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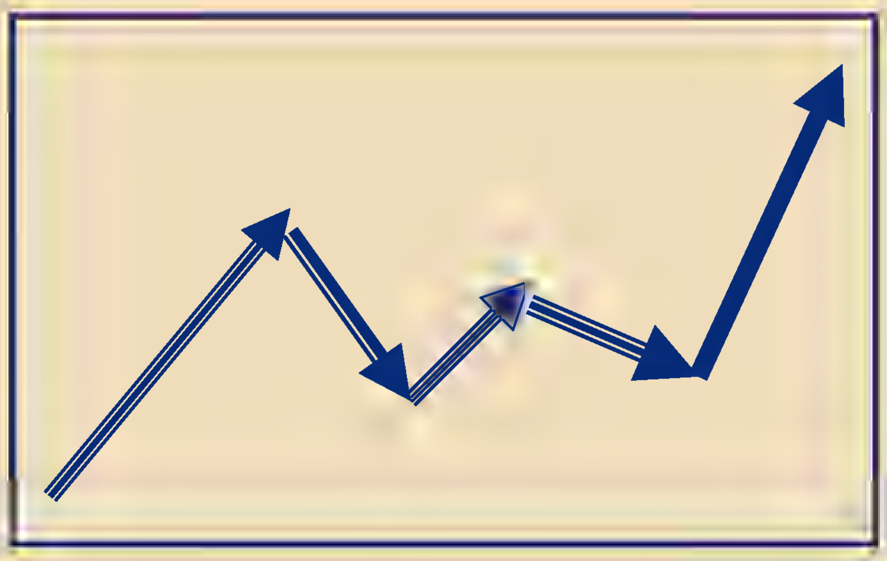
AN INTRODUCTION TO A THEORY OF MARKET COMPETITION

Volume I



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S. PIASECKI, J.W. OWSIŃSKI, AN INTRODUCTION TO A THEORY OF MARKET COMPETITION

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Chapter III
Activity of a local company in the environment of distributed customers

If the sales of products, manufactured by a company, give rise to significant costs, associated with transporting the product to the place, where customer is located, then the price of sale to a distant customer must be increased by the transport cost – provided the same, producing company takes also care of transport at own expense.

If the company does not secure product delivery to the location of the customer, then it is the customer herself that must make use of own or rented transport means. Consequently, the actual cost of purchase of the product is also higher by some transport cost. Hence, transport cost must in any case be accounted for.

1. Activity under even spatial customer distribution

A basic model, which was described in Chapter II, shall now be complemented with the transport cost, which must refer to the nature of spatial distribution of potential customers.

The modifications concern the profit function, Z, in which we should additionally account for transport cost, if it is covered by the company, and the magnitude of demand, Λ, depending upon the extent of the area, over which such sales are carried out. More precisely, the dependence is upon the number of customers within this area and the quantities of the product, purchased by individual customers.

The distribution of customers over the area considered can, of course, be different. For simplicity, we shall assume initially that the density of potential customers (number per area unit) is constant.

In addition, we shall assume that each customer, on the average, has the same, per time unit, needs in terms of purchasing the product considered.

1.i. Basic model 3.i[1] accounting for the cost of delivering the product to customer's storage

Conform to the general assumptions, adopted above, we shall take as given a constant density g of potential customers per area unit. For this assumption, the number of potential customers, L_{mx}, depends upon the radius R of the circle, which constitutes the sales area, namely

$L_{mx} = g\Pi R^2$.

The form of the linear dependence of the magnitude of needs of a potential customer upon product price is assumed to be the same as in the previous models:

$$\lambda = \lambda_0 \cdot q(C) = \lambda_0 \cdot (1 - C/C_{mx}) = a^o \cdot (C_{mx} - C) = \lambda_0 - a^o C,$$

where $a^o = \lambda_0 / C_{mx}$ and $\lambda_0 = 1/T$.

Ultimately, the magnitude of demand, Λ, shall be defined as follows:

$$\Lambda(C,R) = L_{mx} \cdot \lambda = g\Pi R^2(\lambda_0 - a^o C).$$

In order to obtain the function of profit, Z, let us account for the costs of transporting the product to customers, located within the area of the circle of radius R, with the producer in its centre.

[1] Nomenclature taken from Chapter II.

Assume that the cost of transport depends upon the travel distance, *r*, and the number of products transported. Denote by k_T unit transport cost (say, per kilometre per unit of product), so that the cost of transporting one unit of product for the distance *r* is $k_T r$.

Altogether, the costs of transporting products to all customers in the area serviced shall be defined by the formula

$$k_T \cdot g \cdot \lambda \int_{\Omega} r \cdot dS = k_T \cdot g \cdot \lambda \int_0^R r \cdot 2\Pi r \cdot dr = 2\Pi k_T \cdot g \cdot \lambda \frac{1}{3} R^3 =$$
$$= L_{mx} \cdot \overline{\mathrm{K}}_T \cdot \lambda$$

where

$$\overline{\mathrm{K}}_T = k_T \cdot \overline{r} \quad ; \quad \overline{r} = \frac{2}{3} R$$

and

$$L_{mx} = g \pi R^2 \, .$$

Consequently, the function of profit, accounting for transport costs, shall take the form

$$Z(C,R,\mu) = C \cdot \Lambda(C,R) - \mu \cdot \kappa(\mu) - \Lambda(C,R) \cdot \overline{K}_T$$

Where $\kappa(\mu) = b + Q/\mu$.

Since in market equilibrium the equality $\mu = \Lambda(C,R)$ must hold, then as we substitute the value of μ, we get

$$Z(C,R) = [C - b - \overline{K}_T(R)] \cdot \Lambda(C,R) - Q \, .$$

After having transformed this function to the form of $F = Z + Q$, we get

$$F = Z + Q = \left[C - b - \overline{K}_T(R) \right] \cdot (C_{mx} - C) \cdot a \cdot L_{mx}(R) \, .$$

The course of *F* as a function of *C* is shown in Fig. 3.1.

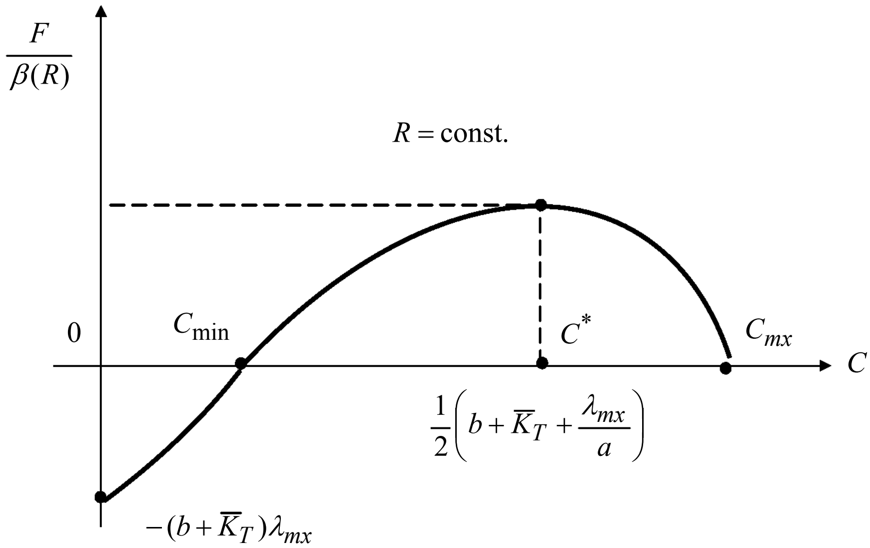

Fig. 3.1. Function F for changing C, model 3.i

If we now take notation

$$C_{\min}(R) = b + \overline{K}_T(R)$$

then we can write down the same function as

$$F = (C\text{-}C_{\min}(R))\cdot(C_{mx}\text{-}C)\cdot a^{o}L_{mx}(R).$$

By taking the zero value of the derivative of F with respect to C, we obtain the solution, maximising the value of F, namely

$$C^{*} = (C_{mx}+C_{\min}) = \frac{1}{2}\left(b+\bar{K}_T(R)+\frac{\lambda_0}{a^{o}}\right)=\frac{1}{2}\left(C_{mx}+b+\frac{2}{3}k_T R\right).$$

The maximum value of F attained for this solution, given a definite value of R, is

$$F^{*} = (C_{mx}\text{-}C_{\min}(R))^{2}\cdot a^{o}L_{mx}(R)/4.$$

By substituting the expressions for $C_{\min}$ and C_{mx} into the function F we get

$$F = \frac{1}{4}\left(\frac{\lambda_0}{a^o} - b - \frac{2}{3}k_T R\right)^2 a^o g\Pi R^2 = a^o\left(\frac{k_T}{3}\right)^2 \Pi g\left(\frac{C_{mx} - b}{\frac{2}{3}k_T} - R\right)^2 R^2 .$$

Then, as we introduce notations
$A = a^o(k_T/3)^2 \Pi g$; $R_{mx} = 3(C_{mx}-b)/2k_T$,

we can write down function F in the form

$$F = A \cdot (R_{mx} - R)^2 \cdot R^2 .$$

The course of this function, having roots
$R_{1,2} = R_{mx}$; $R_{3,4} = 0$

is shown in Fig. 3.2.

After having equated to zero the derivative of function F with respect to R we get the optimum value of the radius of supply area, R^*,

$$R^* = \frac{1}{2}R_{mx} = \frac{3}{4}\frac{C_{mx} - b}{k_T} \qquad (R_{1,2} = 0; R_3 = R_{mx}; R_4 = \frac{1}{2}R_{mx})$$

and, as it is shown in Fig. 3.1, the maximum value of the function is given by

$$F = A \cdot \left(\frac{R_{mx}}{2}\right)^4 .$$

Hence, in order for the production and distribution activity under optimum choice of the supply area and optimum price to be profitable, the following inequality must hold:

$$A \cdot \left(\frac{R_{mx}}{2}\right)^4 - Q > 0$$

In which constant production costs, Q, are accounted for.

By substituting the value of A, we obtain the following form of the inequality:

$$Q < \Pi \cdot a \cdot g \cdot \frac{(C_{mx} - b)^4}{\left(\frac{4k_T}{3}\right)^2}.$$

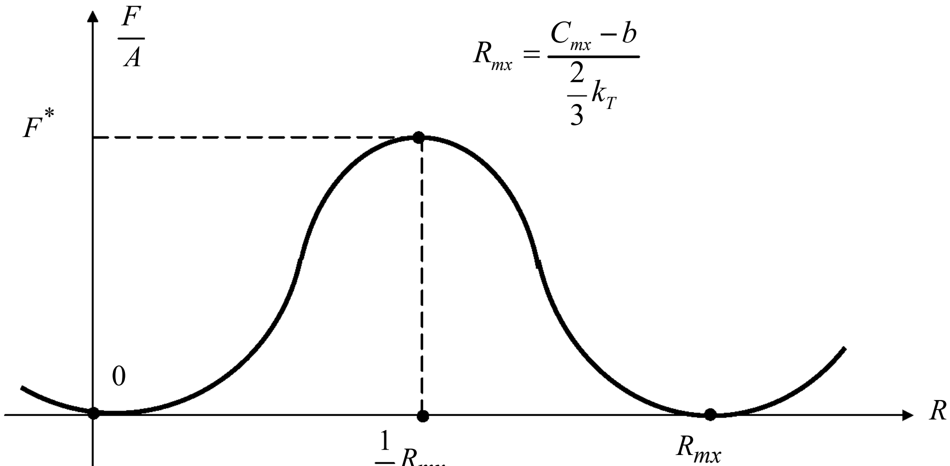

Fig. 3.2. The course of the function F depending upon the radius of the supply area, R

We can now check the course of relations, defining:

Profit per product unit

We, namely, dispose of the cost of manufacturing a unit of product and of transporting it to the customer's inventory:

$$\kappa = \frac{Q}{\mu} + b + \frac{2}{3}k_T R; \ \lambda = a^o (C_{mx} - C).$$

After having introduced the market clearing condition, $\mu = \Lambda$, we obtain

$$\kappa = \frac{Q}{a^o g \Pi R^2 (C_{mx} - C)} + b + \frac{2}{3}k_T R.$$

As we take the derivative of cost to be equal zero (for $R = R^*$), i.e.

$$\frac{\partial \kappa}{\partial R} = 0$$

we obtain

$$R^*(C) = \sqrt[3]{\frac{3Q}{a^o g\Pi k_T (C_{mx} - C)}}\,.$$

Now, since profit on a single unit of product equals $Z^1 = C\text{-}\kappa$, we can equate the derivative of profit with respect to C to zero and get

$$C^*(R) = C_{\mathrm{mx}} - \sqrt[2]{\frac{Q}{a^o g\Pi R^2}}\,.$$

Substituting the expression for C^* into the formula for R^* we obtain

$$R^* = R^*(C) = \sqrt[2]{\frac{3}{k_T}} \cdot \sqrt[4]{\frac{Q}{a^o g\Pi}}\,,$$

and analogously, by substituting R^* into the formula for C^*:

$$C^* = C^*(R^*) = C_{\mathrm{mx}} - \sqrt[2]{\frac{3}{k_T}} \cdot \sqrt[4]{\frac{Q}{a^o g\Pi}}\,.$$

Ultimately, the highest profit per unit of product shall be given by

$$Z^1(C^*,R^*) = C_{\mathrm{mx}}\text{-}b\text{-}4\cdot \sqrt[2]{\frac{3}{k_T}} \cdot \sqrt[4]{\frac{Q}{a^o g\Pi}}\,.$$

Profit shall be positive, when the following inequality is satisfied:

$$(C_{\mathrm{mx}}\text{-}b)^2 > \frac{16}{3} k_T \sqrt[2]{\frac{Q}{a^o g\Pi}}\,.$$

Thus, we can see that when the objective is to maximise profit per unit of product, we obtain a different solution from the one for maximisation of the profit on the sum of sales of all the products.

Returning to the issue of maximising the profit from sales, let us investigate the return on the activity here considered.

Market competition for the basic model accounting for delivery costs

Let us analyse the behaviour of the return ε from production when we account for the cost of supply to all customers. If, again, we assume market clearing condition, $\Lambda = \mu$, then we obtain

$$\varepsilon = \frac{Z(\mu)}{K(\mu)} = \frac{C}{b + \overline{K}_T(R) + \dfrac{Q}{\mu}} - 1$$

In particular, in order to secure the maximum profit of the company, the following relations should be satisfied:

$$C^* = (3C_{\mathrm{mx}}+b)/4;$$

$$\overline{K}_T(R^*) = \frac{2}{3}k_t R^* = \frac{1}{2}(C_{mx} + b)\,;$$

$$R^* = \frac{3}{4k_T}(C_{mx} - b)\,;$$

$$\mu^* = \Lambda^* = g\Pi(R^*)^2(\lambda_0\text{-}a^o C^*) = a^o g\Pi\cdot\left(\frac{3}{k_T}\right)^2\left(\frac{C_{mx} - b}{4}\right)^3;$$

$$L^*_{mx} = \Pi g\left(\frac{3}{4k_T}\right)^2 (C_{mx} - b)^2\,;$$

$$Z^* = Z(C^*,R^*) = \Pi g\left(\frac{3}{16k_T}\right)^2 (C_{mx} - b)^2 - Q\ .$$

The sole possibility for entering the market, dominated by the "resident" producer, selling the product for the price C^* in quantity Λ^*, is to set in motion manufacturing activity of the intensity $\mu_1 > \Lambda$ and to sell the products manufactured for lower prices, $C_1 < C^*$, and at lower production costs:

$\kappa_1 = b + \frac{Q}{\mu_1}$ (assuming the equalities $b_1 = b$ and $Q_1 = Q$),

and, alas much lower profit than the calculated value Z^*.

Consequently, the entering competitor, by selling the products for the lower price C_1 shall give rise to a higher demand, since for $C_1 < C^*$ there is $\Lambda_1 > \Lambda^*$.

Ultimately, the final conclusions are the same as in the preceding case, without consideration of the supply cost. The price war shall ensue, in which the winning side shall be the one disposing of greater financial reserves.

There is also a possibility that another situation shall arise. Namely, the "resident" company might not react to the competitor's entry onto the market. This competitor shall be selling cheaper product to less wealthy customers. In this manner, a new, "poorer" market segment shall arise, leading to a different demand. This competitor, though, shall be gaining much lower profits. In order to increase these profits, the company might decide to lower production costs and product quality, while not losing customers. Thereby, two distinct market segments might arise, with two different products satisfying similar kind of need.

Otherwise, only one producer may stably remain on the market, servicing the area with the radius R^*. According to such a (simplified, definitely) model, the entire surface of the Globe would be covered by the areas, corresponding to monopolistic producer supply regions, the locations of the supply centres being distanced by approximately $2R^*$ (in appropriate units). As the effectiveness of production technologies increases, and the transport costs decrease, with advance of technological progress, the network of producers shall be getting sparser and sparser, with increasing radius R^*.

1.ii. Basic model 3.i without product delivery to customer's inventory

Very often a company selling a product does not ensure transport of the goods purchased to customer's location (e.g. warehouse). In such cases, the customer has to rent and/or pay for transport of the goods purchased to proper location. Naturally, this case is equivalent to the one, when along with the payment for the product, the customer pays for the transport of this product to the location of use or inventory.

In such a case the (ultimate) cost of buying the product ($C^{\#}$) is augmented by the cost of transport ($C^{\#} = C + K_T$), which allows for the following modification of the linear demand function:

$$\Lambda(C,R) =$$

$$g\int_0^R 2\Pi r a^o (C_{mx} - C - k_T r)dr =$$

$$g\left[2\Pi\int_0^R rdr \cdot a^o (C_{mx} - C) \quad - \quad 2\Pi a^o \int_0^R k_T r^2 dr\right] =$$

$$g\left[2\Pi a^o (C_{mx} - C)\int_0^R rdr - 2\Pi a^o k_T \int_0^R r^2 dr\right] =$$

$$2\Pi a^o g\left[(C_{mx} - C)\int_0^R rdr - k_T \int_0^R r^2 dr\right] = \ldots$$

$$= a^o \Pi g R[(C_{mx}-C)R^2/2-k_T R^3/3] = 2a^o \Pi g R^2\ [(C_{mx}-C)/2-k_T R/3] =$$

$$= a^o \Pi g R^2 \, [C_{mx} - C - 2k_T R/3] = AR^2(C_{mx} - C - BR),$$

where $A = a^o g \Pi$ and $B = 2k_T/3$.

As we assume the same form of the relation expressing unit cost of production, i.e.

$$\kappa = \frac{Q}{\Lambda} + b,$$

we obtain the following form of the profit function:

$$Z(R,C) = A \cdot R^2 \cdot (C - b) \cdot (C_{mx} - C - B \cdot R) - Q.$$

By differentiating this function with respect to C, we get

$$\frac{\partial}{\partial C} Z(R,C) = A \cdot R^2 \cdot (C_{mx} - C - B \cdot R) - A \cdot R^2 \cdot (C - b).$$

And now, by zeroing of the derivative, we obtain

$$C^* = \frac{1}{2} \cdot (C_{mx} + b - B \cdot R) \text{ and } Z(R,C^*) = \frac{1}{4} A \cdot R^2 (C_{mx} - b - B \cdot R)^2 - Q.$$

Then, if we differentiate the latter function

$$\frac{\partial}{\partial R} Z(R,C^*) = \frac{1}{2} \cdot A \cdot R \cdot \left[2 \cdot B^2 \cdot R^2 - 3 \cdot B \cdot R \cdot (C_{mx} - b) + (C_{mx} - b)^2 \right]$$

and, again, equate the derivative to zero, then the respective roots are

$$R_1^* = 0, \quad R_2^* = \frac{C_{mx} - b}{B}, \quad R_3^* = \frac{1}{2} \cdot \frac{C_{mx} - b}{B}$$

As we substitute the proper root (R^*_3) to the formula for the value of C^*, we get

$$C^* = (C_{mx} + 3b)/4 \text{ and } Z^* = Z(R^*, C^*) = \frac{1}{64} \cdot \frac{A}{B^2} (C_{mx} - b)^4.$$

If we change the sequence of differentiation of the function $Z(R,C)$, and start from the derivative with respect to R, then we obtain

$$\frac{\partial}{\partial R} Z(R,C) = A \cdot R^2 \cdot (C-b) \cdot \left[2 \cdot (C_{mx} - C) - 3 \cdot B \cdot R\right]$$

As we now equate the derivative to zero, we get

$$R^* = \frac{2}{3} \cdot \frac{C_{mx} - C}{B} \quad \text{and} \quad R_{1,2} = 0,$$

with

$$Z(R^*,C) = \frac{4}{27} \cdot \frac{A}{B^2}(C-b) \cdot (C_{mx} - C) - Q$$

If we now differentiate this function with respect to C, the result is

$$\frac{\partial}{\partial C} Z(R^*,C) = \frac{4}{27} \cdot \frac{A}{B^2}(C_{mx} - C)^2 \cdot (C_{mx} - 4C + 3b),$$

and when we equate the above to zero, we obtain

$$C^*_{1,2} = C_{mx} \qquad ; \qquad C^* = \frac{1}{4} \cdot (C_{mx} + 3 \cdot b)$$

After we then substitute the value of C^* to the formula, defining R^*, we get

$$R^* = \frac{2}{3} \cdot \frac{C_{mx} - C^*}{B} = \frac{2}{3} \cdot \frac{C_{mx} - \frac{1}{4} \cdot (C_{mx} + 3b)}{B} = \frac{1}{2} \cdot \frac{C_{mx} - b}{B}$$

Hence, we obtained the very same formulae for the values of R^* and C^*, so that we can be sure they are correct.

1.iii. Basic model 3.iii with product delivery and nonlinear demand function

Let us consider activity of a company, in which the cost of manufacturing a unit of product is given by the function

$$\kappa(\mu) = \frac{Q}{\mu} + b.$$

The company delivers the product commissioned to the customer's inventory, bearing, in that an additional, average cost of sale:

$$K_T(R) = \frac{2}{3}k_T R = BR.$$

Customer's potential demand ("need") is estimated as

$$\lambda = \lambda_0 \cdot \frac{C_0}{C_0 + C},$$

where C_0 and C are expressed in monetary units per unit of product.

Total demand within the zone of radius R, assuming uniform density g of customers per area unit, is equal

$$\Lambda = g\Pi R^2 \lambda_{mx} \frac{C_0}{C_0 + C}.$$

If we now introduce notation $A = g\Pi R^2 C_0$ and substitute $\mu = \lambda$, then we get the following expression for the value of profit:

$$Z(C,R) = A.R^2.\frac{C - b - \frac{2}{3}k_T R}{C + C_{\square}} - Q$$

The course of the function $Z(C,R)$ is shown, in terms of isoquants, in Fig. 3.3.

In order to assess the shape of the respective function $F(C,R)$ we differentiate it with respect to C and R. We thus get

$$\frac{\partial}{\partial C} Z(R,C) = A.R^2 \frac{C_{\square} + b + \frac{2}{3}k_T.R}{(C + C_{\square})} > 0$$

$$\frac{\partial}{\partial R} Z(R,C) = 2\frac{A \cdot R}{C + C_0} \cdot (C - b - k_T R)$$

.

Hence, we see that the function Z (Fig. 3.3) increases infinitely with the increase of C (for a given value of R). On the other hand, given a value of C, profit attains maximum for $R^* = (C\text{-}b)/k_T$ whatever this value C is.

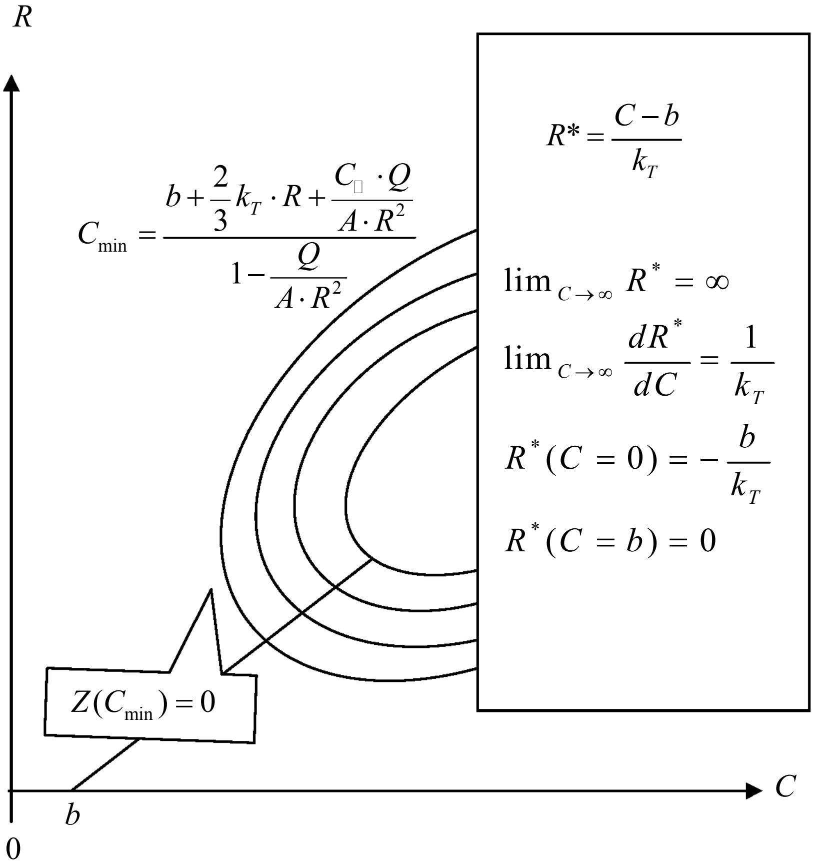

Fig. 3.3. Isoquants of profit function for the model of section 1.iii

If we now substitute into the expression for function Z the optimum value $R = R^*$, then we obtain

$$Z(R^*,C) = \frac{1}{3}\frac{A}{k_T^2}\frac{(C-b)^3}{C+C_0} - Q$$

with, of course, $Z(R^*,0) = -\frac{1}{3}\frac{A}{k_T^2}\frac{b^3}{C_0} - Q$.

If we now determine the derivative of the function $Z(R^*,C)$ with respect to C, we get

$$\frac{\partial}{\partial C} Z(R^*, C) = \frac{1}{3} \cdot \frac{A}{k_T^2} \cdot \frac{(C-b)^2}{(C+C_0)^2} \cdot (2C + 3C_0 + b) \geq 0$$

This function has three roots, namely:

$$C_3^* = -\frac{3C_0 + b}{2} \text{ and } C_{1,2}^* = b,$$

and the value of the function $Z(R^*,C)$, in its dependence upon C (for $C > b$) increases infinitely.

Let us now investigate the behaviour of

Profit per unit of product

Thus:

for $\Lambda = AR^2/(C_0+C)$, where $A = C_0 g \pi \lambda_{\mathrm{mx}}$,

and $\kappa = \dfrac{Q}{\mu} + b + BR$, where $B = \dfrac{2}{3} k_T$

we have $Z^0 = C - \dfrac{Q}{A} \cdot \dfrac{C_0 + C}{R^2} - b - BR$.

Hence

$$\frac{\partial}{\partial C} Z^0 (R, C) = 1 - \frac{Q}{A \cdot R^2}.$$

We can see, therefore, that

$$\frac{\partial Z^0}{\partial C} < 0 \text{ for } R < R_0,$$

$$\frac{\partial Z^0}{\partial C} > 0 \text{ for } R > R_0, \text{ where } R_0 = (Q/A)^{1/2}$$

and at point R_0 profit attains its minimum value.

As we differentiate with respect to R, we obtain

$$\frac{\partial}{\partial R} Z^{\square}(R,C) = 2 \cdot \frac{Q}{A} \cdot \frac{C_{\square} + C}{R^3} - B \; .$$

By equating the derivative to zero, we get

$$R^* = \left(\frac{3Q}{A} \cdot \frac{C_0 + C}{k_T} \right)^{1/3} .$$

At this point the profit function attains its maximum value:

$$Z(R^*, C) = C - b \text{ - } 3\left(\frac{Q(C_0 + C)k_T^2}{9A} \right).$$

It can then be easily seen that profit increases without any limit as C increases, and it is positive for C not less than C_0, fulfilling the inequality

$$C_0 - b - 3 \cdot \sqrt[3]{\frac{Q \cdot (C_{\square} + C^*) \cdot k_T^2}{9 \cdot A}} = 0 \; .$$

<u>1.iv. Basic model 3.iii with nonlinear dependence of demand upon product price without product delivery</u>

In this case, we have

$$\kappa = b + \frac{Q}{\mu} \;\text{ and }\; \lambda = \lambda_0 \frac{C_0}{C_0 + C},$$

$$\text{hence } \Lambda = \frac{AR^2}{C_0 + C + BR} = \frac{L_{mx}(R)\lambda_0}{C_0 + C + BR},$$

where $A = C_0 g \pi \lambda_0$,

and the profit function shall take on the form

$$Z(R,C) = A \cdot R^2 \cdot \frac{C-b}{C_{\square} + C + B \cdot R} - Q$$

As we determine the derivative with respect to C, we get

$$\frac{\partial}{\partial C} Z(R,C) = A \cdot R^2 \cdot \frac{C_{\square} + b + B \cdot R}{\left[C_{\square} + C + B \cdot R\right]^2} > 0$$

Thus, we can see that the function Z increases in C, although at a decreasing rate, up to the limit value AR^2-Q.

By differentiating the profit function with respect to R, we get

$$\frac{\partial}{\partial R} Z(R,C) = A \cdot R \cdot (C-b) \cdot \frac{2 \cdot (C_{\square} + C) + B \cdot R}{\left[C_{\square} + C + B \cdot R\right]^2} > 0$$

which, indeed, means that function Z increases without limit in R.

Let us next analyse the behaviour of

Profit per unit of product

Profit per unit of product is expressed with the formula

$$Z(R,C) = C - \frac{Q}{A} \cdot \frac{C_{\square} + C + B \cdot R}{R^2} - b$$

For $R>(Q/A)^{1/2}$, profit shall be positive when $C>C_0$, where

$$C_0 = \frac{\frac{Q}{A} \cdot \frac{C_{\square} + B \cdot R}{R^2} + b}{1 - \frac{Q}{A \cdot R^2}}$$

Besides, unit profit shall increase without limit along with C, and will increase along with R, tending to the value C-b.

Market competition for the model 3.iii

Since profit from the activity here described is a nondecreasing function of C and R, we deal with a situation exactly opposite to the one, under which perfect competition arises, and demonstrates the absolute domination of the "resident" firms, which took over the most lucrative part of the market of the "rich" customers.

Entry of a competitive company with an equivalent product, sold at lower price, may at most lead to the "detachment" of a separate market of less wealthy customers, as this was described for the preceding model.

It is, of course, also possible that a competitor enters the market with an even more expensive equivalent product, taking advantage of the attitude of some customers of gaining higher prestige by using a more expensive product, distinguished by the insignificant ornaments. We shall not deal with such cases, since, according to the initial assumptions, we only consider the rationally behaving customers.

2. A complex model with product delivery costs

In this model we shall assume that, similarly as before, product transport costs are deducted from the proceeds of the company, but that the density of customers per surface unit is not constant, decreasing proportionally to the distance r from the centre of the area, where the manufacturer, selling the produce, is located. According to the classification from Chapter II, this is basic model 2.i, with product delivery under uneven spatial distribution of customers.

The remaining assumptions shall be kept unchanged, including the one concerning the magnitude of customers' demand, linearly decreasing with the increase of product price.

Changes, therefore, concern the profit function Z, in which we must additionally account for the variable transport costs. We also have to determine again the value of demand, Λ, depending upon the radius of the area, over which product is sold – or, more precisely, upon the number of customers on the area and the quantity of product, purchased by every customer.

If we designate with symbol λ_0, as before, the maximum (say: annual) demand for the product of a single customer, then demand at price C shall be equal $\lambda = \lambda_0 - a^{o}C$, with $a^{o} = \lambda_0/C_{mx}$.

Next, denote with g_{mx} the maximum density of customers, purchasing the product considered, for $r \to 0$, per unit area of sale.

For an $r>0$ the density of potential customers, interested in the product, per unit area at distance r from the centre, shall decrease to the value $g = g_{mx} - \varphi r$ [in, e.g., customers per square kilometre], with $0 \le r \le R_{mx}$, and, of course, $R_{mx} = g_{mx}/\varphi$ and $\varphi = g_{mx}/R_{mx}$.

Consequently, the expected sale of the product per area unit, distanced by r from the centre, during a year, shall be equal

$$\lambda(r) = (\lambda_0 - a^{o}C)(g_{mx} - \varphi r).$$

With this, the profit z_{mx} per product unit, must be decreased by the cost of product delivery to the customer's location,i.e. over distance r:

$$z = z_{mx} - k_T r, \text{ where } z_{mx} = C - b.$$

In the above, k_T is, as before, the transport tariff per unit of distance and of product. The value of z_{mx} corresponds to the maximum profit per unit of product for $r=0$. Denote by l_{mx} the distance of product delivery, for which transport cost absorbs the entire profit from product sale, under a given price C:

$$l_{mx} = \frac{z_{mx}}{k_T}.$$

In order to avoid any potential misunderstandings, let us put the three relations, presented before, in a unified framework:

$$\lambda = \lambda_0 - a^{\circ}C = \lambda_0(1 - \frac{C}{C_{mx}}) = a^{\circ}(C_{mx} - C);$$

$$g = g_{mx} - \varphi r = g_{mx}(1 - \frac{r}{R_{mx}}) = \varphi(R_{mx} - r);$$

$$z = z_{mx} - k_T r = z_{mx}(1 - \frac{r}{l_{mx}}) = k_T(l_{mx} - r),$$

where: $a^{\circ} = \lambda_0/C_{mx}$, $\varphi = g_{mx}/r_{mx}$, $k_T = z_{mx}/l_{mx}$, $z_{mx} = C - b$.

Note that the area, over which the sales are carried out, is equal πR^2, where R is the radius of the circle, in whose centre the producer is located. Hence, the total demand $\Lambda(R)$ for the product considered within the zone of radius R is given by the expression

$$\Lambda(R) = \int_{\Omega} \lambda(r) \cdot dS = \int_0^R \lambda(x) \cdot 2\Pi x \cdot dx = 2 \cdot \Pi \cdot (\lambda_0 - \overset{\circ}{a} \cdot C) \cdot \left(\frac{1}{2} g_{mx} - \frac{1}{3} \varphi \cdot R \right) \cdot R^2 .$$

Let us now determine the transport costs to all the customers, distanced by less than R_{mx} from the producer (or the seller):

$$K_T(R) = 2 \cdot \Pi \cdot k_T \int_0^R \lambda(x) \cdot x^2 \cdot dx = 2 \cdot \Pi \cdot k_T \cdot (\lambda_0 - \overset{\circ}{a} \cdot C) \cdot \left(\frac{1}{3} g_{mx} - \frac{1}{4} \varphi \cdot R \right) \cdot R^3$$

Consequently, costs of manufacturing and delivering the product to the customers shall be equal

$$\Lambda\left(b + \frac{Q}{\Lambda}\right) + K_T = 2 \cdot \Pi \cdot (\lambda_0 - \overset{\circ}{a} \cdot C) \cdot \left[\left(\frac{1}{2} g_{mx} - \frac{1}{3} \varphi \cdot R \right) R^2 \cdot b + \left(\frac{1}{3} g_{mx} - \frac{1}{4} \varphi \cdot R \right) R^3 \cdot k_T \ \right] + Q$$

As the sales value is given by the formula

$$C \cdot 2\Pi \cdot (\lambda_0 - \overset{\circ}{a} \cdot C) \cdot \left(\frac{1}{2} g_{mx} - \frac{1}{3} \varphi \cdot R \right) R^2$$

then the profit function, assuming demand-supply equilibrium (market clearing), takes on the following form:

$$Z = 2\Pi \cdot (\lambda_0 - \overset{\circ}{a} \cdot C) \cdot k_T \cdot d \cdot \left[\frac{C-b}{k_T} \left(\frac{1}{2} \frac{g_{mx}}{\varphi} - \frac{1}{3} R \right) R^2 - \left(\frac{1}{3} \frac{g_{mx}}{\varphi} - \frac{1}{4} R \right) R^3 \right] - Q$$

Let us now investigate the behaviour of the function $F = Z+Q$ and introduce the following notation for this purpose

$$A = 2\pi k_T \varphi a^{o};$$

$$D = (C-b)/k_T;$$

$$R_{mx} = g_{mx}/\varphi.$$

Then, expression in the square brackets in the last formula can be rewritten in the form

$$f_0(R,C) = \frac{1}{4} R^2 - \frac{1}{3}(R_{mx} + D) \cdot R + \frac{1}{2} D \cdot R_{mx}$$

and the function F in the form

$$F = A \cdot (C_{mx} - C) \cdot f_0(R,C) \cdot R^2$$

As we differentiate function F with respect to R, we get (see Fig. 3.4):

$$\frac{\partial F}{\partial R} = A \cdot \left\{ R^2 - (D + R_{mx}) \cdot R + D \cdot R_{mx} \right\} \cdot R = A \cdot f_1(R,C) \cdot R$$

The roots of the function f_1 are $R_1 = D$ and $R_2 = R_{mx}$.

Hence, the optimum value of the variable R is

$$R^* = \min\{D, R_{mx}\}.$$

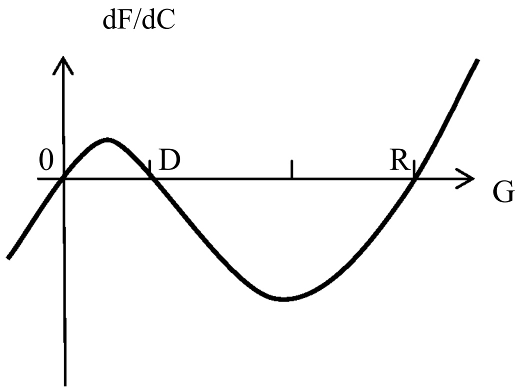

Fig. 3.4. The derivative of function *F* (here "R" denotes R_{mx})

<u>Case of $R_{mx}<D$</u>

In this case, we have $R^* = R_{mx}$, and

$$F = A\cdot(C_{mx}-C)\cdot f_0(R_{mx},C)\cdot R_{mx}^2 = A\frac{k_T}{6}(D_{mx}-D)\cdot(D-\frac{R_{mx}}{2})\cdot R_{mx}^3$$

since $D = (C-b)/k_T$, hence: $C = Dk_T+b$ and $C_{mx}-C = k_T(D_{mx}-D)$, where we used the notation $D_{mx} = (C_{mx}-b)/k_T$.

The roots of this latter variant of function *F* are: $D_1 = R_{mx}/2$ and $D_2 = D_{mx}$.

We treated function *F* as a function of the variable *D*, depending upon *C*. Let us find the value D^*, maximizing *F*. For this purpose we determine the derivative:

$$\frac{dF}{dD} = 2A_0\left[\frac{1}{2}\left(D_{mx}+\frac{R_{mx}}{2}\right)-D\right], \text{ where } A_0 = \frac{A}{6}k_T R_{mx}^3.$$

So, as we can see, function *F* attains its maximum at the point

$$D^* = \frac{1}{2}\left(D_{mx}+\frac{R_{mx}}{2}\right).$$

If now we introduce this value to the expression for the function *F*, we obtain

$$F(D^*) = \frac{A_0}{4}(D_{mx} - \frac{R_{mx}}{2})^2$$

But $D^* = \dfrac{C^* - b}{k_T}$, so that $\dfrac{1}{2}(D_{mx} + \dfrac{R_{mx}}{2}) = \dfrac{C^* - b}{k_T}$, hence

$$C^* = \frac{1}{2}k_T\left(D_{mx} + \frac{R_{mx}}{2}\right) + b\,.$$

Since $D_{mx} = \dfrac{C_{mx} - b}{k_T}$, then, ultimately, function F attains its maximum at the point

$$R^* = R_{mx}$$

$$C^* = \frac{1}{2}\left(C_{mx} + b + \frac{1}{2}R_{mx}\cdot k_T\right)$$

provided that the initially assumed inequality $R_{mx} < \dfrac{C^* - b}{k_T}$ holds, or, after substitution of the value of C^*, the inequality $R_{mx} < \dfrac{2}{3}D_{mx}$, or yet $\dfrac{3}{2}R_{mx}k_T + b < C_{mx}$.

<u>Case of $D < R_{mx}$</u>

In this case $R^* = D^*$, and as we substitute $(C-b)/k_T = D$, we get

$$F = A(C_{mx} - C)\cdot f_0(D)\cdot D^2 = A\frac{k_T}{12}(D_{mx} - D)\cdot(2R_{mx} - D)\cdot D^3$$

The roots of this function are

$$D_1 = 0\ ;\quad D_2 = D_{mx}\ ;\quad D_3 = 2R_{mx}$$

All of these roots are located outside of the interval of variability of D that is of interest for us, i.e. $0 < D < D_{mx}$. In order, therefore, to find the optimum value of D, maximizing the function of profit (within the range that is of interest to us), we shall determine the derivative of the function F with respect to D. Thus,

$$\frac{dF}{dD} = \frac{A}{12} \cdot k_T \cdot \left\{5D^2 - 4(2R_{mx} + D) + 6D_{mx}R_{mx}\right\} \cdot D^2$$

The roots of this function are

$$D_{1,2} = \frac{2}{5}\left\{(2R_{mx} + D_{mx}) \cdot \left[1 \pm \sqrt{1 - \frac{15}{8}\frac{D_{mx}R_{mx}}{(2R_{mx} + D_{mx})^2}}\right]\right\}$$

and $D_3 = 0$.

The shape of the function F implies that

$$D^* = \frac{2}{5}(2R_{mx} + D_{mx}) \cdot \alpha,$$

where $\alpha = 1 - \left(1 - \frac{15}{8}\frac{D_{mx}R_{mx}}{(2R_{mx} + D_{mx})^2}\right)^{1/2}$, and $0 < \alpha < 1$.

Now, as we substitute the optimum value D^* to the function F, we obtain

$$F(D^*) = \frac{A}{12}k_T(D_{mx} - D^*) \cdot (2R_{mx} - D^*) \cdot (D^*)^3 =$$

$$= \frac{A}{12}k_T\left(\frac{2}{5}\alpha\right)^3 \cdot \left[D_{mx}(1 - \frac{2}{5}\alpha) - \frac{4}{5}R_{mx}\alpha\right]\left[R_{mx}(2 - \frac{4}{5}\alpha) - \frac{2}{5}D_{mx}\alpha\right](2R_{mx} + D)^3$$

From the equality $D^* = \frac{C^* - b}{k_T}$ we deduce that

$$C^* = \frac{2}{5}(2R_{mx} + D_{mx}) \cdot \alpha \cdot k_T + b$$

while from the equality $R^* = \frac{C^* - b}{k_T}$ we get $R^* = \frac{2}{5}(2R_{mx} + D_{mx}) \cdot \alpha$.

The formulae for the values of C^* and R^* are valid, if the assumed inequality, $D^* < R_{mx}$, holds, this inequality taking on, after the substitution of the value for D^*, the following form

$$\beta D_{mx} < R_{mx} \text{ or } \beta(C_{mx} - b) < R_{mx}k_T, \text{ where } \beta = \frac{2\alpha}{5-4\alpha}.$$

In particular, when α<5/7, condition 2/3 D_{mx}<R_{mx} is fulfilled.

The values of R^* and C^*, determined for both cases, define the optimum zone and price of product sales, in terms of profit maximisation.

Ultimately, then, profit Z, accounting for the necessity of covering constant costs, Q) shall be expressed through the formula $Z = F - Q$, where

$$F(D^*) = \frac{A_0}{4}(D_{mx} - \frac{R_{mx}}{2})^2$$

for R_{mx} < 2/3 D_{mx}, or

$$F(D^*) = \frac{A}{12}k_T(D_{mx} - D^*)\cdot(2R_{mx} - D^*)\cdot(D^*)^3 =$$

$$= \frac{A}{12}k_T\left(\frac{2}{5}\alpha\right)^3 \cdot \left[D_{mx}(1-\frac{2}{5}\alpha) - \frac{4}{5}R_{mx}\alpha\right]\left[R_{mx}(2-\frac{4}{5}\alpha) - \frac{2}{5}D_{mx}\alpha\right](2R_{mx} + D)^3$$

for R_{mx} > 2/3 D_{mx}.

Market competition for the basic model with uneven density of customers, accounting for delivery cost

We analyse the function of return on production,

$$\varepsilon(\mu) = \frac{C}{b + \frac{D(R)+Q}{\mu(R)}} - 1$$

where, for an existing producer, the optimum sales price of the product manufactured is equal

$$C^* = \frac{C_{mx} + C_T}{2} = \frac{1}{2}\left(\frac{\lambda_0}{\overset{\circ}{a}} + \frac{1}{2}\frac{g_{mx}}{\varphi}k_T - b\right)$$

in conditions of optimum production volume

$$\Lambda(R^*) = \mu(R^*) = \frac{\Pi}{6}\frac{g_{mx}^4}{\varphi^3}\left[\lambda_{mx} + \overset{\circ}{a}\cdot\left(b - \frac{1}{2}\frac{g_{mx}}{\varphi}k_T\right)\right]$$

for

$$D^* = D(R^*) = \frac{1}{2k_T}\left(\frac{\lambda_0}{\overset{\circ}{a}}\frac{1}{2}\frac{g_{mx}}{\varphi}k_T - 3b\right)$$

with $R^* = g_{mx}/\varphi$.

Under the optimum values of *C*, *R* and μ, the value of the return function, ε, depends uniquely upon the parameters of the model, and so is the same for all the competitors.

Hence, we see that a newcomer, in order to push the resident company from the market, has to decrease product price, entering the market with higher production volume and servicing the area of greater radius. All these controls, though, bring the newcomer (usually) a lower profit than the one enjoyed by the resident company. The values of the respective decision variables, *C*, *R* and μ shall, namely, be different from the optimal ones.

Consequently, when the resident company, defending its market share, shall also lower the price of the product in question, the price war shall begin. If, however, the resident company does not react to the action of the competitor, this may lead to the appearance of a separate segment of the market, with less wealthy customers and – supposedly – lower quality products.

This book presents a complete exposition of a coherent and far-reaching theory of market competition. It is based on simple precepts, does not require deep knowledge of either economics or mathematics, and is therefore aimed primarily at undergraduate students and all those trying to put in order their vision of how the essential market mechanisms might work. Volume II, now in preparation, shall bring the theory to further problems and results.

The logic of the presentation is straightforward; it associates the microeconomic elements to arrive at both more general conclusions and at concrete formulae defining the way the market mechanisms work under definite assumed conditions.

Some may consider this exposition too simplistic. In fact, it is deliberately kept very simple, for heuristic purposes, as well as in order to make the conclusions more clear. Adding a lot of details that make theory more realistic – these details, indeed, changing from country to country, and from sector to sector – is mainly left to the Reader, who is supposed to be able to design the more accurate image on the basis of the foundations, provided in the book.

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