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Dedicated to Professor Beloslav Riečan on his 75th anniversary

Fuzzy pricing of catastrophe bond with a stepwise payoff function

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Abstract

In this paper we discuss the problem of catastrophe bond pricing with a stepwise payoff function. An approach based on the martingale method is applied. In order to price the catastrophe bond we use fuzzy parameters and apply the Vasicek interest rate model. We assume replicability of interest rate changes by financial instruments existing in the market as well as independence between catastrophe occurrence and behaviour of financial market. Then the Monte Carlo simulations based on the obtained fuzzy pricing formula are carried out. The presented fuzzy sets approach may incorporate expertise knowledge to overcome lack of precise data in the discussed case.

Keywords: catastrophe bond, Vasicek model, martingale method, Monte Carlo simulations, fuzzy numbers.

1 Introduction

During last years, the insurance industry face overwhelming risks caused by natural catastrophes. Losses from a single catastrophic event could reach tens of billion \$ (see e.g. [16]). Therefore new insurance mechanisms have been developed.

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The classical insurance mechanisms are not adequate to deal with extreme losses caused by natural catastrophes. Even one, single catastrophe could cause problems with reserves for many insurers or even bankruptcy of these enterprises. In case of natural catastrophes the sources of losses are strongly dependent in terms of time and localization. Additionally, losses from such events are extremely huge.

In contrary, classical insurance approach assumes that losses are modelled by independent and identically distributed (iid) random variables. Additionally, value of each claim is small comparing to whole portfolio of insurer, like in case of car accidents. Therefore it may be useful to use new insurance instruments in order to stabilize the behaviour of portfolio for insurance company or even budget of government.

As it was mentioned, a single catastrophic event could result in damages measured in billions of dollars. Because daily fluctuations on worldwide financial markets reach also the same scale, securitization of losses (i.e. ”packaging” losses into form of tradable assets) may be helpful for dealing with results of extreme natural catastrophes (see e.g. [6, 10, 11, 13]). One of possible instruments of this type is known as catastrophe bonds (Act-of-God bonds, cat bonds, see e.g. [9, 21, 25]).

In the financial literature dedicated to catastrophe bonds the problem of their pricing is not widely discussed. Many authors underline advantages of investing in catbonds. Pricing methods in [1, 11] are very simplified and probabilistic model proposed in [11] has its limitations. [14] is devoted to cat bond pricing, using behavioral finance method. In [27] probability transforms and the Sharpe ratio are used to evaluate the risk-adjusted performance of cat bonds. There are several advanced stochastic pricing models with discrete and continuous time. In some of them utility function is incorporated to the pricing model (see [5, 24, 8]). However, choosing a well-suited utility function can be an additional problem in practice. The first interesting approach was presented by Vaugirard in [26]. The author applied the arbitrage approach for cat bonds pricing. He overcome the problem of non completeness of the market and non-traded insurance-linked underlyings in Merton's manner (see [15]).

This paper is dedicated to the problem of catastrophe bond pricing with a stepwise payoff function. We continue and extend the Vaugirard's approach. In our approach the martingale method is applied. In order to price the catastrophe bond we use fuzzy parameters and apply the Vasicek interest rate model. We assume no arbitrage, replicability of interest rate changes by financial instruments existing in the market as well as independence between catastrophe occurrence and behaviour of financial market. Then the Monte Carlo simulations based on the obtained fuzzy pricing formula are carried out.

There is a need to take into account possible errors and uncertainties which arise from estimation of rare events with serious, catastrophic consequences like natural catastrophes. Therefore we apply the approach based on fuzzy sets which may also incorporate expertise knowledge of assumptions about future behaviour of catastrophic events or to overcome lack of precise, historical data.

This paper is organized as follows. In Section 2 we present some preliminaries for fuzzy sets. In Section 3 we discuss the general features of catastrophe bonds, present catastrophe bond with a stepwise payoff function, price this type of catastrophe bond and conduct simulations in order to find appropriate price for fuzzy approach. We conclude the paper in Section 4.

2 Preliminaries

2.1 Fuzzy and interval arithmetics

In this section we recall some basic facts about fuzzy sets and numbers.

Let X be a universal set and $\tilde{A}$ be a fuzzy subset of X. We denote by $\mu_{\tilde{A}}$ its membership function $\mu_{\tilde{A}} : X \to [0,1]$, and by $\tilde{A}_\alpha = \{x : \mu_{\tilde{A}} \geq \alpha\}$ the α-level set of $\tilde{A}$, where $\tilde{A}_0$ is the closure of the set $\{x : \mu_{\tilde{A}} \neq 0\}$.

In our paper we assume that $X = \mathbb{R}$.

Let $\tilde{a}$ be a fuzzy number. Then the α-level set $\tilde{a}_\alpha$ is a closed interval, which can be denoted by $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ (see e.g. [29]).

We can now introduce the arithmetic of any two fuzzy numbers. Let $\odot$ be a binary operator $\oplus$, $\ominus$, $\otimes$ or $\oslash$ between fuzzy numbers $\tilde{a}$ and $\tilde{b}$, where the binary operators correspond to $\circ$: $+$, $-$, $\times$ or $/$, according to the "Extension Principle" in [29]. Then the membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\mu_{\tilde{a}\odot\tilde{b}}(z) = \sup_{(x,y):x\circ y=z} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\} . \tag{1}$$

Let $\odot_{int}$ be a binary operator $\oplus_{int}$, $\ominus_{int}$, $\otimes_{int}$ or $\oslash_{int}$ between two closed intervals $[a,b]$ and $[c,d]$. Then

$$[a,b] \odot_{int} [c,d] = \{z \in \mathbb{R} : z = x \circ y, \forall x \in [a,b], \forall y \in [c,d]\} , \tag{2}$$

where $\circ$ is an usual operation $+, -, \times$ and $/$, if the interval $[c,d]$ does not contain zero in the last case.

Therefore, if $\tilde{a}, \tilde{b}$ are fuzzy numbers, then $\tilde{a} \odot \tilde{b}$ is also the fuzzy number and its α-level set is given by

$$(\tilde{a} \oplus \tilde{b})_\alpha = \tilde{a}_\alpha \oplus_{int} \tilde{b}_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha] ,$$

$$(\tilde{a} \ominus \tilde{b})_\alpha = \tilde{a}_\alpha \ominus_{int} \tilde{b}_\alpha = [\tilde{a}^L_\alpha - \tilde{b}^U_\alpha, \tilde{a}^U_\alpha - \tilde{b}^L_\alpha] \ ,$$

$$(\tilde{a} \otimes \tilde{b})_\alpha = \tilde{a}_\alpha \otimes_{int} \tilde{b}_\alpha =$$

$$= [\min\{\tilde{a}^L_\alpha \tilde{b}^L_\alpha, \tilde{a}^L_\alpha \tilde{b}^U_\alpha, \tilde{a}^U_\alpha \tilde{b}^L_\alpha, \tilde{a}^U_\alpha \tilde{b}^U_\alpha\}, \max\{\tilde{a}^L_\alpha \tilde{b}^L_\alpha, \tilde{a}^L_\alpha \tilde{b}^U_\alpha, \tilde{a}^U_\alpha \tilde{b}^L_\alpha, \tilde{a}^U_\alpha \tilde{b}^U_\alpha\}] \ ,$$

$$(\tilde{a} \oslash \tilde{b})_\alpha = \tilde{a}_\alpha \oslash_{int} \tilde{b}_\alpha =$$

$$= [\min\{\tilde{a}^L_\alpha / \tilde{b}^L_\alpha, \tilde{a}^L_\alpha / \tilde{b}^U_\alpha, \tilde{a}^U_\alpha / \tilde{b}^L_\alpha, \tilde{a}^U_\alpha / \tilde{b}^U_\alpha\}, \max\{\tilde{a}^L_\alpha / \tilde{b}^L_\alpha, \tilde{a}^L_\alpha / \tilde{b}^U_\alpha, \tilde{a}^U_\alpha / \tilde{b}^L_\alpha, \tilde{a}^U_\alpha / \tilde{b}^U_\alpha\}] \ ,$$

if α-level set $\tilde{b}_\alpha$ does not contain zero for all $\alpha \in [0,1]$ in the case of $\oslash$.

Triangular fuzzy number $\tilde{a}$ with membership function $\mu_{\tilde{a}}(x)$ is defined as

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{for} \quad a_1 \le x \le a_2 \\ \frac{x-a_3}{a_2-a_3} & \text{for} \quad a_2 \le x \le a_3 \\ 0 & \text{otherwise} \end{cases} \ , \tag{3}$$

where $[a_1, a_3]$ is the supporting interval and the membership function has peak in a_2. Triangular fuzzy number $\tilde{a}$ is denoted as

$$\tilde{a} = (a_1, a_2, a_3) \ .$$

Triangular fuzzy numbers are special cases of Left-Right (or L-R) fuzzy numbers (e.g. see [2, 7]), where linear functions used in the definition are replaced by monotonic functions. For such numbers the membership function may be calculated as

$$\mu_{\tilde{a}}(x) = \begin{cases} L\left(\frac{a_2-x}{a_2-a_1}\right) & \text{for} \quad a_1 \le x \le a_2 \\ R\left(\frac{x-a_2}{a_3-a_2}\right) & \text{for} \quad a_2 \le x \le a_3 \\ 0 & \text{otherwise} \end{cases} \ , \tag{4}$$

where L and R are continuous strictly decreasing function defined on $[0,1]$ with values in $[0,1]$ satisfying the conditions

$$L(x) = R(x) = 1 \text{ if } x = 0 \ , L(x) = R(x) = 0 \text{ if } x = 1 \ .$$

The L-R fuzzy number $\tilde{a}$ is denoted as

$$\tilde{a} = (a_1, a_2, a_3)_{LR} \ .$$

Next we turn to fuzzy estimation based on statistical approach (see [4]). This approach may be seen as a way to obtain L-R numbers based on statistical data.

Let X be a random variable with probability density function $f_\theta(.)$. Assume that parameter θ is unknown and are to be estimated from a sample $X_1, X_2, \ldots X_n$. Let $\hat{\theta}$ be a statistics based on $X_1, X_2, \ldots X_n$ which is used for such estimation.

Then for the given confidence level $0 \le \beta \le 1$ we have the $\beta \cdot 100\%$ confidence interval $[\theta_L(\beta), \theta_R(\beta)]$ for θ which is established by the condition

$$\mathrm{P}_{f_\theta}(\theta_L(\beta) \le \theta \le \theta_R(\beta)) = \beta \ . \tag{5}$$

If we suppose that $[\theta_L(0), \theta_R(0)] = [\hat{\theta}, \hat{\theta}]$ then we could construct fuzzy estimator $\hat{\tilde{\theta}}$ of $\tilde{\theta}$. We place the confidence intervals, one on top of the other, to produce a triangular shaped fuzzy $\tilde{\theta}$ whose α-cuts are the confidence intervals on $\beta = (1-\alpha)$ confidence levels (see [4] for additional details).

2.2 Stochastic and financial preliminaries

We begin with notations and basic definitions concerning catastrophe bonds and their pricing. We define stochastic processes describing dynamics of the spot interest rate and aggregated catastrophe losses.

We apply stochastic models with continuous time and time horizon of the form $[0, T']$, where $T' > 0$. Date of maturity of catastrophe bonds T is not later than T', i.e. $T \le T'$. We consider two probability measures: P and Q and we denote by E^P and E^Q the expectations with respect to them.

We define stochastic processes and random variables with respect to probability P.

Let $(W_t)_{t\in[0,T']}$ be Brownian motion. It will be used in stochastic model of the risk-free interest rate.

Let $(U_i)_{i=1}^{\infty}$ be a sequence of identically distributed random variables. We treat U_i as value of losses during i-th catastrophic event.

We also define compound Poisson process by formula

$$\tilde{N}_t = \sum_{i=1}^{N_t} U_i, t \in \left[0, T'\right],$$

where N_t is Poisson process with an intensity $\kappa > 0$.

For each $t \in [0, T']$ value of process N_t is equal to the number of catastrophic events till the moment t. In particular,

$$\begin{aligned}
&N_0 = 0 \ \text{ P - a.s.,} \\
&E^P N_t = \kappa t \ \text{ for } \ t \in \left[0, T'\right] \ \text{ and} \\
&P\left(N_t - N_s = k\right) = e^{-\kappa(t-s)} \frac{\left[\kappa\left(t-s\right)\right]^k}{k!}, \ \ k = 0, 1, 2, ...
\end{aligned}$$

Moments of a jumps of process $(N_t)_{t\in[0,T']}$ are interpreted as moments of catastrophic events.

For each $t \in [0, T']$ process $\tilde{N}_t$ describes the aggregated catastrophe losses till the moment t. $\left(\tilde{N}_t\right)_{t\in[0,T']}$ is a nondecreasing stochastic process, with right - continuous trajectories of a stepwise form. Heights of its jumps are equal to values of losses during catastrophic events.

All the above processes and random variables are defined on a filtered probability space $\left(\Omega, F, (F_t)_{t\in[0,T']}, P\right)$. The filtration $(F_t)_{t\in[0,T']}$ is given by formula

$$F_t = \sigma\left(F_t^0 \cup F_t^1\right), F_t^0 = \sigma\left(W_s, s \le t\right),$$
$$F_t^1 = \sigma\left(\tilde{N}_s, s \le t\right), t \in \left[0, T'\right].$$

We assume that

$$F_0 = \sigma(\{A \in F : P(A) = 0\})$$

and that $(W_t)_{t\in[0,T']}$, $(N_t)_{t\in[0,T']}$ and $(U_i)_{i=1}^{\infty}$ are independent. Then the probability space with filtration satisfies standard assumptions, i.e. σ-algebra F is P-complete, filtration $(F_t)_{t\in[0,T']}$ is right continuous and F_0 contains all the sets from F of P-probability zero. Moreover, we assume that random variables U_i, $i = 1, 2, ...$ have bounded second moment.

We denote by $(B_t)_{t\in[0,T']}$ banking account satisfying the following equation:

$$dB_t = r_t B_t dt,\ B_0 = 1,$$

where $r = (r_t)_{t\in[0,T']}$ is a risk-free spot interest rate.

We assume that zero-coupon bonds are traded in the market. We denote by $B(t, T)$ the price at the time t of zero-coupon bond with maturity date $T \le T'$ and with the face value equal to 1.

We price catastrophe bonds under the assumption of no possibility of arbitrage in the market. We also make two additional assumptions. We first assume that investors are neutral toward nature jump risk (Assumption 1). This assumption has practical confirmations in the market (see e.g. [1], [26]). Secondly (Assumption 2), we assume that changes in interest rate r can be replicated by existing financial instruments (especially zero-coupon bonds).

3 Catastrophe bonds

3.1 Features of catastrophe bonds

As it was mentioned before, there are many problems with classical insurance mechanisms. We may mention troubles with coverage of losses by insurance enterprises, dependencies among sources of risks, potentially unlimited losses, problems with adverse selection, moral hazard and reinsurance pricing cycles. Therefore alternative financial or insurance instruments may be useful for insurers. The problem is to ”package” risks caused by natural disasters and appropriate losses into classical forms of tradable financial assets, like bonds or options. One of the most popular catastrophe-linked security is the catastrophe bond, known also as *cat bond* or *Act-of-God* bond (see [9, 12, 21, 25]).

There is one important difference among cat bonds and more classical types of bonds. The payment function of cat bond depends on additional random variable, i.e. occurrence of some natural catastrophe in the specified region and the fixed time interval. Such event is called *triggering point* (see [12]). For example, the A-1 USAA bond introduced in 1997 by insurer from Texas, was connected with hurricane on the east coast of USA between July 15, 1997 and December 31, 1997. If there had been a hurricane in mentioned above region with more than $1 billion loses against USAA, the coupon of the bond would have been lost. Catastrophe bonds may be related to various kinds of triggering points — e.g. to magnitude of earthquake, the losses from flood, insurance industry index of losses, some parameters of catastrophe event, etc.

As in case of classical bonds, the structure of payments for cat bonds depends also on some primary underlying asset. For mentioned A-1 USAA bond, the payment equalled LIBOR (London Interbank Offered Rate) plus 282 basis points.

The main aim of cat bonds is to transfer *risk* from insurance markets or governmental budgets to financial markets. Apart from transferring capital, a liquid catastrophe derivatives market allow insurance and reinsurance companies to adjust their exposure to natural catastrophic risk dynamically through hedging with those contracts at lower transaction costs. If the triggering point is connected with industry loss indices or parametric triggers, the moral hazard exposure of bond investors is greatly reduced or eliminated. Cat bonds are often rated by an agency such as Standard & Poor’s, Moody’s, or Fitch Ratings.

3.2 Catastrophe bond with stepwise payoff function

Let

$$0 < K_1 < ... < K_n,\ n > 1$$

be a sequence of constants.
Let $\tau_i : \Omega \to [0, T'],\ 1 \leq i \leq n$ be a sequence of stopping times defined as follows

$$\tau_i(\omega) = \inf_{t \in [0,T']} \left\{ \tilde{N}(t)(\omega) > K_i \right\} \wedge T',\ 1 \leq i \leq n.$$

Let

$$w_1 < w_2 < ... < w_n$$

be a sequence of nonnegative constants, for which $\sum_{i=1}^{n} w_i \leq 1$. Let

$$\Phi = \sum_{i=1}^{n} w_i \Phi_i,$$

where Φ_i are cumulative distribution functions of τ_i.

Definition 1. *We denote by $IB_s(T, Fv)$ a catastrophe bond satisfying the following assumptions:*

a) If the catastrophe does not occur in the period $[0, T]$, i.e. $\tau_1 > T$, the bondholder is paid the face value Fv;

b) If $\tau_n \leq T$, the bondholder receives the face value minus the sum of write-down coefficients in percentage $\sum_{i=1}^{n} w_i$.

c) If $\tau_{k-1} \leq T < \tau_k$, $1 < k \leq n$, the bondholder receives the face value minus the sum of write-down coefficients in percentage $\sum_{i=1}^{k-1} w_i$.

d) A cash payments are done at date of maturity T.

3.3 Pricing of catastrophe bond

The first step in our considerations is to obtain the valuation formula for $IB_s(T,Fv)$, assuming the Vasicek model of the spot interest rate. This model is very popular and often used for modeling of the risk-free interest rate in the market. We apply the following theorem from [19] for a general form of catastrophe bond $IB_{cat}(T, Fv)$ with a payoff function $\nu_{IB_{cat}(T,Fv)}$ depended on T, Fv and the compound Poisson process $\tilde{N}$.

Theorem 1. *Let $IB(t)$ be the price of a $IB_{cat}(T, Fv)$ at time t. Then*

$$IB(t) = E^Q\left(\exp\left(-\int_t^T r_u u du\right)|F_t\right) E^Q\left(\nu_{IB_{cat}(T,Fv)}|F_t\right). \tag{6}$$

In particular,

$$IB(0) = E^Q\left(\exp\left(-\int_0^T r_u u du\right)\right) E^Q \nu_{IB_{cat}(T,Fv)}. \tag{7}$$

In the above theorem measure Q is defined by the Radon-Nikodym derivative:

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda_u dW_u - \frac{1}{2}\int_0^T \lambda_u^2 du\right) P\text{-a.s.}$$

for a predictable process λ_u, connected with the risk premium for risk-free bonds. The interest rate satisfies the following equation

$$dr(t) = a(b - r(t))\, dt + \sigma dW_t \tag{8}$$

for positive constants a, b and σ.

The following theorem (proved in [19]) gives the pricing formula for $IB_s(T, Fv)$.

Theorem 2. *Let* $IB(0)$ *be the price of a* $IB_s(T, Fv)$ *at time* 0. *Let*

$$\Phi = \sum_{i=1}^{n} w_i \Phi_i,$$

where Φ_i *are cumulative distribution functions of* τ_i. *Then*

$$IB(0) = Fve^{-T\cdot R(T,r(0))}\left\{1 - \Phi(T)\right\}, \tag{9}$$

where

$$R(\theta, r) = R_\infty - \frac{1}{a\theta}\left\{(R_\infty - r)\left(1 - e^{-a\theta}\right) - \frac{\sigma^2}{4a^2}\left(1 - e^{-a\theta}\right)^2\right\}$$

and

$$R_\infty = b - \frac{\lambda\sigma}{a} - \frac{\sigma^2}{2a^2}.$$

Sketch of the proof. From Theorem 1 it follows that

$$IB(0) = E^Q\left(\exp\left(-\int_0^T r_u u du\right)\right) Fv E^Q\left\{1 - \sum_{i=1}^{n} w_i I_{\tau_i \le T}\right\}.$$

From zero-coupon bond pricing formula for the Vasicek interest rate model (see e.g. [26]) it follows that

$$E^Q\left(\exp\left(-\int_0^T r_u u du\right)\right) = e^{-T\cdot R(T,r(0))}.$$

Since τ and W are independent, $I_{\tau_i \le T}$ and $\frac{dQ}{dP}$ are independent. Therefore

$$E^Q\left\{1 - \sum_{i=1}^{n} w_i I_{\tau_i \le T}\right\} 1 - \sum_{i=1}^{n} w_i E^P\left(I_{\tau_i \le T}\right) = 1 - \Phi(T).$$

Finally, the pricing formula at time $t = 0$ has the form (9). $\square$

The lemma below (formulated in [18]) gives the form of the cumulative distribution functions of τ_i and can be applied to computations of the catastrophe bond price.

Lemma 1. *The value of the cumulative distribution function* Φ_i*,* $1 \leq i \leq n$*, at the moment* T *has the form*

$$\Phi_i(T) = 1 - \sum_{j=0}^{\infty} \frac{(\kappa T)^j}{j!} e^{-\kappa T} \Phi_{\tilde{U}_j}(K_i),$$

where $\Phi_{\tilde{U}_j}$ *is the cumulative distribution function of the sum* $\tilde{U}_j = \sum_{p=0}^{j} U_p$*. In the above formula we assume that* $U_0 \equiv 0$*.*

Our aim is to present the catastrophe bond pricing formula in case when the parameters of the spot interest rate are not precisely known. To model this uncertainty we introduce fuzzy numbers $\tilde{a}$, $\tilde{b}$, $\tilde{\sigma}$ and $\tilde{r}_0$ in place of a, b, σ and $r(0)$. We also treat the market price of risk as a small fuzzy number. Therefore we replace the parameter λ by its negative fuzzy counterpart $\tilde{\lambda}$.

Let $\mathcal{F}(R)$ the set of all fuzzy numbers. The proposition below was proved in [28].

Proposition 1. *Let* $f : R \to R$*, for which the inverse image of any value is compact, induces a fuzzy-valued function* $\tilde{f} : \mathcal{F}(R) \to \mathcal{F}(R)$ *via the extension principle and the* α*-level set of* $\tilde{f}(\tilde{\Lambda})$ *is* $\tilde{f}(\tilde{\Lambda})_\alpha = \{f(x) : x \in \tilde{\Lambda}_\alpha\}$*.*

Applying Proposition 1, we obtain the following fuzzy version of the pricing formula.

Theorem 3.

$$\tilde{IB}(0) = \mathrm{Fv} \otimes e^{-T \otimes \tilde{R}(T)} \otimes E^P \nu_{IB_s(T,Fv)}, \tag{10}$$

where

$$\begin{aligned} \tilde{R}(T) = \tilde{R}_\infty \ominus \Big\{ \Big(\tilde{R}_\infty \ominus \tilde{r}_0\Big) \otimes \left(1 \ominus e^{-\tilde{a} \otimes T}\right) \ominus \tilde{\sigma} \otimes \tilde{\sigma} \otimes \left(1 \ominus e^{-\tilde{a} \otimes T}\right) \\ \otimes \left(1 \ominus e^{-\tilde{a} \otimes T}\right) \oslash (4 \otimes \tilde{a} \otimes \tilde{a}) \Big\} \oslash (\tilde{a} \otimes T) \end{aligned} \tag{11}$$

and

$$\tilde{R}_\infty = \tilde{b} \ominus \tilde{\lambda} \otimes \tilde{\sigma} \oslash \tilde{a} \ominus \tilde{\sigma} \otimes \tilde{\sigma} \oslash (2 \otimes \tilde{a} \otimes \tilde{a}). \tag{12}$$

To calculate the α-level sets of $\tilde{IB}(0)$ we use formulas similar to (10), (11) and (12), replacing the operators $\oplus$, $\ominus$, $\otimes$, $\oslash$ by $\oplus_{int}$, $\ominus_{int}$, $\otimes_{int}$, $\oslash_{int}$.

3.4 Numerical examples of fuzzy approach

In order to find the price of the model of catastrophe bond described in Section 3.3 according to Theorem 3, the appropriate simulations are conducted.

In our simulations we use two sets for parameters. First set is used for modelling of losses. In this case we assume that quantity of losses is modelled by Poisson process with intensity κ and the value of each loss is given by random variable from Gamma distribution with scale parameter ζ and shape parameter β. Other types of distributions for modelling the value of losses are also possible, e.g. Weibull distribution, lognormal distribution.

The second set of parameters is used for generation of the risk-free spot interest rate trajectories for the Vasicek model. In this case we use fuzzy parameters, i.e. we assume that parameters are described by α-sets which may be derived e.g. from triangular fuzzy numbers or L-R numbers (see Section 2). The more detailed approach for applying Monte Carlo simulation in case of intervals based on α-sets was presented in [22].

We assume that for each considered example of catastrophe bond the trading horizon is set on 5 years and we generate 1000 interest rate trajectories for 10000 simulations of Poisson process.

In case of *Example I – III* we analyse the estimators of cat bond price if the limits of α-sets are extended. We assume that $\kappa = 0.05$, $\zeta = 5$, $\beta = 10$, therefore the generated losses have catastrophic nature, i.e. they are rare, but with high value. The catastrophe bond is described by parameters

$$\mathrm{Fv} = 1 \, , \lambda = -0.1 \, , w_1 = 0.1 \, , w_2 = 0.2 \, , w_3 = 0.3 \, , \\ K_0 = 10 \, , K_1 = 30 \, , K_2 = 60 \, . \quad (13)$$

Then based on equation (10) we obtain estimators for price of catastrophe bond presented in Table 1. As we could see, the average and median tend to decreasing. Minimum, first quartile, 1% quantile and 5% quantile decrease. Maximum, 95% quantile, 99% quantile and third quartile increase. The same applies for standard deviation which means that the error caused by uncertainty, i.e. widen intervals for α-sets, is also higher.

Then (*Example I* and *Example IV – V*) we analyse the estimators of cat bond price if values of ζ and β are increased, i.e. expected value of single catastrophe and its variance are higher. We assume that α-set are described by intervals

$$\tilde{a}_\alpha = [0.02, 0.03] \, , \tilde{b}_\alpha = [0.05, 0.06] \, , \\ \tilde{\sigma}_\alpha = [0.01, 0.02] \, , \tilde{r}_\alpha(0) = [0.05, 0.07] \quad (14)$$

Table 1: Numerical estimators for price of catastrophe bond in *Example I, II, III*

	Example I	*Example II*	*Example III*
$\tilde{a}_\alpha$	[0.02,0.03]	[0.015,0.035]	[0.01,0.04]
$\tilde{b}_\alpha$	[0.05,0.06]	[0.045,0.065]	[0.04,0.07]
$\tilde{\sigma}_\alpha$	[0.01,0.02]	[0.005,0.025]	[0.005,0.03]
$\tilde{r}_\alpha(0)$	[0.05,0.07]	[0.045,0.075]	[0.04,0.08]
Average	0.838605	0.840329	0.830122
First quartile	0.828917	0.820693	0.805097
Median	0.837279	0.840307	0.830691
Third quartile	0.848429	0.860893	0.853687
Standard deviation	0.0129151	0.024689	0.0305631
Minimum	0.80987	0.788846	0.769381
1% quantile	0.812674	0.793393	0.773218
5% quantile	0.817523	0.800249	0.781488
95% quantile	0.85927	0.880068	0.879264
99% quantile	0.864803	0.890673	0.894962
Maximum	0.870979	0.901888	0.91128

and the catastrophe bond is the same as in previous scenarios, i.e. parameters are given by (13). The obtained estimators may be found in Table 2. As we could see, estimators for higher values of ζ and β tend to be lower.

In case of *Example I* and *Example VI – VII* we analyse the estimators of cat bond price for increasing values of triggering points K_1, K_2, K_3. We assume that α-set are described by intervals (14) and the parameters of catastrophe bond are given by (13) (apart from values of triggering points). The obtained estimators may be found in Table 3. As we could see, all of the estimators, including average, are higher for higher values of triggering points.

4 Conclusions

In this paper we discuss the problem of catastrophe bond pricing with a stepwise payoff function. The stochastic approach based on martingale method is applied. We use fuzzy parameters and apply Vasicek interest rate model. In order to obtain catbond valuation formula we assume no arbitrage, replicability of interest rate changes by financial instruments existing in the market as well as independence between catastrophe occurrence and behaviour of financial market. Because of possible errors and uncertainties which arise from estimation of rare events with

Table 2: Numerical estimators for price of catastrophe bond in *Example I, IV, V*

	Example I	*Example IV*	*Example V*
ζ	5	10	20
β	10	20	40
Average	0.838605	0.792859	0.792998
First quartile	0.828917	0.783671	0.783463
Median	0.837279	0.792791	0.793391
Third quartile	0.848429	0.80131	0.802
Standard deviation	0.0129151	0.0122199	0.0123183
Minimum	0.80987	0.766055	0.823057
1% quantile	0.812674	0.768511	0.768826
5% quantile	0.817523	0.774009	0.772971
95% quantile	0.85927	0.813689	0.813814
99% quantile	0.864803	0.820323	0.818425
Maximum	0.870979	0.826507	0.823057

Table 3: Numerical estimators for price of catastrophe bond in *Example I, VI, VII*

	Example I	*Example VI*	*Example VII*
K_1	10	10	10
K_2	30	40	50
K_3	60	80	100
Average	0.838605	0.856291	0.868028
First quartile	0.828917	0.846251	0.85794
Median	0.837279	0.856378	0.867994
Third quartile	0.848429	0.866102	0.878155
Standard deviation	0.0129151	0.0129288	0.013136
Minimum	0.80987	0.827617	0.837584
1% quantile	0.812674	0.831855	0.841295
5% quantile	0.817523	0.835124	0.846545
95% quantile	0.85927	0.877042	0.889667
99% quantile	0.864803	0.883296	0.894722
Maximum	0.870979	0.887066	0.898591

serious, catastrophic consequences and lack of precise, historical data, the fuzzy set approach is applied. Then appropriate simulations for the obtained fuzzy pricing formula are generated. We analyse output from some numerical experiments for various sets of parameters.

References

[1] Anderson R.R., Bendimerad F. , Canabarro E. and Finkemeier M., Analyzing insurance-linked securities. Journal of Risk Finance **1 (2)**, 49 - 78 (2000).

[2] Bardossy A., Duckstein L. (eds.), Fuzzy Rule-Based Modeling with Applications to Geophysical, Biological and Engineering Systems (Systems Engineering), CRC-Press, 1995

[3] Borch K., The Mathematical Theory of Insurance, Lexington Books, Lexington, 1974

[4] Buckley J. J., Fuzzy Statistics, Springer, 2004

[5] Cox S.H., Pedersen H.W.: Catastrophe Risk Bonds. North American Actuarial Journal **4 (4)**, 56 – 82 (2000)

[6] Cummins J.D., Doherty N., Lo A., Can insurers pay for the ”big one”? Measuring the capacity of insurance market to respond to catastrophic losses. Journal of Banking and Finance 26, 2002

[7] Dubois D., Prade H., Fuzzy Sets and Systems – Theory and Applications. Academic Press, New York, 1980

[8] Egamia M., Young V.R.: Indifference prices of structured catastrophe (CAT) bonds. Insurance: Mathematics and Economics **42**, 771 – 778 (2008)

[9] Ermolieva, T., Romaniuk, M., Fischer, G., Makowski, M., Integrated model-based decision support for management of weather-related agricultural losses. In: Enviromental informatics and systems research. Vol. 1: Plenary and session papers - EnviroInfo 2007, Hryniewicz, O., Studziński, J., Romaniuk, M. (eds.), Shaker Verlag, IBS PAN, 2007

[10] Freeman, P, K., Kunreuther, H., Managing Environmental Risk Through Insurance. Boston, Kluwer Academic Press, 1997

[11] Froot K. A., The market for catastrophe risk: A clinical examination. Journal of Financial Economics, 60 (2), 2001

[12] George J. B., Alternative reinsurance: Using catastrophe bonds and insurance derivatives as a mechanism for increasing capacity in the insurance markets. CPCU Journal, 1999

[13] Harrington S. E., Niehaus G., Capital, corporate income taxes, and catastrophe insurance. Journal of Financial Intermediation, 12 (4), 2003

[14] Kai Y., Zhong-ying Q., Shang-zhi Y.: Study on Catastrophe Bond Pricing Model Based on Behavioral Finance. in International Conference on Management Science and Engineering, 1725 – 1736 (2007)

[15] Merton R.C.: Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, **3**, 125 - 144 (1976)

[16] Muermann A., Market Price of Insurance Risk Implied by Catastrophe Derivatives. North American Actuarial Journal, 12 (3), pp. 221 – 227, 2008

[17] Niedzielski J., USAA places catastrophe bonds, National Underwriter, Jun 16, 1997

[18] Nowak P., Romaniuk M., Ermolieva T., Evaluation of Portfolio of Financial and Insurance Instruments Simulation of Uncertainty, *to be published*

[19] Nowak P., Romaniuk M., Pricing and simulations of catastrophe bonds, *to be published*

[20] Nowak P., Romaniuk M., Ermolieva T., Integrated management of weather - related agricultural losses - computational approach. In: Information Systems Architecture and Technology, Wilimowska, E., Borzemski, L., Grzech, A., witek, J. (eds.), Wrocaw 2008

[21] Nowak P., Romaniuk M., Portfolio of financial and insurance instruments for losses caused by natural catastrophes. In: Information Systems Architecture and Technology. IT Technologies in Knowledge Oriented Management Process (Wilimowska Z., Borzemski L., Grzech A., witek J., Eds.), Wrocław 2009

[22] Nowak P., Romaniuk M., Computing option price for Levy process with fuzzy parameters, European Journal of Operational Research, 201 (1), 2010

[23] Nowak P., Romaniuk M., Analiza własności portfela złożonego z instrumentów finansowych i ubezpieczeniowych (in Polish), Studia i Materiały Polskiego Stowarzyszenia Zarzadzania Wiedza 31, pp. 65 – 76, 2010

[24] Reshetar G.: Pricing of Multiple-Event Coupon Paying CAT Bond. Working Paper, Swiss Banking Institute, University of Zurich (2008)

[25] Romaniuk M., Ermolieva, T., Application EDGE software and simulations for integrated catastrophe management. International Journal of Knowledge and Systems Sciences, 2(2), pp. 1 - 9, 2005

[26] Vaugirard V. E., Pricing catastrophe bonds by an arbitrage approach. The Quarterly Review of Economics and Finance, 43, pp. 119 – 132, 2003

[27] Wang S.S.: Cat Bond Pricing Using Probability Transforms. in: Geneva Papers: Etudes et Dossiers, special issue on Insurance and the State of the Art in Cat Bond Pricing **278**, 19 – 29 (2004)

[28] Wu H.-Ch. (2004). Pricing European options based on the fuzzy pattern of Black-Scholes formula. Computers & Operations Research, 31, 1069 – 1081.

[29] Zadeh L. A., Fuzzy sets. Information and Control, 8, 338 – 353, 1965

[30] Wu, Xueping, A New Stochastic Duration Based on the Vasicek and CIR Term Structures Theories. Journal of Business Finance and Accounting, **27**, 91193 (2000)

The papers presented in this Volume 2 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) organized in Warsaw on September 30, 2011 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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