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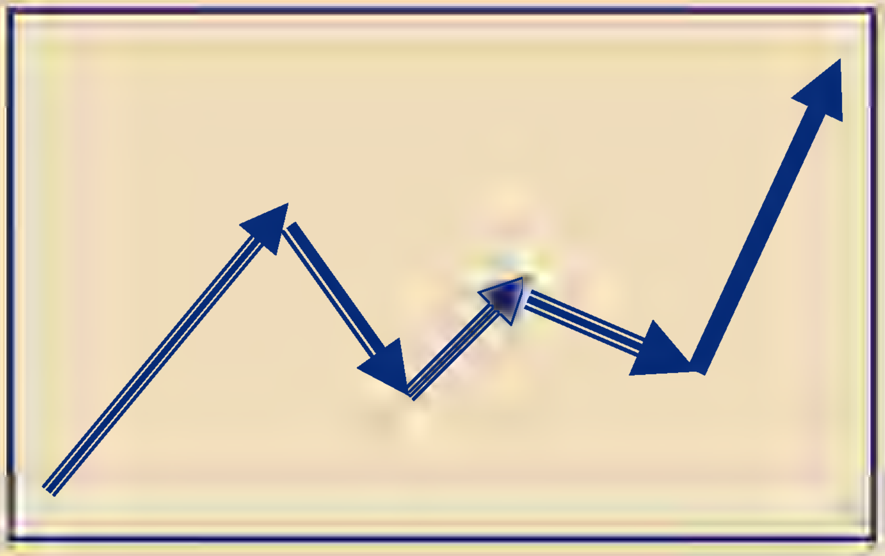
AN INTRODUCTION TO A THEORY OF MARKET COMPETITION

Volume I



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S. PIASECKI, J.W. OWSIŃSKI, AN INTRODUCTION TO A THEORY OF MARKET COMPETITION

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Chapter IV

Optimising the activity of a global company

1. Models of companies for constant spatial densities of potential customers

We assume that a company sells its products in a certain zone at price C per unit of product. Let R be the radius of this zone, and g the average density of spatial distribution of the potential customers (number of customers per area unit). Then, the total number of customers catered to is defined by the formula

$L_{mx} = g\pi R^2$.

We shall also assume that the average annual potential demand from a potential customer for the products in question depends upon the price C in the following manner:

$$\lambda = \lambda_0 q(C) = \lambda_0\left(1 - \frac{C}{C_{mx}}\right) = \frac{\lambda_0}{C_{mx}}(C_{mx} - C) = a^o(C_{mx} - C),$$

expressed in product units per customer per time unit. Here, λ_0 is the demand generated by an individual (potential) customer over a time unit (e.g. a year), while C_{mx} is the price, for which this demand tends to zero, $\lambda \to 0$, and, of course, $a^o = \lambda_0/C_{mx}$.

In the above expression $q(C)$ could be interpreted as "probability" that a potential customer shall, in fact, become an actual customer.

Consequently, the overall demand for the products of the company, sold within the zone considered, shall be defined as

$\Lambda = L_{mx}\lambda = g\pi a^o R^2(C_{mx}-C)$.

We shall assume, next, without any detriment to the general character of the considerations, that the product is delivered to the purchasing customer by the company at its expense. Note that if the delivery cost were to be put on the customer, this would actually amount to increasing the final price for the product, as perceived by the customer, by the transport cost, and the ultimate effect would be the same. It is assumed that the cost of delivery of the product by the company varies with distance r, over which the product is transported. If the customers are distributed over a circular zone of radius R, and delivery is effected from the centre, the value of r shall vary from 0 to R.

Thus, the average cost of transporting the product to the evenly distributed customers would amount to $K_T = \frac{2}{3} R k_T$, per product unit, with k_T being the transport tariff applied.

Assume, next, that the cost of turning out a unit of product is defined by the following function, accounting for the "scale effect":

$$\kappa = \frac{Q}{\mu} + b,$$

where μ is the scale (intensity) of production in product units per time unit, Q represents constant production costs, and b is the direct cost of production.

Production intensity μ can vary from 0 to a certain μ_{mx}, the maximum intensity, for which the respective production line was designed. The constant costs of maintaining production, irrespective of the value of μ, encompass amortization of productive assets, technical maintenance cost, cost of technical personnel, air conditioning, lighting and maintenance of buildings, estate tax, etc. The direct costs of production, which depend upon μ, include the cost of directly used labour and the costs of materials used (in some cases – also energy), when manufacturing the product. Direct costs are usually standardized in the sense of calculated and verified rates of use of materials, machine times, operator times etc., as well as wage rates per unit product.

Based on the notations introduced, we derive the following expression for the value of profit of the company:

$$Z(C,R,\mu) = C\Lambda - \mu \cdot \kappa(\mu) - K_T\Lambda - Q.$$

By making use of the market clearing assumption of $\mu = \Lambda$, we get

$$Z(C,R) = (C - b - K_T)\Lambda - Q.$$

As the company wishes to maximize the profit, it will try to sell its product for the optimum price (see Chapter III), i.e.

$$C^* = \frac{1}{2}(C_{mx} + b + \frac{2}{3}k_T R^*),$$

while ensuring delivery of the product to all customers, located within the circular zone of optimum radius, i.e.

$$R^* = \frac{3}{4k_T}(C_{mx} - b).$$

Further extension of the sales zone, beyond the value of radius R^*, shall bring diminishing profit. It is easy to note that the optimum radius of the sales zone increases as the transport tariff decreases. Hence, in particular, the sales zone may even encompass the entire globe, when transport of products costs very little in comparison with the unit price of products (like in the case of computer processors). Note also that as the transport costs decrease, due to the technological advances in construction of transport means and improvement of transport routes, the number of companies, manufacturing a given product, should, according to the here presented perspective, decrease. The intensity of this phenomenon is associated closely with the rate of globalisation processes.

The here determined value of R^* does not necessarily have to be the ultimate limit of the sales zone. There is namely an organisational possibility of lowering yet the costs of transport, since the "scale effect" exists also in transport.

In order to take advantage of this possibility, the company would establish a network of the sales outlets of its products, these sales outlets, indexed i, $i = 1, 2, \ldots$, servicing local customers within the sales zones of respective radiuses R_i. These sales outlets may bear various names, like wholesale outlets, local branches, logistic centres, etc. Yet, the price of the product sold from an outlet i, C_i, must account for the cost of transport between the main centre of the company and the local outlet. If the distance between the main centre and the local outlet is d_i, then one could expect unit transport cost of $d_i k_T$, but, actually, the cost may be much lower, owing to the fact that large quantities of the product are transported, $V_i >> 1$, allowing for the use of cheaper means of mass transport.

When transport means of very high capacity are used, carrying very large quantities of product, it is frequent that the cost of using the transport means does not depend proportionally upon the distance travelled. Besides, when containers are used, cost does not depend upon the degree of filling of the containers. The transporting company calculates the cost of moving and renting a container taking into account the actual route followed (very often not the shortest one at all), as well as – or even first of all – the period the container is occupied, and assuming it is fully used (no fee difference for various degrees of filling).

We shall simplify the calculation of transport costs, taking that it is composed of:

- constant cost K_M,
- cost proportional to the distance, over which goods are transported, d_i, and the transport tariff, k_M (per distance unit).

Consequently, the cost of transporting a single product to the branch location indexed i shall be expressed as

$$K_i = \frac{K_M + k_M d_i}{V_M},$$

where V_M is the capacity of the transport means, taken here as the volume of goods transported. In order for the mass transport to be profitable, the following relation must hold:

$$K_i = \frac{K_M + k_M d_i}{V_M} < k_T d_i,$$

which is, of course, satisfied, when

$$d_i > \frac{K_M}{k_T V_M - k_M}.$$

In the above, k_T is, as before, the transport tariff per single item of product and unit of distance.

Hence, using large-scale transport means is profitable, therefore, for sufficiently large freight scales and distances of transport. So, we see here again the "scale effect", known from production, namely – the cost of transporting a unit of product is lower, for the same distance, when larger quantities are transported.

In what follows we shall be assuming that transport of products to far-off branch locations takes place by means of large scale mass transport equipment, while transport from the branch location to the final customer is done with the transport means for carrying single product items.

2. Model of a global company with product delivery and variable sales price in distant branch facilities

In this case, it is ultimately the customers who have to cover the cost of transport, K_i, of the quantities V_M of products from the producer to the distant branches. Consequently, the price C_i, of the product sold at branch i, must be increased by the value of cost K_i:

$$C_i = C + K_i.$$

At the same, according to the assumption, the company considered has to secure delivery to the customers within the area of the circle of the radius R_i, serviced by the branch. This entails bearing of the additional average cost of transporting the product to the customer, namely

$$K_{iT} = \frac{2}{3} R_i k_T.$$

We shall further assume that the price of the product "loco inventory of producer" should be determined so as to maximise profit. If the sales network is owned by the producer, the value of C ought to maximise the global profit of the company – production and sales network, along with transport.

When producer is not the owner of the sales network, then the sales price ought to maximise the profit of the network, under the assumption that producer sells its products for a higher price:

$$C' = \frac{Q}{\Lambda} + b', \text{ with } b' > b,$$

the difference, b'-b being the profit per unit of product, gained by the producer.

As we know, producer lowers the price of product as the network increases order volumes, as with price decreases the sales increase. Consequently, producer and the network are linked by the common interest of increasing their profits. Thereby, the two cases can be reduced to one model of costs and revenues with maximisation of joint profit, leaving aside the issue of dividing the joint profit as the secondary one. This can, of course, be proposed only if we assume that producer and seller (the network) act rationally.

In what follows we shall be using exclusively the symbol b, understanding that if the producer is not the owner of the sales network, then b shall, in fact, mean b'.

Now, let us determine the global profit of the company, selling products manufactured within the "company" sales zone, indexed i =0, and in some distant sales zone, with $i \neq 0$. We get

$$Z = (C - \kappa - \mathrm{K}_{0T}) \cdot \Lambda_0 + (C_i - \kappa - K_i - \mathrm{K}_{\mathrm{iT}}) \cdot \Lambda_i$$

where $C_i = C + K_i$ and $\kappa = \dfrac{Q}{\Lambda_0 + \Lambda_i}$.

Then, as we introduce the notation:

$\Lambda_0 = A_0(C_{\mathrm{mx}}\text{-}C)$; $\Lambda_I = A_i(C^i_{\mathrm{mx}}\text{-}C)$, where $C^i_{\mathrm{mx}} = C_{\mathrm{mx}}\text{-}K_i$,

$A_0 = \pi R_0^2 g a^{\mathrm{o}}$; $A_i = \pi R_i^2 g a^o$;

$D_0 = \dfrac{2}{3} k_T R_0 A_0$; $D_i = \dfrac{2}{3} k_T R_i A_i$, we obtain the following formula for the profit of the producer and the seller:

$$Z(C,R_0,R_i)=(C-b)\cdot\left[A_0\cdot(C_{mx}-C)+A_i\cdot(C^i_{mx}-C)\right]-\left[D_0\cdot(C_{mx}-C)+D_i\cdot(C^i_{mx}-C)\right]-Q$$

Now, as we determine the derivative of the function Z with respect to C and equate this derivative to zero, we obtain the following equation:

$$-2\cdot(A_0+A_1)\cdot C+(A_0+A_1)\cdot(b+C_{mx})-A_0\cdot K_0+D_0+D_1=0\,.$$

Therefrom, ultimately, the formula for the optimum value of the sales price in the producer zone, indexed i=0, will become

$$C^*=\frac{1}{2}\cdot\left[C_{mx}+b+\frac{2}{3}\cdot k_T\cdot\frac{R_0^3+R_i^3}{R_0^2+R_i^2}-\frac{K_M+d_i\cdot k_M}{V_M}\cdot\frac{R_0^2}{R_0^2+R_i^2}\right]$$

while for the sales zone $i > 0$ the respective expression will be:

$$C_i^*=C^*+\frac{K_M+d_i\cdot k_M}{V_M}$$

If we now introduce these expressions into the one for the "global" profit function Z, then we get

$$Z(C^*,R_0,R_i)=\frac{1}{4}\cdot\pi\cdot g\cdot a\cdot\frac{\left[(C_{mx}-b-K_i)\cdot R_0^2+(C_{mx}-b-K_0-K_i)\cdot R_i^2\right]^2}{R_0^2+R_i^2}-Q$$

As we now differentiate this expression with respect to the variables R_0 and R_i, and then equate the derivatives to zero, we can solve the resulting system of nonlinear equations, so as to obtain the optimum values of the respective radiuses, $R^*{}_0$ and $R^*{}_i$.

We can also proceed in a different manner, by using the previously derived formulae (in Chapter III) for the optimum radius of individual sales zones, i.e.:

$$R^*{}_0=\frac{3}{4k_T}(C_{\mathrm{mx}}\text{-}C)\text{ and }R^*{}_i=\frac{3}{4k_T}(C_{\mathrm{mx}}\text{-}K_i\text{-}C)$$

where $C = C^*$.

It can easily be seen that we would obtain, thereby, the problem of simultaneous optimization of the two kinds of unknowns in the form of an implicit function.

It appears that it would be much simpler to use two sets of formulae:

- for the values of C^* and C^*_i, and
- for the values of R^*_0 and R^*_i,

in the process of successive approximations.

Generally, then, a global company may have a number of branches in various regions of the world, under the condition, of course, that the sales zones, $i = 0, 1, 2, 3,\ldots$, do not overlap.

Naturally, the more distant a zone is (bigger K_i), the smaller the values of R_i.

Consequently, around a producer, manufacturing a product in question, there develops a logistic network of branches, selling the products of this producer. Automobile industry is, as of now, organized in just this manner. In case the logistic network becomes independent (since under definite conditions this is more profitable, see Piasecki, 2005a,b), a single network shall be selling cars of various producers, and this network shall dictate the prices of cars, not only the ultimate sales prices, but also the negotiated prices of purchase from the producers. Thereby, the profits of the producers shall also depend upon the policies of the network.

The issue of designing optimum parameters of the sales network, and, in particular, of the determination of the magnitude and shape of the zones, gets even more complicated, when we account for the variable population density and the existing transport (road) infrastructure. Then, the "honeycomb" structure, characterized by the diminishing magnitudes of cells along with increasing distance from the producer, shall change into a much more complex structure, "fingerlike" or "archipelago-like".

3. Model of a company selling its product for a constant price in the entire network irrespective of distance from the centre to a branch

We maintain all the previous notations, and the profit function shall take now on the form (for the centre and a branch no. i):

$$Z = (C-\kappa-K_{0T})\Lambda_0 + (C-\kappa-K_{iT}-K_i)\Lambda_i, \text{ where } \kappa = \frac{Q}{\Lambda_0 + \Lambda_i} + b.$$

After we substitute the value for Λ, we get

$$Z = [(C-b)A-K-B](C_{\mathrm{mx}}-C)-Q$$

where: $A = A_0 + A_i$, $K = K_{0T}A_0 + K_{iT}A_i$, $B = K_iA_i$.

Upon determination of the derivative of the function Z with respect to C and equating it to zero, we get:

$$-2 \cdot A \cdot C + K + B + A\left(C_{mx} + b\right) = 0$$

Hence, therefrom:

$$C^* = \frac{1}{2} \cdot (C_{mx} + b + \frac{K+B}{A})$$

After we introduce the value of $C = C^*$ into the profit function, we get

$$Z(C^*, R_0, R) = \frac{1}{2} \cdot E \cdot \left\{ \frac{1}{2} \cdot (C_{mx}^2 - b^2) \cdot R^2 + (C_{mx} + b - \frac{G}{R^2}) \cdot (\frac{3}{2} \cdot G + \frac{2}{3} \cdot k_T \cdot R_0^3) - \frac{1}{2} \cdot (C_{mx} - b) \cdot G \right\}$$

where

$$E = \pi g a^{\mathrm{o}},\ G = \frac{2}{3} k_T R_i^3 + \frac{K_M + d_i k_M}{V_M} R_i^2,\ R = R_0^2 + R_i^2.$$

Let us note that the constant sales price in the entire network is approximately equal the average sales price in the network with variable prices, depending upon the distance from the centre to the branch.

If, namely, we introduce the notation

$$C_{\otimes} = \frac{1}{2}\left(C_{mx} + b + \frac{2}{3} k_T \frac{R_0^3 + R_i^3}{R_0^2 + R_i^2} \right)$$

then we obtain:

$$C^{*}{}_{\text{CONST}} = C_{\otimes} + \frac{1}{2} K_{0T} \frac{R_0^2}{R_0^2 + R_i^2}.$$

On the other hand, assuming variable prices, we get the values of C^* (for i=0) and $C^*{}_i$, as follows:

$$C^{*} = C_{\otimes} - \frac{1}{2} K_0 \frac{R_0^2}{R_0^2 + R_i^2}, \text{ and}$$

$$C^{*}{}_i = C^{*} + K = C_{\otimes} + \frac{1}{2} K_{0T} \frac{R_0^2}{R_0^2 + R_i^2} + K_{iT} \frac{R_0^2}{R_0^2 + R_i^2}.$$

It can, therefore, be easily noted that we have

$$C^{*}{}_{\text{CONST}} = C_{\otimes} + \frac{1}{2} K_i \frac{R_i^2}{R_0^2 + R_i^2} \approx C_{\otimes} + \frac{1}{2} K_i \frac{R_0^2}{R_0^2 + R_i^2} = \frac{1}{2}(C^{*} + C_i^{*}) = C_{\text{AVERAGE}}.$$

Similarly as before, we can determine the optimum values of R_0 and R_i by equating the partial derivatives of the function $Z(C,R_0,R)$ to zero and solving the resulting system of equations, or iteratively approximating the solution with a method analogous to previously presented.

4. Basic model of a global company with variable density of customers

We shall assume in this model that – similarly as before – the cost of transporting the products are charged on the revenue of the company, but we shall also assume that the density of customers is not constant and that it decreases proportionally to the distance r

from the centre of the area, in which a producer or the wholesale outlet is located, selling the product.

The remaining assumptions are not changed, including the demand of the customers for the products, which decreases linearly with the increase of product price.

Thus, changes concern the profit function, Z, in which we must consider, additionally, the change of the transport costs, covered by the company, and the change in demand, Λ, whose magnitude depends upon the radius of the area, the variable density of customers on the area, and the average quantity of the product purchased by the customers.

If we denote by λ_0 the annual – potential – demand for the product from one potential customer, then the effective demand of a customer (disposing of a definite income) shall be equal, as before:

$$\lambda = \lambda_0 q(C) = \lambda_0\left(1-\frac{C}{C_{mx}}\right) = a^{o}(C_{mx}\text{-}C), \text{ where } a^{o} = \frac{\lambda_0}{C_{mx}}.$$

Let us next denote with g_{mx} the maximum density of the potential customers (for $r \to 0$) per unit of surface. For $r > 0$ the density of the potential customers, g, per unit of surface, at the distance r from the centre, shall decrease down to

$$g(r) = g_{mx} - \varphi r;\ 0 < r < R_{mx}, \text{ with } \varphi = g_{mx}/R_{mx}.$$

Hence

$$g(r) = g_{mx}\left(1-\frac{r}{R_{mx}}\right) = \varphi(R_{mx} - r).$$

Consequently, the expected sales of the product per unit of surface at the distance r from the centre, over a unit of time (a year), shall be equal

$$\lambda(r) = a^{o}(C_{mx} - C)(g_{mx} - \varphi r).$$

Since the area, encompassed by the sales zone of the product, is equal πR^2, where R is the radius of the circle, in whose centre the seller is located, total demand Λ for the product shall be equal

$$\Lambda(R)=\int_{\Omega}\lambda(r)dS=\int_0^R\lambda(r)\cdot 2\pi r\cdot dr=2\pi a^o(C_{mx}-C)\left(\frac{1}{2}g_{mx}-\frac{1}{3}\varphi R\right)R^2 .$$

As we determine, next, the cost of transport to all customers located at the distance of r from the producer, with, of course, $r\leq R$ (and $R\leq R_{\mathrm{mx}}$), we get

$$K_T(R)=\ 2\pi k_T\int_0^R\lambda(r)r^2dr\ =\ 2\pi k_T a^o(C_{mx}-C)\left(\frac{1}{3}g_{mx}-\frac{1}{4}\varphi R\right)R^3 .$$

Consequently, costs of producing and delivering the goods to the customers, shall be equal

$$\Lambda\left(b+\frac{Q}{\Lambda}\right)+K_T\ =$$

$$=2\pi a^{\mathrm{o}}(C_{\mathrm{mx}}\text{-}C)\left[\left(\frac{1}{2}g_{mx}-\frac{1}{3}\varphi R\right)R^2 b+\left(\frac{1}{3}g_{mx}-\frac{1}{4}\varphi R\right)R^3 k_T\right]+Q .$$

Since the value of sales is expressed through the formula

$$C\cdot 2\pi a^{\mathrm{o}}(C_{\mathrm{mx}}\text{-}C)\left(\frac{1}{2}g_{mx}-\frac{1}{3}\varphi R\right)R^2 ,$$

then the profit function, under market clearance conditions (μ=Λ), shall take on the form

$Z=$

$$= 2\pi a^{o}(C_{mx}\text{-}C)k_T q\left[\frac{C-b}{k_T}\left(\frac{1}{2}\frac{g_{mx}}{\varphi}-\frac{1}{3}R\right)R^2-\left(\frac{1}{3}\frac{g_{mx}}{\varphi}-\frac{1}{4}R\right)R^3\right]-Q\,.$$

Let us next consider the behaviour of the function

$F = Z + Q =$

$$= 2\pi k_T\varphi a^{o}(C_{mx}\text{-}C)\left[\frac{C-b}{k_T}\left(\frac{1}{2}R_{mx}-\frac{1}{3}R\right)-\left(\frac{1}{3}R_{mx}-\frac{1}{4}R\right)R\right]R^2 =$$

$$2\pi H(D_{mx}\text{-}D)\left[\frac{1}{4}R^2-\frac{1}{3}(D+R_{mx})R+\frac{1}{2}DR_{mx}\right]R^2$$, since $R_{mx}= g_{mx}/\varphi$,

where: $D = (C\text{-}b)/k_T$; $D_{mx} = (C_{mx}\text{-}b)/k_T$; $H = a^{o}\varphi k_T^2$.

Hence, function F can be written down as follows:

$$F(D,R) = 2\pi H\cdot(D_{mx}-D)\cdot f_0(R)\cdot R^2$$

where

$$f_0(R)=\frac{1}{4}R^2-\frac{1}{3}(R_{mx}+D)\cdot R+\frac{1}{2}D\cdot R_{mx}\,.$$

Let us note that D is the distance to a customer, for which the sum of direct costs, b, and transport costs, becomes equal the sales price C.

In order to facilitate the analysis of function $F(D,R)$ we shall present it in yet another form,

$$F_1(D,R) = 2\pi H(D_{mx}\text{-}D)(D\text{-}f(R))\left(\frac{1}{2}R_{mx}-\frac{1}{3}R\right)R^2,$$

where $f(R) = \dfrac{\frac{1}{3}R_{mx} - \frac{1}{4}R}{\frac{1}{2}R_{mx} - \frac{1}{3}R}R$, and the equality $F_1(D,R) = F(D,R)$ holds.

The course of the function $f(R)$ is shown in Fig. 4.1, in which notation $x = R/R_{mx}$ has been adopted.

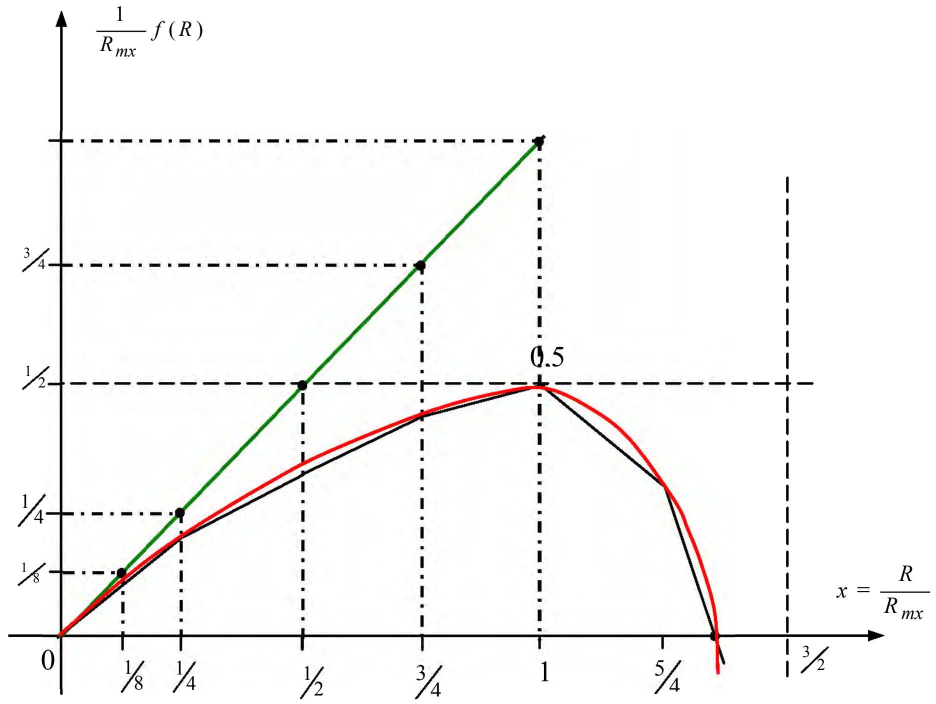


Fig. 4.1. Course of the function $f(R)$

We shall start the determination of the values D^* and R^* from the establishment of the value of D^* for a given value of R = const., where we use the notations

$$\max_D F_1(D,R) = F_1(D^*,R) = F_1^*(R)$$

We obtain immediately from the form of the function $F_1(D,R)$ that

$$D^* = \frac{1}{2}\left[D_{mx} + f(R)\right]$$

Hence, as we substitute for D_{mx}, we obtain

$$C^* = \frac{1}{2}\left[C_{mx} + k_T \cdot f(R) + b\right]$$

the optimum sales price in the sales zone of the radius R. Then, as we substitute $D = D^*$ in the function $F_1(D,R)$, we get

$$F_1^*(R) = \frac{1}{6}\pi H \cdot (\frac{3}{2}R_{mx} - R) \cdot [D_{mx} - f(R)]^2 \cdot R^2 =$$

$$= \frac{3}{2}\pi H \cdot \frac{\left[\frac{1}{4}R^2 - \frac{1}{3}\left(D_{mx} + R_{mx}\right) \cdot R + \frac{1}{2}D_{mx} \cdot R_{mx}\right]^2}{\frac{3}{2}R_{mx} - R} \cdot R^2$$

Let us note that both the denominator and the squared distance (R^2), are non-negative, in view of $0 \le R \le R_{mx}$. In particular, the value of the ratio

$$\frac{R^2}{\frac{3}{2}R_{mx} - R}$$

increases sharply in R. It remains, therefore, to check, whether the polynomial appearing in the nominator of the last expression for the function $F_1^*(R)$ has two roots. We have, namely,

$$r_{1,2} = \frac{2}{3}\left(D_{mx} + R_{mx}\right) \cdot \left\{1 \pm \sqrt{1 - \frac{9}{2} \cdot \frac{D_{mx} \cdot R_{mx}}{\left(D_{mx} + R_{mx}\right)^2}}\right\}$$

and the two roots are real, if inequality

$$1 > \frac{9}{2} \cdot \frac{R_{\max} \cdot D_{mx}}{(R_{\max} + D_{mx})^2}$$

is fulfilled. We can write down this inequality in the form

$$\frac{2}{9} > \frac{\delta}{(1+\delta)^2}\text{, where } \delta = \frac{D_{mx}}{R_{mx}} > 0.$$

The diagram of the function $y = \dfrac{\delta}{(1+\delta)^2}$ is shown in Fig. 4.2.

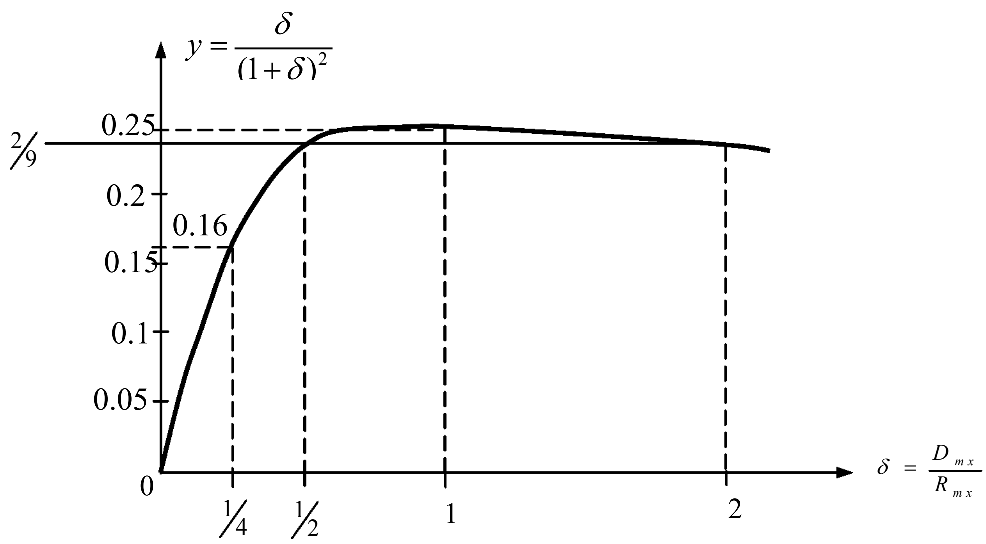


Fig. 4.2. Diagram of the auxiliary function y

As can be seen in Fig. 4.2, the inequality $y(\delta) < 2/9$ is **not** satisfied, for $\delta > 0$, over the segment of values $\delta_1 < \delta < \delta_2$, where, here, $\delta_1 = ½$, and $\delta_2 = 2$. Hence, the polynomial in the nominator of the expression for $F_1^*(R)$ shall **not** have real roots, when the inequality $½ < D/R_{mx} < 2$ is fulfilled. Otherwise, when $D_{mx}/R_{mx} \le ½$, function F_1 takes on the form

$$F_1^*(R) = \frac{3}{2}\pi H \cdot \frac{(R-r_1)^2 \cdot (R-r_2)^2}{\frac{3}{2}R_{mx} - R} \cdot R^2 .$$

Let us consider this case. If we introduce the notations

$$x = \frac{R}{R_{mx}} \;; \qquad x_1 = \frac{r_1}{R_{mx}} \;; \qquad x_2 = \frac{r_2}{R_{mx}}$$

then the expression, defining $F_1^*(R)$ for $D_{mx}/R_{mx} < ½$ can be written down as follows

$$F_1^*(x) = B\frac{(x-x_1)^2 \cdot (x-x_2)^2}{\frac{3}{2} - x} \cdot x^2 \text{, where } B = \frac{3}{2}\pi H R_{mx}^5 .$$

As we have $\delta = D_{mx}/R_{mx}$, so the area, where the function F_1^* of the argument x is defined, is given by inequalities $0 < x \le \delta \le ½$.

Thus, for instance, the values of the roots considered are, for particular cases, as follows:

for $\delta \to 0$ we have $x_1 \to 0$, and $x_2 \to 4/3$;

for $\delta = ¼$ we have $x_1 = 1/6(5-\sqrt{7})$; $x_2 = 1/6 \cdot (5+\sqrt{7})$;

for $\delta = ½$ we have $x_1 = 1$ and $x_2 = 1$.

Value of $\delta = 0$ corresponds to the situation, when $k_T \to 0$ or C_{mx}-$b \to 0$. As can be concluded from the range of values of x_1 and x_2, these roots are located outside of the interval, where function F is defined, i.e. $0 < x \le \delta \le ½$.

Figs. 4.3, 4.4 and 4.5 show changing shapes of the function F, the values of variables R and C being expressed in a relative manner through $x = R/R_{mx}$ and $y = C/C_{mx}$, for different b and k_T.

The value of H, having no influence on the shape of the function, was selected in order to make the three-dimensional diagrams more easily legible.

The role of the value of k_T for the shape of the function, for constant $b = 0.1$, can be seen in: Fig. 4.3a – for $k_T = 0.1$, Fig. 4.5b – for $k_T = 0.5$, Fig. 4.5b – for $k_T = 1.0$, and Fig. 4.3b – for $k_T = 2.0$.

The influence, exerted on the shape of the function by the value of b, for constant $k_T = 0.1$, can be seen in Fig. 4.3a – for $b = 0.1$ and Fig. 4.4a – for $b = 0.5$.

On the other hand, for constant value $k_T = 2.0$ the influence of b can be seen in Fig. 4.3b – for $b = 0.1$, and Fig. 4.4b – for $b = 0.5$.

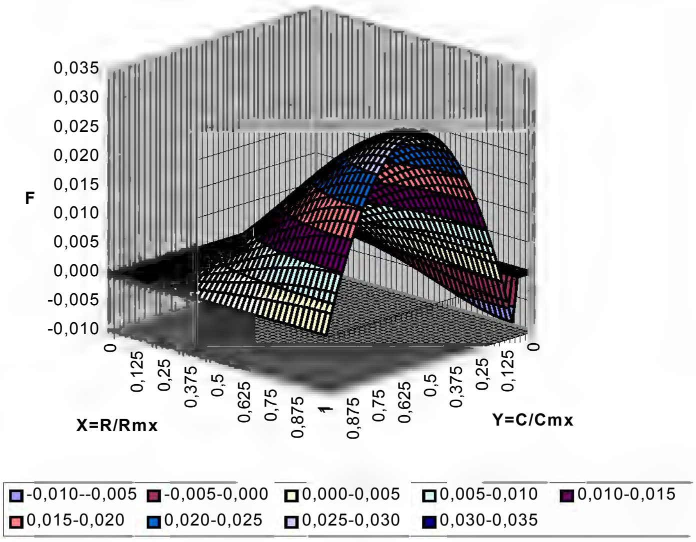
After we substitute the expression for $D^*(R)$ to the original form of the function $F(D,R)$, we obtain the formula for $F^*(R)$:

$$F^*(R) = 6\pi H\left[\frac{1}{4}R^2 - \frac{1}{3}(D_{mx} + R_{mx})\cdot R + \frac{1}{2}D_{mx}\cdot R_{mx}\right]^2 \cdot \frac{R^2}{\frac{3}{2}R_{mx} - R},$$

and then, after having introduced the substitution $x = R/R_{\mathrm{mx}}$, we get the expression for the function $F^*(x)$:

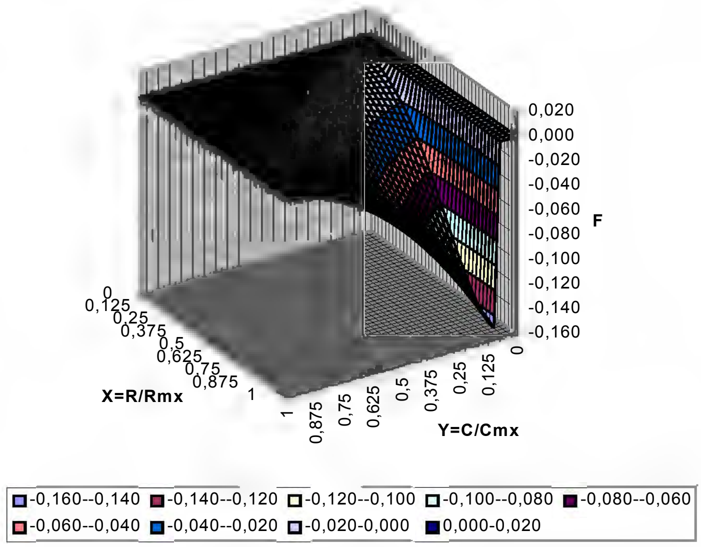
$$F^*(x) = \frac{3}{8}\pi H\left[x^2 - \frac{4}{3}(\delta + 1)\cdot x + 2\delta\right]^2 \cdot \frac{x^2}{\frac{3}{2} - x}, \text{ for } 0 \le x \le \min\{\delta, 1\},$$

whose shape is shown in Fig. 4.6.



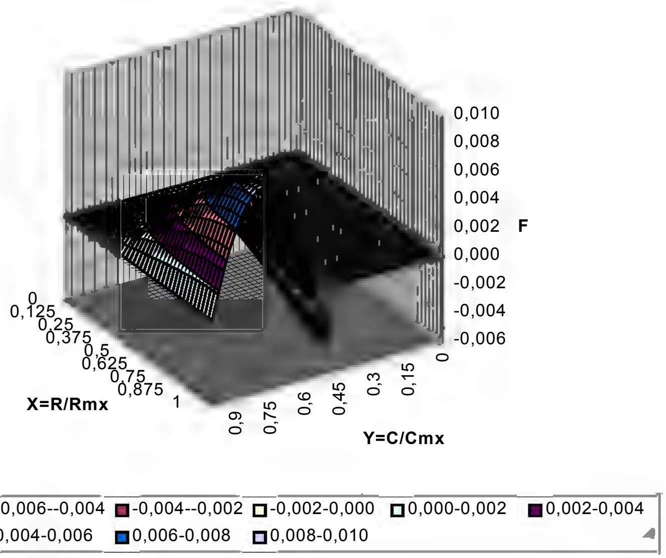
Ymin = b = 0,1	k1 = 0.1	H = 0.1
Fmax = 0.0301	Xopt = 1	Yopt = 0.575

Figure 4.3a



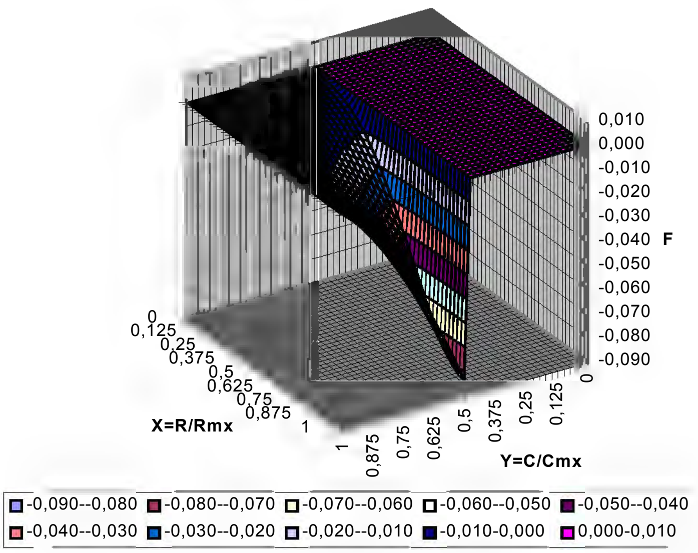
Ymin = b = 0.1	k1 = 2	H = 2
Fmax=0.0024	Xopt=0.3316	Yopt=0.7632

Figure 4.3b



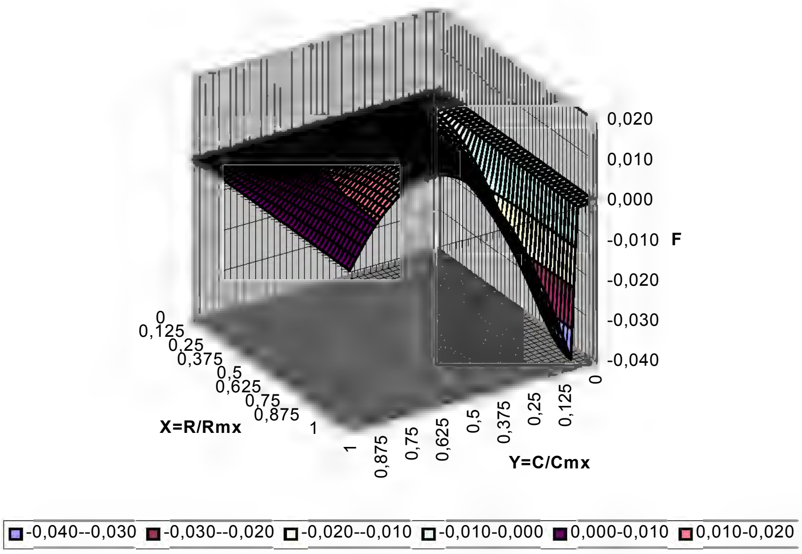
Ymin = b = 0.5	k1 = 0.1	H=0.1
Fmax = 0.0084	Xopt = 1	Yopt = 0.775

Figure 4.4a



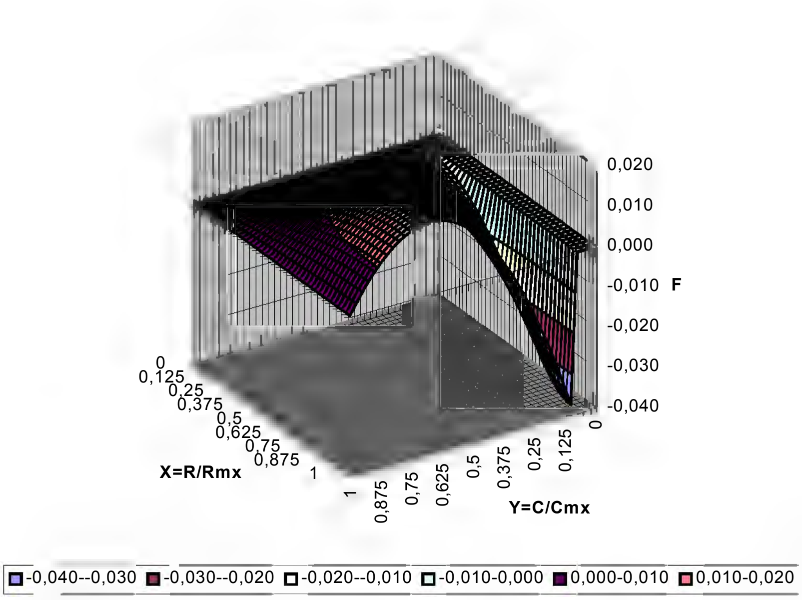
min = b = 0.5	k1 = 2	H = 2
Fmax = 0.0002	Xopt = 0.1859	Yopt = 0.8717

Figure 4.4b



Ymin = b = 0.1	k1 = 0.5	H = 0.5
Fmax = 0.0176	Xopt = 1	Yopt = 0.675

Figure 4.5a



Ymin = b = 0.1	k1 = 1		H= 1
Fmax = 0.0077	Xopt = 0.6446		Yopt = 0.7446

Figure 4.5b

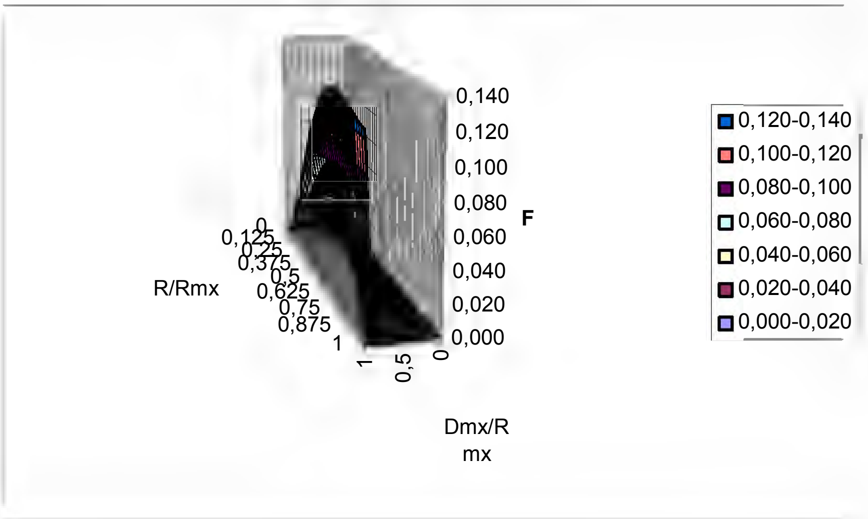


Figure 4.6a

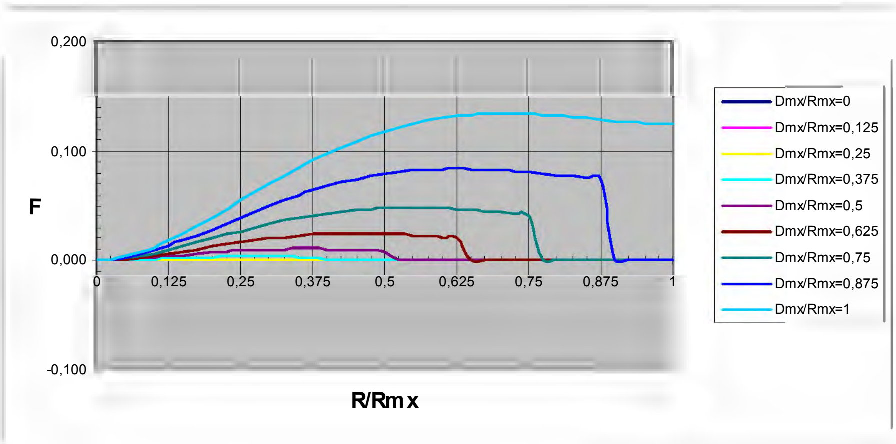


Figure 4.6b

A complete image of the function $F^*(x)$, with $x = R/R_{mx}$, in 3D space, can be seen in Fig. 4.6a, while in the form of a flat image in Fig. 4.6b, for a number of values of $\delta = D_{mx}/R_{mx}$. This latter figure makes apparent the dependence of the optimum radius of the zone, R^*, upon δ.

The second order polynomial, which appears in the function $F^*(x)$, has two real-valued non-negative roots, x_1 and x_2, only when $\delta \leq ½$.

It can be easily checked that they are bigger than δ (except for $\delta = 0$), and hence are outside of the range of values of x that are of interest for us.

At the same time, this polynomial attains the minimum value for

$$x_{\min} = \frac{2}{3}(1+\delta)$$

that is – also outside of the range of values of x we are interested in (for $x \geq ½$ as well).

Consequently, function $F^*(x)$ can be represented as the product of two functions, y_1 and y_2, the decreasing one,

$$y_1 = \left[x^2 - \frac{4}{3}(\delta+1)\cdot x + 2\delta \right]^2$$

and the increasing one:

$$y_2 = \frac{x^2}{\frac{3}{2} - x}$$

These two functions are illustrated in Fig. 4.7.

By differentiating the function $F^*(x)$, we get

$$\frac{d}{dx}F^*(x) = \frac{x^2 - \frac{4}{3}(\delta+1)\cdot x + 2\delta}{(\frac{3}{2} - x)^2} \cdot \left\{ -5x^3 + 4(\delta+1)\cdot x^2 - (10\delta+8)\cdot x + 6\delta \right\} \cdot x$$

As established before, the expression outside of the square bracket does not attain zero in the range that is of interest to us. Hence, only the third-order expression in this bracket might provide the roots of which at least one would correspond to the sought value x^*, for which $F^*(x)$ attains its maximum.

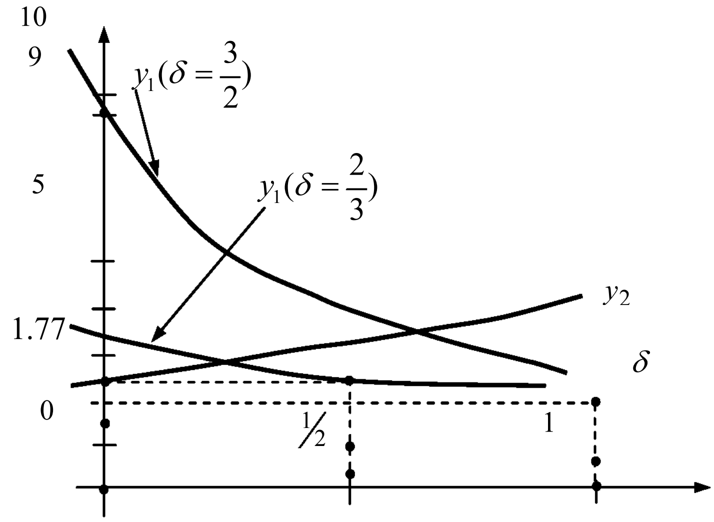


Fig. 4.7. Functions y_1 and y_2 composing F^*

A general course of this function is shown in Fig. 4.8. The pattern, appearing there, results from the fact that the characteristic parameters of the 3rd order polynomial ax^3+bx^2+cx+d take in this case the values: $a = -5 < 0$, and $\Delta = 3ac-b^2 = -16\delta^2+46\delta-49 < 0$.

Since analytic formulae for the roots of the 3rd order polynomial are quite cumbersome, it may be much easier to calculate the values of the polynomial for a sequence of values of the argument x and select the one, for which the polynomial attains the maximum.

Another technique consists in differentiating the 3rd order polynomial, so that we obtain the 2nd order polynomial, in this case having the form

$$-15x^2+(8\delta+26)x-(10\delta+8).$$

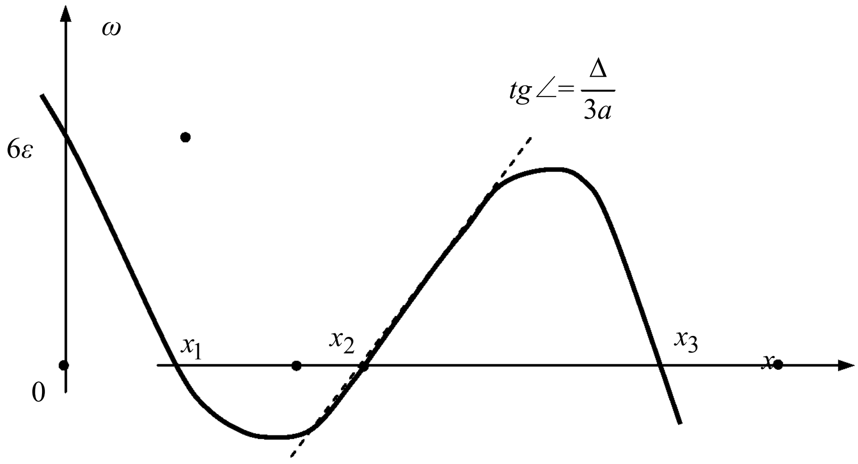


Figure 4.8. The shape of the function $F^{*}(x)$

After we equate this polynomial to zero, we get

$$x_{1,2} = \frac{1}{30}(8\delta + 26) \cdot \left\{ 1 \pm \sqrt{1 - 60\frac{10\delta + 8}{(8\delta + 26)^2}} \right\}$$

As this can be easily verified, the values of x_1 and x_2 (in the range of $0.1 \le \delta \le 1$), change within the boundaries

$0.448 \le x_1 \le 0.842$
$1.337 \le x_2 \le 1.424$.

Since the minimum of the polynomial occurs at the point $x_{\min} = \frac{1}{30}(8\delta + 26)$, we can limit the table of values of the function to the range $0 < x < x_{\min}$.

In our situation yet another way to determine the value of x^* (and R^*) is to derive an approximate formula for the value of x^*. This is convenient, especially in the case, when the maximum of the function is “flat”, like in our example. Then, even a significant deviation of the approximate location of x^* from the real one has a very limited impact on the value of $F^*(x)$ in the vicinity of the maximum.

Assuming that for practical purposes the accuracy of ±0.01 suffices, the formula for the approximate value of x^* may have the form

$$x^* \approx \begin{cases} 0.74\delta + 0.002 \ \textit{for}\ \delta \le 0.5 \\ 0.686\delta + 0.024 \ \textit{for}\ \delta > 0.5 \end{cases}.$$

Ultimately, we get $R^* = R_{\text{mx}}x^*$, and

$$C^* = \frac{1}{2}\left[C_{mx} + k_T \cdot f(R^*) + b\right]$$

where

$$f(R^*) = \frac{\frac{1}{3}R_{mx} - \frac{1}{4}R^*}{\frac{1}{2}R_{mx} - \frac{1}{3}R^*} \cdot R^* = R_{mx}\frac{\frac{1}{3} - \frac{1}{4}x^*}{\frac{1}{2} - \frac{1}{3}x^*} \cdot x^*$$

If we admit the following, very rough approximation:

$$x^* \approx \frac{3}{4}\delta, \text{ for } \delta \le ½,$$

then, after we substitute this value into the respective expressions above, we finally obtain

$$C^* \approx \frac{1}{2}\left[C_{mx} + b + R_{mx}k_T\frac{1 - \frac{9}{16}\delta}{2 - \delta} \cdot \delta\right],$$

$$R^* \approx \frac{3}{4}R_{\text{mx}},$$

$$F^* \approx \frac{9}{32}\pi H R_{mx}^5\left(1 - \frac{7}{16}\delta\right)^2 (2-\delta)^{-1}\delta.$$

These formulas, determining in an approximate manner the optimum values of the respective quantities, are, of course, valid for $\delta \leq 0.5$. Approximate formulas for the values of C^*, R^* and F^* in the case of $\delta > 0.5$ can be derived in an analogous manner.

Let us note, next, that the activity of the company shall be profitable, i.e. profit will be positive, when the inequality $F^* > Q$ is satisfied, or, after we substitute the values of C^*, R^*, the inequality

$$\frac{9}{32} \cdot \frac{\pi \cdot a \cdot \varphi \cdot k_T^2 R_{mx}^5}{Q} > \frac{2-\delta}{\left(1 - \frac{7}{16}\delta\right)^2 \cdot \delta}$$

If the highest value of the operational profit, F^*, after having deduced the constant cost, Q, is positive, then our undertaking shall be bringing benefits.

5. Network optimisation for a global company functioning on several markets

Let us consider an arbitrary sales zone (of a branch of a global company), indexed, i, situated at the distance d_i from the place, where the product of interest is manufactured. We shall assume, further, that the sales outlet (the i^{th} branch) is located at the point of maximum density of potential customers, $g_{\text{mx},i}$, this density decreasing linearly with the distance from the branch office, r.

We look for the optimum sales price C_i of the product and the radius R_i of the zone, within which the customers are offered the delivery of the product purchased at the expense of the company.

We already know (see the preceding section) that the maximum value of the operational profit for such a zone would be equal

$$F^* = F^*(R) = 6\pi H\left[\frac{1}{4}R^{*2} - \frac{1}{3}(D_{mx} + R_{mx})R^* + \frac{1}{2}D_{mx}R_{mx}\right]^2 \frac{R^{*2}}{\frac{3}{2}R_{mx} - R^*},$$

where

$$C^*(R^*) = \frac{1}{2}\left[C_{mx} + k_T \cdot f(R^*) + b'\right];$$

$$R^* = R_{mx}x^*; f(R^*) = \frac{\frac{1}{3}R_{mx} - \frac{1}{4}R^*}{\frac{1}{2}R_{mx} - \frac{1}{3}R^*}R^*;$$

$$x^* = \begin{Bmatrix} 0.74\delta + 0.002 \ for\, \delta \le 0.5 \\ 0.686\delta + 0.024 \ for\ \delta > 0.5 \end{Bmatrix}.$$

In the above formulas, whenever this would not imply an error, index "*i*" was omitted for the sake of notational simplicity.

It can be easily noticed that the value of F^*_i depends only upon the parameters of the function F^*_i:

$$\lambda_{mx,i},\ C_{mx,i},\ g_{mx,i},\ R_{mx,i},\ k_{T,i},\ b'_i,$$

with $b'_i = b+K_i$,

where K_i is the cost of transporting the product from the place, where it is manufactured, to the location of branch *i*. This cost refers to mass transport, and is defined by the function

$$K_i = K(V^*_i, d_i),$$

with the choice of the value V^*_i (capacity of the transport means) depending upon the magnitude of demand for transport, which is assumed equal demand for product, that is, Λ^*_i:

$$\Lambda^*_i(R^*_i, C^*_i) = 2\pi a^o_i (C_{mx,i} - C^*_i)\left(\frac{1}{2}g_{mx,i} - \frac{1}{2}\varphi_i R^*_i\right)R^{*2}_i,$$

where $a_i^o = \dfrac{\lambda_{0,i}}{C_{mx,i}}$, $\varphi_i = \dfrac{g_{mx,i}}{R_{mx,i}}$.

Consequently, we are capable of determining the values of F^*_i for all the local branches (or: authorised dealers), located in places numbered $i = 0,1,2,\ldots,I$, where index value i=0 denotes the zone of sale in the place of manufacture (where equality b'_0=b holds).

Knowing the values of F^*_i we can determine also the summary profit of the global company:

$$F^* = \sum_{i=0}^{I} F_i^*$$

as well as the summary sales value:

$$\Lambda = \sum_{i \in \mathrm{I}} \Lambda_i^* = \mu$$

where, of course, I = {0,1,2,…,*i*,…,*I*} is the set of numbers (labels) of branches of the global company.

With production lines of the above capacity, $\mu = \Lambda$, constant maintenance costs are associated, Q. Hence, inequality $F^* > Q$ must hold. Otherwise, the entire activity is unprofitable. If it is profitable, though, then in the next step of calculations we determine

$$Q_i = \frac{\Lambda_i^*}{\Lambda} \cdot Q$$

i.e. the parts of the constant cost of maintaining the production lines, which ought to be charged on particular local branches.

Next, we determine $K_{0,i}$, of maintaining the storage facilities of the branches i, which sell Λ^*_i of the product, and verify, whether the local inequality $F^*_i > Q_i + K_{0,i}$ holds for each i.

Let us remind that the constant cost Q of maintaining the common production lines is composed of amortisation, costs of maintaining the production surface and the storage, taxes on land and buildings, costs of repair of the fixed assets of the branch, costs of protection, etc.

If for some local branches the respective inequality is not fulfilled (and cannot be fulfilled by simple organisational measures), then these branches ought to be liquidated, and their indices removed from the set I = {0,1,2,…,*I*}. Due to such an overview of the profitability of functioning of particular branches, we obtain a new set of indices (branch labels), $\mathrm{I}^{(1)}$.

For this new set of branches, we calculate again the values

$$\Lambda^{(1)} = \sum_{i \in \mathrm{I}^{(1)}} \Lambda_i^* \quad ; \quad F^{*(1)} = \sum_{i \in \mathrm{I}^{(1)}} F_i^*$$

Then, for the newly determined value

$$\mu = \sum_{i \in I^1} \Lambda_i^* = \Lambda^{(1)}$$

we calculate the value of $Q^{(1)}$ and then the values

$$Q_i^{(1)} = \frac{\Lambda_i^*}{\Lambda} \cdot Q^{(1)}$$

and again verify, whether for every $I = 0,1,2,\ldots$ the inequality

$$F_i^* > Q_i^{(1)} + K_{0,\,i}$$

holds. If for any $i \in \mathrm{I}^{(1)}$ this inequality is not fulfilled, then, again, this branch is eliminated from $\mathrm{I}^{(1)}$ and thereby a new, different set of branches, $\mathrm{I}^{(2)}$, is obtained.

Once we have to specify a new set of branches, $\mathrm{I}^{(2)}$, we also have to go through the same procedure of verification of profitability, that is, first, calculation of:

$$\Lambda^{(2)} = \sum_{i \in \mathrm{I}^{(2)}} \Lambda_i^* = \mu^{(2)} \quad ; \qquad F^{(2)} = \sum_{i \in \mathrm{I}^{(2)}} F_i^*$$

followed by the determination of the values of $Q^{(2)}$ and $Q^{(2)}{}_i$.

If, upon verification of the inequalities

$$F_i^* > Q_i^{(2)} + K_{0,\,i}$$

a new set of branches, $\mathrm{I}^{(3)}$, differing from $\mathrm{I}^{(2)}$, is obtained, the entire procedure has to be repeated once more (at least). This is done until two consecutive sets $\mathrm{I}^{(t)}$ become equal.

Then, we will be able to treat the respective values obtained as truly optimal:

$\Lambda^*{}_i$ and $C^*{}_i$, $R^*{}_i$, $K^*{}_{0,i}$; along with $\Lambda^* = \mu^*$, Q^* and I^*,

for the given values of characteristics concerning:

- demand: $C_{\mathrm{mx},i}$ and $\lambda_{\mathrm{mx},i}$,
- density of customers: $g_{\mathrm{mx},i}$, $R_{\mathrm{mx},i}$,
- transport costs: K_i, $k_{T,i}$,
- spatial distribution of the branches: d_i,
- production: b, Q.

The thus determined values guarantee the highest positive profit that can be gained from the entire activity: production and distribution of goods.

Yet, not any profit ensures overall positive return on the undertaking. This depends, namely, upon the capital, involved in the undertaking. And so, if the entire cost of purchasing and installing the production line, and establishing all the branches is estimated by the value *CAP*, then the ultimate condition of profitability is

$$F^{*}\text{-}Q^{*}\text{-}\Sigma_{i\in I*}K_{0,i} > \rho\cdot CAP,$$

where ρ is the interest on deposit.

When the above condition is not satisfied, then, despite the potentially positive profit, it would be more beneficial to keep the capital in the bank.

6. Market competition for the basic model with delivery cost and uneven density of customers

Let us analyse the function of profitability of manufacturing a product, for one, definite zone:

$$\varepsilon = \frac{revenue - \cos ts}{\cos ts} = \frac{revenue}{\cos ts} - 1.$$

After we introduce the previously established expressions to the above formula, we get

$$\varepsilon = \frac{C}{b + k_T f(R) + \dfrac{Q}{\Psi(R)\cdot(C_{mx} - C)}} - 1,$$

where $\Psi(R) = 2\pi a^{o}\varphi R^{2}\left(\frac{1}{2}R_{mx} - \frac{1}{3}R\right)$.

It can be easily noticed that the above expression takes the value -1 for $C = 0$ and for C tending to C_{mx}. It is within this segment that the value of C is contained, securing the maximum value of the return indicator, ε, for a given value of R.

For the optimum values of C and R the value of ε attains the maximum, which, in these conditions, shall depend uniquely upon the assumed parameters of the model, and its value shall be equal for all the competitors (since we have initially assumed common access to the technologies of production and transport).

Hence, we can see that a competitor, in order to push away, in a given sector, the resident producer, has to go down with the price below the one quoted by the resident company, entering the market with an appropriately higher production and providing service over a broader area. All these steps, though, do not guarantee for the competitor a bigger profit than the one achieved by the resident company, and bring about a lower profitability ε, if the prices quoted by the resident company were optimal.

So, the competitor has to dispose of adequately ample financial reserves, in order to be able to take the place of the resident company, by causing losses resulting from the occupation of a part of the market, which brings about serious underuse of the production capacity, and, consequently, increase of unit cost, κ.

Otherwise, if the resident company acted non-optimally, by selling product for the price higher than the optimum one, the competitor could enter the market with lower – optimum – price and more easily push the resident company from the market, while gaining higher profit than the resident company.

If, however, the resident company act optimally, the competitor can only push them away by bearing significant "entry cost".

Taking into account the fact that the resident producer can also lower the price in the particular sales zones, the price war can break out – a global one, in this case.

In order to avoid this kind of war of attrition, in which often the respective governments are involved, an agreement is sometimes concluded, according to which the global market is split into the exclusive spheres of influence.

This book presents a complete exposition of a coherent and far-reaching theory of market competition. It is based on simple precepts, does not require deep knowledge of either economics or mathematics, and is therefore aimed primarily at undergraduate students and all those trying to put in order their vision of how the essential market mechanisms might work. Volume II, now in preparation, shall bring the theory to further problems and results.

The logic of the presentation is straightforward; it associates the microeconomic elements to arrive at both more general conclusions and at concrete formulae defining the way the market mechanisms work under definite assumed conditions.

Some may consider this exposition too simplistic. In fact, it is deliberately kept very simple, for heuristic purposes, as well as in order to make the conclusions more clear. Adding a lot of details that make theory more realistic – these details, indeed, changing from country to country, and from sector to sector – is mainly left to the Reader, who is supposed to be able to design the more accurate image on the basis of the foundations, provided in the book.

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