organized as follows: Section 2 introduces and discusses the problem of poverty analysis at a conceptual level; Section 3 provides some basic mathematical definitions concerning fuzzy theory, needed in the following analysis; Section 4 describes the EU-SILC database and the data used for poverty analysis; Section 5 discusses FMCA; Section 6 describes its application to the Sicily case while some concluding comments are eventually provided in Section 7.

## 2. Analysing poverty: its measure and structure

It is hard to provide a definition of poverty that can be directly turned into a set of objective statistical indicators. As a matter of fact, assessing poverty implies some economical, statistical and even ethical choices that can be argued in many respects. Historically, poverty has been often reduced to a single monetary dimension, usually household income. Perhaps, the most relevant limitation of the monetary approach is that only a single aspect of poverty is taken into account, while it is widely accepted that poverty is a multidimensional concept (Lemmi and Betti, 2006). For this reason, poverty surveys tend to supplement income data with additional qualitative information about living conditions. Such data allow for a more faithful representation of poverty, however they pose some statistical problems when trying to single out consistent poverty indicators. In fact, the problem of classifying people in terms of poverty is even harder when more attributes, often qualitative, are considered in the evaluation space, due to the lack of a clear scaling and aggregating criterion. There are many concepts to combine indicators of poverty into a single one, see for example considering the axioms proposed by Tsui (2002).

Realizing that poverty is a vague and complex phenomenon, great interest has arisen towards the use of fuzzy set theory (Lemmi and Betti, 2006). The basic idea is to combine different indicators (quantitative or qualitative) into a membership function that measures the degree an individual belongs to the set of 'poor', developing subsequent analysis using fuzzy reasoning. Even if these approaches represent a step towards more realistic poverty measurement, they still tend to reduce complexity. They produce one-dimensional indicators which, on one hand, may be easily communicated and understood, but on the other hand are not suitable for describing the structure of poverty. More details on poverty measurements are discussed in a forthcoming paper.

As one can see, assessing poverty is a very difficult process even from a conceptual point of view. A more appropriate representation of poverty requires dealing with its complexity, without treating it with oversimplified conceptual schemes, thus poverty is here described in terms of the relational pattern across its attributes, that is our concept of 'structural representation'. To this aim we employ some tools of fuzzy theory to explore and represent the network of implications among different poverty facets.

## 3. Preliminaries

The analysis presented in this paper is mainly established on some basic tools of fuzzy and partial order theory. Given the technical nature of the present volume, in this section we collect some basic definitions pertaining fuzzy sets only, which are helpful for the comprehension. More details can be found in the references given in the text and in the forthcoming publication (Annoni et al., 2008). The reader who is already familiar with these tools may easily skip this part.

Full details on partial order tools used may be found for example in Brüggemann et al. (2001), Neggers and Kim (1998) or Trotter (1992).

### 3.1. Fuzzy theoretic basic tools (Zimmermann, 1991)

Def. 3.1. Fuzzy set: if $X$ is a collection of objects, a *fuzzy set* $\tilde{A}$ in $X$ is a collection of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

where $\mu_{\tilde{A}}(x)$ is called the degree of membership of $x$ in $\tilde{A}$ and assumes values in [0,1]. In the special case of $\mu_{\tilde{A}}(x)$ taking only values 0 or 1, the set $\tilde{A}$ would be the classical, crisp set $A$.

Often, one is interested in those elements of $X$ that belongs to the fuzzy set $\tilde{A}$ with a degree at least equal to a chosen value $\alpha$. This introduces the notion of $\alpha$-cut of a fuzzy set.

Def. 3.2. $\alpha$-cut set: let $X$ be a set, $\tilde{A}$ a fuzzy set in $X$ and choose $0 \leq \alpha \leq 1$. The $\alpha$-cut set $A_\alpha$ is the set of elements of $X$ such that $\mu_{\tilde{A}}(x) \geq \alpha$.

Note that $A_\alpha$ is a crisp set. Most of the following analysis about poverty relies on the concepts of fuzzy binary relations and $\alpha$-cut relations.

Def. 3.3. Fuzzy relation: let $X$ be a set. A fuzzy relation $\tilde{R}$ is a fuzzy subset of $X \times X$. If $x, y \in X$, then $\mu_{\tilde{R}}(x, y)$ measures the degree of membership of the pair $(x, y)$ to the binary relation $\tilde{R}$ (also the strength of the relation between $x$ and $y$).

If $X$ is a finite set with cardinality $n$ and if its elements are labeled $x_1,\dots,x_n$, then a fuzzy relation $\tilde{R}$ on $X$ can be represented as a square matrix $M_{n\times n}$, where

$$M_{ij} = \mu_{\tilde{R}}(x_i, x_j), \quad x_i, x_j \in X. \tag{1}$$

As for ordinary fuzzy sets, the notion of $\alpha$ -cut can be given for fuzzy relations:

Def. 3.4. $\alpha$ -cut relation: let $0 \le \alpha \le 1$ and $\tilde{R}$ be a fuzzy relation on a set $X$. An $\alpha$ -cut set $R_\alpha$, of $\tilde{R}$ is the crisp relation defined by

$$xR_\alpha y \Leftrightarrow \mu_{\tilde{R}}(x,y) \ge \alpha, \quad x,y \in X. \tag{2}$$

## 4. EU-SILC data-base

The European Statistics on Income and Living Condition (EU-SILC) panel is the most important source of data about income, welfare and material deprivation in EU countries. The following section describes the EU-SILC variables selected for the subsequent analysis.

### 4.1. Selected variables

Among variables (attributes) included in EU-SILC data-set, we selected some which shape different aspects of poverty (Table 1). They are all variables of categorical type and relate to: housing appropriateness (HH040, HH050 and HH090), arrears on different types of payments (HS010, HS020 and HS030), capacity to afford expenses not strictly necessary (HS040, HS050 and HS060), and the deprivation of basic appliances and durable goods (HS070, HS080, HS100 and HS110). Variable categories have all been modified in order to show the same orientation towards the poverty level: the higher the score the higher the poverty. This recoding procedure leads to variables all of dichotomous type.

For the case under study, selected variables represent different descriptors of poverty in a particular Italian region, Sicily, which belongs to historically economically depressed areas of the country.

Table 1. Selected criteria

| **Name** (assigned name) | **Original categories** | **Recoded categories** | **Description** |
|---|---|---|---|
| HH040 (House) | 1 = Yes<br>2 = No | 0 = No<br>1 = Yes | Leaking roof, damp walls, floors, foundation, or rot in window frames or floor |
| HH050 (Warm) | 1 = Yes<br>2 = No | 0 = Yes<br>1 = No | Ability to keep home adequately warm |
| HS020 (Bills) | 1 = Yes<br>2 = No | 0 = No<br>1 = Yes | Arrears on utility bills |
| HS040 (Holiday) | 1 = Yes<br>2 = No | 0 = Yes<br>1 = No | Capacity to afford paying for one week annual holiday away from home |
| HS050 (Food) | 1 = Yes<br>2 = No | 0 = Yes<br>1 = No | Capacity to afford a meal with meat, chicken, fish (or vegetarian equivalent) every second day |
| HS060 (Extra) | 1 = Yes<br>2 = No | 0 = Yes<br>1 = No | Capacity to face unexpected financial expenses |
| HS070 (Phone) | 1 = Yes<br>2 = No cannot afford<br>3 = No other | 0 = Yes or no other<br>1 = No cannot afford | Do you have a telephone (including mobile phone)? |
| HS080 (TV) | 1 = Yes<br>2 = No cannot afford<br>3 = No other | 0 = Yes or No other<br>1 = No cannot afford | Do you have a color TV? |
| HS100 (Wash) | 1 = Yes<br>2 = No cannot afford<br>3 = No other | 0 = Yes or No other<br>1 = No cannot afford | Do you have a washing machine? |
| HS110 (Car) | 1 = Yes<br>2 = No cannot afford<br>3 = No other | 0 = Yes or No other<br>1 = No cannot afford | Do you have a car? |

## 5. The structure of poverty: a fuzzy multiple-criteria analysis FMCA on poverty attributes

FMCA here employed was originally proposed by Van de Walle et al. (1995) and recently mathematically refined by De Baets and De Meyer (2003a). In our case we want to analyze poverty attributes on the basis of their incidence on poor households.

The starting point for our analysis is matrix $S_{m\times n}$, with poverty attributes on rows and households on columns. $S_{ij}$ denotes the score of household $j$ $(j=1,...,n)$ on poverty attribute $i$ $(i=1,...,m)$. The selection of 'poor' households is performed on an income basis and is described in Section 6.

### 5.1. The fuzzy subsethood relation

The starting point for FMCA is the measurement of the level of subsethood of one fuzzy set into another, as introduced by Kosko (Van de Walle et al., 1995):

Def. 5.1. let $\tilde{A}$ and $\tilde{B}$ are two fuzzy sets in $X$. The Kosko's degree of subsethood of $\tilde{A}$ in $\tilde{B}$ is a binary fuzzy relation defined as:

$$SH(\tilde{A},\tilde{B}) = \frac{\sum_{x\in X}\min(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x))}{\sum_{x\in X}\mu_{\tilde{A}}(x)} \quad \text{if} \quad \tilde{A}\neq\emptyset \qquad (3)$$

$$SH(\tilde{A},\tilde{B}) = 1 \quad \text{if} \quad \tilde{A}=\emptyset$$

From this definition follows that $SH(\tilde{A},\tilde{A})=1$ and that $A\subseteq B \Longleftrightarrow SH(A,B)=1$ for any two crisp sets $A,B$.

Kosko's measure has been already employed for empirical purposes within different fields (Van de Walle et al., 1995; Haven, 1998). To adapt the procedure to our specific case, a short explanation is due at this point.

Let the set $X=\{x_1,x_2,...x_n\}$ be the set of poor households and $\tilde{S}_i$ be the fuzzy set in the set $X$ defined as $\tilde{S}_i=\{x_j,\mu_{\tilde{S}_i}(x_j)\}$, where $\mu_{\tilde{S}_i}(x_j)$ is the numerical value scored by household $x_j$ on attribute $i$ $(i=1,...m)$. Since in our case $\mu_{\tilde{S}_i}(x_j)$ may only assume value 0 or 1, we are dealing with crisp sets $S_i$ representing row vectors of values (0/1) scored by the set of 'poor' families on attribute $i$. For example, if on attribute $i$ the first household scores '1', the second household scores '1', ... and household $n$ scores '0', the crisp set which corresponds to attribute $i$ is: $S_i = \{(1,1),(2,1)...(n,0)\}$.

In this particular 'crisp' case it is straightforward to show that, starting from the definition of fuzzy and crisp sets (Def. 3.1), Kosko's subsethood measure (3) reduces to:

$$SH(S_i, S_k) = \frac{|(S_i \cap S_k)|}{|S_i|} \tag{4}$$

so it is a kind of normalized measure of the set intersection.

In the case of poverty data Kosko's measure indicates how much an attribute is implied by another, over 'poor' households. The higher the value of $SH(S_i, S_k)$, the higher the level of 'fuzzy inclusion' of $S_i$ in $S_k$, so that the more likely is the statement that "being poor on attribute $i$ implies being poor on attribute $k$."

## 5.2. Turning the subsethood relation into a fuzzy quasi order

To be used for comparison purposes, the fuzzy binary relation $SH$ has to be reflexive and $T$-transitive with respect to a chosen $T$-norm (Klement et al., 2000), that is relation $SH$ has to be transformed into a quasi-order. As aforementioned, reflexivity of $SH$ simply comes from its definition. As far as $T$-transitivity is regarded, its general definition is due:

Def. 5.2. $T$-transitive fuzzy relation (De Baets and De Meyer, 2003a):

A fuzzy relation $\tilde{R}$ is called *T-transitive* if:

$$T(\tilde{R}(x,y), \tilde{R}(y,z)) \leq \tilde{R}(x,z), \quad \forall (x,y,z) \in X^3 \tag{5}$$

where $T$ stands for a triangular norm. The most important example of a $T$-norm is the *min* operator and the $T$-transitivity with respect to the *min* norm is called the *min*-transitivity.

The $SH$ relation is not *min*-transitive. Recently it has been proven that any fuzzy relation on an arbitrary universe $X$ has a $T$-transitive closure ($T$-closure) for any $T$-norm, that is the smallest $T$-transitive fuzzy relation including it (De Baets and De Meyer, 2003a). For finite universe $X$, as for the present paper, De Baets and De Meyer (2003a) propose an operative method, known as *matrix method*, for computing the *min*-closure, which also is applied in the PyHasse module (see Section 6.1) that leads to a fuzzy relation $\tilde{Q}$ which is a fuzzy quasi-order relation and allows for performing a consistent comparison among attributes based on it.

A comment is due at this point. The $min$ -transitive closure of a fuzzy relation may not adequately approximate the original fuzzy relation. So if the *min*-transitive closure is to be used instead of the original fuzzy relation, it should be verified that the former is a good approximation of the latter. Usually, the distance between two fuzzy relations can be simply measured in terms of the $L_1$ norm of the difference between matrices representing them (De Baets and De Meyer, 2003b). This metric is adopted here to verify the distance between the *min*-transitive closure and the original fuzzy relation. In this particular case results are encouraging as shown in Section 6.2.

## 5.3. Extracting a crisp partial order from the fuzzy quasi order

The next step is, in a sense, a step backward from the fuzzy quasi-order relation $\tilde{Q}$ to a crisp quasi order relation $Q_\alpha$ that is an $\alpha$ -cut of $\tilde{Q}$. $Q_\alpha$ is more easily interpreted and provides a rationale for comparing poverty attributes. In view of constructing a consistent comparison among attributes, the fundamental result is that any $\alpha$ -cut $Q_\alpha$ of a fuzzy quasi-order relation $Q$ is a crisp quasi-order relation (Bandler and Kohout, 1988). Thus, for each different value of $\alpha$, different order relations on equivalence classes of attributes are obtained (Haven, 1998). Each equivalence class comprises a group of attributes which shall be considered as equivalent, in the sense that they have the same crisp relational pattern with respect to all other attributes. These equivalence classes become larger as $\alpha$ values decrease and melt together gradually (Van de Walle et al., 1995).

Allowing the value of $\alpha$ to be less than one is equivalent to state that an implication between two attributes is true on a general basis even if it is not perfectly verified in the available data. Many notable contributions for addressing the problem of noisy data in partial order theory are available (Brüggemann et al., 2001; Luther et al., 2000; Pudenz et al., 2000). The fuzzy relation approach allows for circumventing the problem of discarding the noise translating it into the choice of the proper $\alpha$ -cut value. It may be argued that this is only a way of dislocating the issue: as the value of $\alpha$ increases, incomparability relations among attributes are assigned more and more significance. Hence the problem is shifted to the proper choice of the $\alpha$ -cut which assures an adequate trade-off between discarding non informative incomparability relations and keeping all relevant order relations. In Section 6.2 we employ a method which implies as little subjectiveness as possible in the choice of the 'critical' $\alpha$ -cut.

Once an $\alpha$ -cut has been selected, the resulting crisp relation is a quasi order ($\leq$). For $\leq$ to be transformed into a partial order ($\leq'$) it suffices to introduce equivalence classes (Van de Walle et al., 1995) and to analyze the partially ordered quotient set.

## 6. Analyzing poverty attributes in Sicily

We focused our analysis on the structure of poverty among households classified as poor in terms of income. As ten poverty attributes are examined for three hundred and fiftyfive households the analysis starts with a data matrix of dimensions 10 x 335.

### 6.1. The fuzzy module of the software package PyHasse

The Fuzzy partial order analysis has been performed with a new software package named PyHasse, recently implemented and still under development using Python® language by one of the authors. A general introduction will be found in another paper, this issue; here we restrict ourselves on those part of PyHasse, which is specifically applied in this poverty study.

The fuzzy module of PyHasse implements the matrix method described in De Baets and De Meyer (2003a) to compute the *min*-transitive closure of a finite fuzzy relation. The different steps which may be performed using the fuzzy module of PyHasse are:

1. Open the data file.
2. Normalize the data matrix.
3. Apply Kosko's subsethood method.
4. Find the transitive closure by the "matrix method".
5. Extract a crisp partial order relation.
6. Calculate the Hasse Diagram (HD) and some partial order properties.

Optional the data matrix can be looked upon, different interim matrices can be shown by the "ScrolledText"-method of Python. See Fig. 1 for the Graphical User Interface of the fuzzy partial order module of PyHasse.

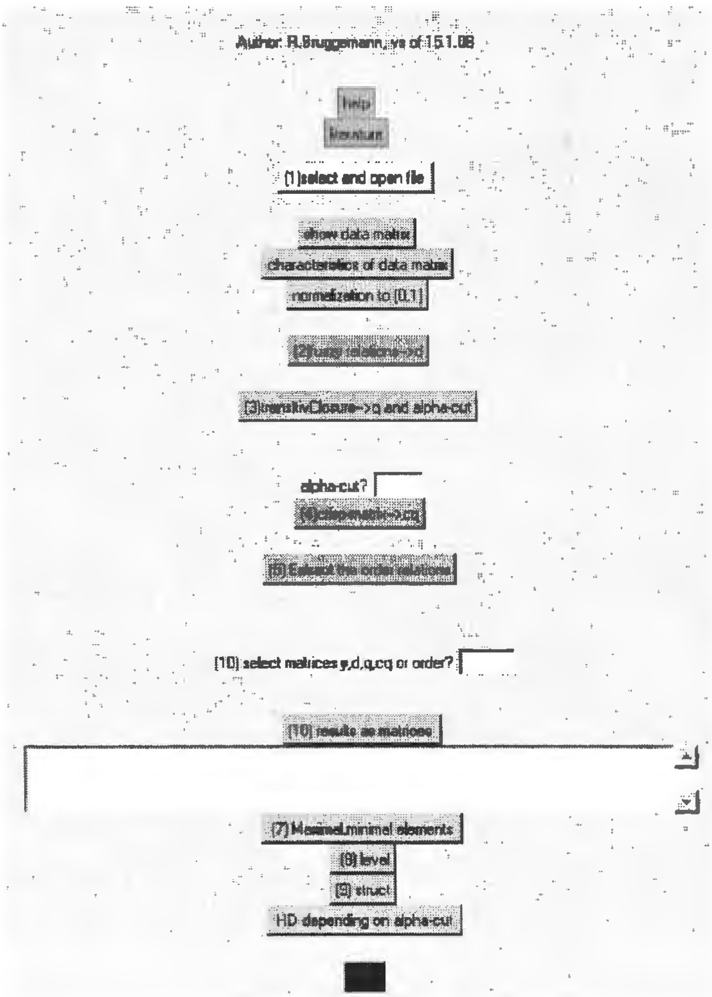



Figure 1. The PyHasse interface of the fuzzy module

## 6.2. FMCA for Sicilian households

Applying Kosko's subsethood measure to Sicily data, the matrix $\tilde{D}$ results (not shown here) and from that the *min*-transitive closure $\tilde{Q}$ turns out to be:

$$\tilde{Q}=\begin{pmatrix} 1 & 0.952 & 0.507 & 0.784 & 0.279 & 0.171 & 0.171 & 0.279 & 0.616 & 0.552 \\ 0.506 & 1 & 0.506 & 0.706 & 0.279 & 0.171 & 0.171 & 0.279 & 0.526 & 0.506 \\ 0.558 & 0.968 & 1 & 0.863 & 0.279 & 0.171 & 0.171 & 0.279 & 0.611 & 0.726 \\ 0.506 & 0.93 & 0.506 & 1 & 0.279 & 0.171 & 0.171 & 0.279 & 0.526 & 0.506 \\ 0.509 & 0.943 & 0.507 & 0.887 & 1 & 0.171 & 0.171 & 0.396 & 0.679 & 0.717 \\ 0.75 & 0.938 & 0.625 & 0.938 & 0.563 & 1 & 0.375 & 0.438 & 0.75 & 0.875 \\ 0.571 & 0.929 & 0.643 & 0.857 & 0.643 & 0.429 & 1 & 0.429 & 0.714 & 0.714 \\ 0.537 & 0.951 & 0.512 & 0.951 & 0.512 & 0.171 & 0.171 & 1 & 0.732 & 0.659 \\ 0.506 & 0.943 & 0.506 & 0.766 & 0.279 & 0.171 & 0.171 & 0.279 & 1 & 0.506 \\ 0.507 & 0.978 & 0.507 & 0.824 & 0.279 & 0.171 & 0.171 & 0.279 & 0.588 & 1 \end{pmatrix} \quad (8)$$

To verify that matrix $\tilde{Q}$ is a good approximation of matrix $\tilde{D}$, we compute the following indicator $\beta$:

$$\beta = \frac{\left\|\tilde{Q}-\tilde{D}\right\|_1}{\left\|\tilde{D}\right\|_1} \quad (9)$$

where $\|\cdot\|_1$ stands for the $L_1$ matrix norm. For Sicily data the $\beta$ coefficient equals $0.057$, showing that the transitive closure approximates very well the original $\tilde{D}$ matrix. In view of the $\alpha$-cut analysis, we compute $\beta$ also for the sequence of $\tilde{D}$ matrix obtained assigning the value $0$ to elements less than cut-off values 0.05, 0.1, 0.15… and so on until 0.95. The maximum value of the sequence of $\beta$ is 0.128, obtained for a cut-off value of 0.5. Matrix $\tilde{Q}$ and all its $\alpha$-cuts can thus be used instead of matrix $\tilde{D}$ for our comparison goal.

The series of $\alpha$-cuts is composed of $45$ different values, that is all the different values in matrix $\tilde{Q}$. Not all Hasse Diagrams may be presented here (see, however, (Fig. 3), instead we prefer to plot a relevant parameter of the Hasse diagram in relation to the corresponding $\alpha$-cut that provides a criterion for selecting the value of $\alpha$.

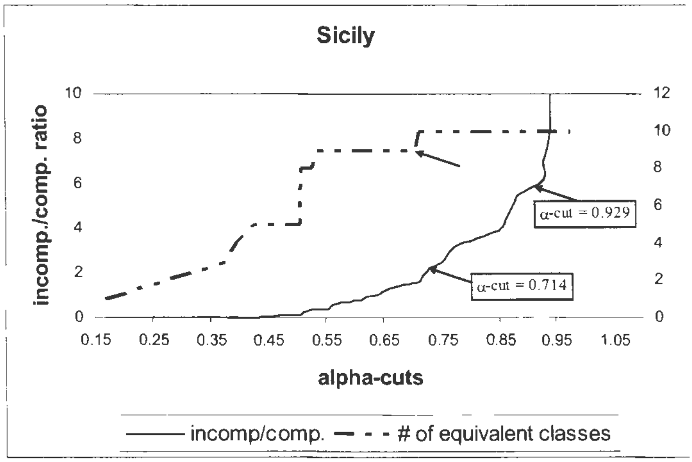


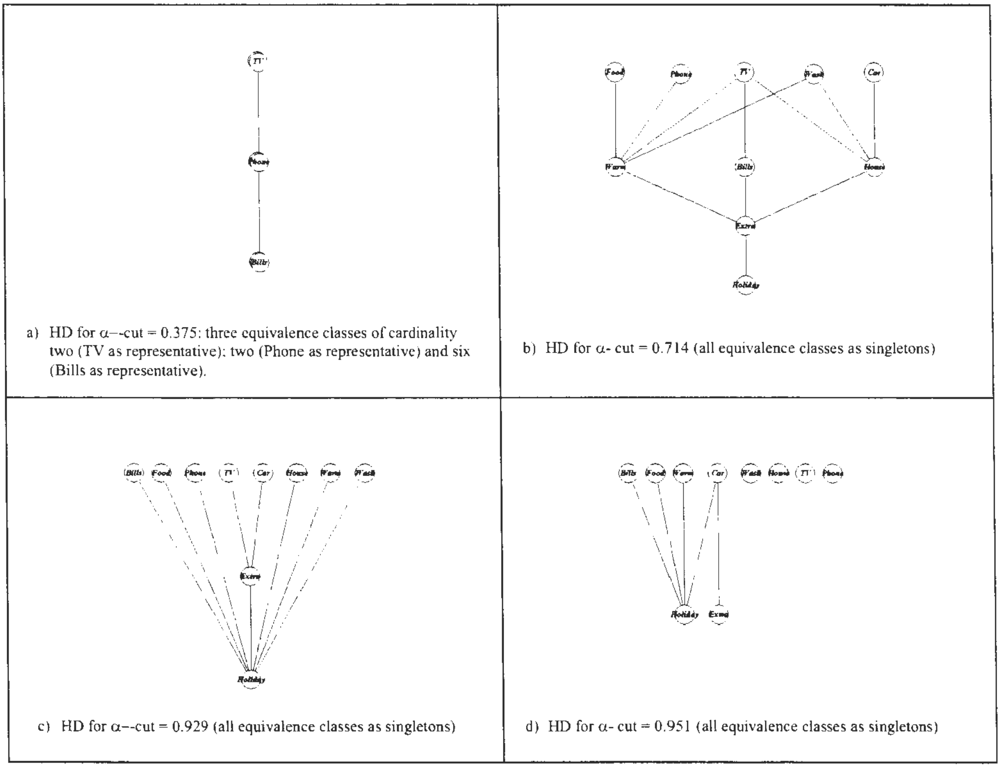

Figure 2. $I/C$ and number of equivalent class as a function of $\alpha$ -cuts

The chosen parameter is the ratio between the number of incomparabilities and the number of comparabilities $I/C$ of the Hasse diagram which correspond to the particular $\alpha$ -cut. The purpose is to find out a simple indicator of Hasse diagram 'complexity' whose trend can be associated to the different $\alpha$ -cuts. Fig. 2 shows the curve $I/C$ vs $\alpha$ -cuts for Sicily.

The least $\alpha$ -cut is 0.171 thus meaning that until 0.171 all attributes are indistinguishable, belonging to the same equivalence class. As $\alpha$ increases, an implication pattern emerges and the partial order induced on the poverty attributes becomes more and more complex: the number of equivalence classes increases and, correspondingly, the number of elements within each class decreases.

The analysis of Fig. 2 shows two 'break-points' seem to characterize the curve: one is for $\alpha = 0.714$ and the other is for $\alpha = 0.929$ (see arrows in the figure.).

To illustrate the evolution of the partial order as a function of $\alpha$ -cut we are plotting four different Hasse Diagrams which correspond to $\alpha$ -cut values equal to 0.375, 0.714, 0.929 and 0.951 (Fig. 3). As can been easily seen for low values of $\alpha$ -cut (Fig. 3a) most elements are included in equivalence classes, which therefore have a high cardinality, and no incomparability relations appear. The two HD shown in Fig. 3b and c correspond to the two 'critical' $\alpha$ -cuts discussed shortly above, whilst the last HD (Fig. 3d) features many incomparabilities, being associated to a high value of $\alpha$ .



a) HD for α–-cut = 0.375: three equivalence classes of cardinality two (TV as representative); two (Phone as representative) and six (Bills as representative).

b) HD for α- cut = 0.714 (all equivalence classes as singletons)

c) HD for α–-cut = 0.929 (all equivalence classes as singletons)

d) HD for α- cut = 0.951 (all equivalence classes as singletons)

Figure 3. Hasse Diagrams for different $\alpha$ -cut values

We nevertheless chose the first one as 'critical' $\alpha$ since it is the lower $\alpha$ -cut where all attributes are separated (at this point the number of equivalence classes reaches the number of attributes) while the number of incomparabilities is not too large. From this value on the $I/C$ value rapidly increases and this means that from this point on small changes of $\alpha$ values (i. e. small changes in the subsethood relations between attributes) induce many incomparabilities in the poset. At $\alpha = 0.714$ an implication pattern is thus revealed that is not trivial being not too sensible to noise or small changes in the subsethood relation. Taking the point $\alpha = 0.714$ as the critical point, we can plot the corresponding Hasse diagram, revealing the implication pattern among poverty attributes. Therefore the critical Hasse Diagram shown in Fig. 3b graphically represents what we mean with the term 'structure' when analyzing the concept of poverty.

## 6.3. Interpretation of results

The critical poset of Figure 3b may be interpreted with the help of basic tools of partial order theory (Trotter, 1992). It has length three and width five, with maximal anti-chain including *Food, Phone, Wash, TV* and *Car*. Moreover, it has a bottom element, *Holiday*, which is directly related to and only to attribute *Extra*. A direct interpretation of the location of these two attributes is that they are associated, in a fuzzy sense, to the lowest values of Kosko's subsethood measure (Def. 5.1). From a social-economic point of view this means that, consistently with the discussion of Section 5.1, having difficulties on any other poverty attribute implies, in a fuzzy sense, having difficulties in affording a one-week holiday and facing extra expenses. In other words, if a household suffers from some form of poverty (among those considered in this study), then it is likely it cannot afford holidays nor extra expenses. Another simple interpretation of the critical poset is as follows. The attribute *TV*, as an instance, is connected downwards with many other aspects: *Warm*, *Bills*, *House*, *Extra* and *Holidays*. This means that not affording a TV fuzzy-implies that the household has difficulties in paying utility bills and extra expenses, in living in a proper home and in affording holidays. This attribute may hence be interpreted as a symptom of deep-rooted poverty.

Moreover, a further insight into the number and length of maximal chains may also be of help in getting some results. Sicily critical poset has eight chains of length three. Apart from particular attributes included in the chains, this means that the poset is characterized by many relational implications. In other words this means that being poor in Sicily means being deficient on various aspects, thus in turn indicating a deep-rooted poverty. This is also confirmed by the analysis of simultaneous incomparable attributes: in the critical poset of Figure 3b there are only two attributes *Food* and *Phone* which are simultaneously not comparable to five others, suggesting again an articulated structure of poverty in this area of Italy.

Some general remarks may be outlined from the Hasse Diagram of Fig. 3b.

Generally speaking, difficulties in facing unexpected expenses or in allocating money for holidays can be suffered also by people near the poverty line, who are not likely to suffer from other forms of material deprivation. In Sicily deprivation of durable goods seems to be at the basis of other forms of poverty. This fact may be interpreted as follows. Nowadays, durable goods, like car, washing machine or television, are retained as essential assets of a household. People deprived of them are really away from the common accepted level of well-being. Moreover, deprivation on durable goods probably reveals a prolonged state of poverty.

A durable good has by definition a 'long' life, so that a household may possess it even if, at present, it has economic problems. On the contrary, lacking of durable goods is likely to reveal that hard economic conditions have been lasting for a long time.

## 7. Concluding remarks

We focus on the relationships among some attributes characterizing poverty and present a procedure to represent such a relational pattern. Implication patterns produced by FMCA are posets of attributes and are represented by means of Hasse diagrams. As such, they are analyzed in terms of poset theory and different structural parameters are computed on them (dimension, width, length).

An application to data from EU-SILC survey (year 2004) is provided which regards the most southern Italian region, Sicily, which is historically depressed. It shows how the proposed approach is capable to get new insight into poverty patterns and can be used to characterize regional poverty in structural terms.

Interesting results come out from the analysis of the case study. Deprivation of some durable goods seems to be a serious symptom of poverty, since it implies deprivation of many other attributes. The implication pattern among attributes of Sicily shows a 'rich' structure thus suggesting a deep-rooted poverty condition of households.

Even if the analysis leads to appealing results, further research is certainly needed both at methodological and applicative level. As an instance, at the methodological level, more research should be devoted to the choice of proper $\alpha$ -cuts, since the relational structure extracted by our procedure depends upon such a choice. Our approach is here to choose the critical $\alpha$ -cut that allows for discarding non informative differences while keeping all relevant information of the poset. Even if it seems to be quite effective, this choice is affected by some degree of subjectiveness. From the applicative point of view, more attention should be paid to the choice of poverty attributes to guarantee a sound economic interpretation of results.

Furthermore it would be interesting to replicate such analysis on more data-sets, pertaining different countries and referring to different times. This would give the chance to assess its economic meaning and its capability to trace differences and similarities of poverty structures both spatially and temporally. In this perspective, official EU-SILC data-sets are very promising and we are going to devote some of our research to their study.

## References


Annoni P., Fattore M., Brüggemann R. (2008) A multi-criteria fuzzy approach for analyzing poverty structure. Submitted to *Social Choice and Welfare*.

Atkinson A. B., Marlier E., Nolan B. (2004) Indicators and Targets for Social Inclusion in the European Union. *JCMS 2004,* **42**(1), 47-75.

Bandler W., Kohout L. (1988) Special properties, closures and interiors of crisp and fuzzy relations. *Fuzzy Sets and Systems*, 26, 317-331.

Brüggemann R., Halfon E., Welzl G., Voigt K., Steinberg C. (2001) Applying the Concept of Partially Ordered Sets on the Ranking of Near -Shore Sediments by a Battery of Tests. *Journal of Chemical Information and Computer Sciences*, 41, 918-925.

Cerioli A. and Zani S. (1990) A fuzzy approach to the measurement of poverty. In: C. Dagum and M. Zenga, eds., *Income and Wealth Distribution, Inequality and Poverty.* Springer-Verlag, Berlin. 272-284.

De Baets B. and De Meyer H. (2003a) On the existence and construction of *T*-transitive closures. *Information Sciences,* 152, 167-179.

De Baets B. and De Meyer H. (2003b) Transitive approximation of fuzzy relations by alternating closures and openings. *Soft Computing*, 7, 210-219.

Haven E.E. (1998) The fuzzy multicriteria analysis method: an application on NPV ranking. *International Journal of Intelligent Systems in Accounting, Finance & Management*, 7, 243-252.

Klement E.P., Mesiar R., Pap E. (2000) *Triangular Norms*. Kluwer Academic Publishers, Dordrecht.

Lemmi A. and Betti G., eds. (2006) *Fuzzy Set Approach to multidimensional Poverty Measurement.* Springer, New York.

Luther B., Brüggemann R., Pudenz S. (2000) An Approach to Combine Cluster Analysis with Order Theoretical Tools in Problems of Environmental Pollution. *MATCH – Commun. Math. Comput. Chem.*, 42, 119-143.

Neggers J. and Kim H.S. (1998) *Basic Posets.* World Scientific, Singapore.

Pudenz S., Brüggemann R., Luther B., Kaune A., Kreimes K. (2000) An algebraic/graphical tool to compare ecosystems with respect to their pollution V: cluster analysis and Hasse diagrams. *Chemosphere*, 40, 1373-1382.

Trotter W.T. (1992) *Combinatorics and Partially Ordered Sets: Dimension Theory.* The John Hopkins University Press, Baltimore and London.

Tsui K. (2002) Multidimensional Poverty Indices. *Social Choice and Welfare*, 19, 69-93.

Van de Walle B., De Baets B., Kerre E.E. (1995) Fuzzy multi-criteria analysis of cutting techniques in a nuclear reactor dismantling project. *Fuzzy Sets and Systems* 74, 115-126.
Zimmermann H.J. (1991) *Fuzzy Set Theory and its Applications.*2nd ed. Kluwer Academic Publishers, Boston/Dordrecht/London.

This book is a collection of papers, prepared in connection with the 8th International Workshop on partial orders, their theoretical and applied developments, which took place in Warsaw, at the Systems Research Institute, in October 2008. The papers deal with software developments (PYHASSE and other existing software), theoretical problems of ranking and ordering under various assumed analytic and decision-making-oriented conditions, as well as experimental studies and down-to-earth pragmatic questions.

ISBN 83-894-7521-9
EAN 9788389475213