ANALIZA SYSTEMOWA I ZARZĄDZANIE

Książka jubileuszowa
z okazji
50-lecia pracy naukowej

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ISBN 83-85847-34-0

Druk: "ARGRAF" Agencja Poligraficzno-Wydawnicza, Warszawa
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A TRANSFORMATION FOR SOLVING A SINGULAR LINEAR-QUADRATIC PROBLEM[1]

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The interpretation of the columns of the controllability matrix as the directions of state-jumps caused by some control "impulses" is utilised. Using this the subspace of the state-jumps with zero-costs is determined. The basis of the subspace is used for creating a novel state transformation which in one step converts the singular Linear-Quadratic (LQ) problem with given control time and free end state to a non-singular one. It is shown that the transformed state equations have partially the Luenberger-Brunovský controllable canonical form. Owing to this the formulas determining the singular strips on which lie the trajectories described by the solutions of the non-singular problem can be derived.

Keywords. Optimal control; linear systems; canonical forms; singular problems.

1 Introduction

In engineering, the cases in which to some components of the control are related no costs are rather frequent. In such cases the corresponding to them the Linear-Quadratic (LQ) problems are singular. Therefore the singular control problems were considered in many papers and books [1,4,6-8,10-12]. An excellent survey of the singular control problems and the methods of their solving are described by Bell and Jacobson [1], while the monograph of Clements and Anderson [4] is devoted to the singular LQ problems.

[1]The paper was partially supported by the Science Researche Committee (KBN), Grant No. 8 T11A 006 014.

In the methods of solving the singular LQ problems the important role play the transformation approaches designed to convert the singular problem to a non-singular one [8,12,7,10,11]. The Kelley's transformation technique [8,12] converts the singular problem into a non-singular one in a state space of reduced dimension, while the Goh's transformation [7,10,11] preserves the dimensionality of the original state space.

In [4] the Kelley's transformation is developed for the general case of vector controls. Owing to applying the appropriate transformation of state and control the original problem is converted to the so called standard form. After rearranging the state and control variables the singular problem is transformed to a non-singular one with reduced dimension of the state. In general case, in order to obtain a non-singular problem by means of all the mentioned methods the transformations must be applied successively several times.

The approach of the present paper results from past experiences of the author [5], concerning some distributional solutions of linear differential equations related with time optimal control problems. A novel transformation of the state is proposed, which in one step converts the singular problem to non-singular one. It is shown that a part of transformed state equations have the Luenberger-Brunovský [9,2] controllable canonical form. The convenient form of these equations makes it possible to derive the formulas determining the so called singular strip on which the non-singular trajectories lie.

2 The Singular LQ Problem

The considered problem is defined by the differential equation and the performance index in the form

$$\dot{x} = Ax + Bu \tag{1}$$

$$I = \int_0^{t_f} (x^T Qx + 2x^T Gu + u^T Hu)dt \tag{2}$$

where x and u are n and r-dimensional vectors of state and control respectively; A, B are appropriate constant matrices and the $rankB$ is maximum ; the block-matrix $\begin{bmatrix} Q, & G \\ G^T, & H \end{bmatrix}$ and the matrix H is symmetric, non negative definite and $rankH = d = r - l$, where $0 < l \leq r$, $l < n$; t_f is a given stopping time. The initial state $x(0) = x_0$ is given and the final state $x(t_f)$ is free.

The singularity of the problem results from the assumption that the matrix H is non negative definite. This assumption denotes that no "costs" are assigned to some components of the control or to some linear combinations of these components.

We assume that for $t > 0$ the admissible control laws have the form of some functions defining the dependence of the control u on the state x and time t. At time $t = 0$ the admissible control laws can generate the "impulses" in the form of Dirac function and/or its derivatives.

The solved problem is as follows : Among admissible control laws the optimal control law is to be found for which the performance index (2) takes a minimal value.

3 A Jump-Wise Transfer of the State

Let $b_i, i = 1, 2, .., r$ be the i-th column of the matrix B, and $u^T = [u_1, u_2, .., u_r]$. Let us consider the case in which

$$u_i = \delta^{(k)}(t)\,,\; u_j = 0\,,\; j \neq i\,,\; x(0^-) = x_0 = 0 \qquad (3)$$

Here $\delta^{(k)}(t)$ denotes the k-th distributional derivative of the Dirac function $\delta(t)$ with respect to time and $x(0^-)$ denotes the left-hand side limit at time $t = 0$. From Lemma 1 of Willems [13] it results that the solution of (1) in the form of distribution has the form

$$x(t) = b_i\delta^{(k-1)} + Ab_i\delta^{(k-2)} + .. + A^{k-1}b_i\delta^0 + A^k e^{At} b_i.\mathbf{1}(t) \qquad (4)$$

where $\delta^{(j)} = \delta^{(j)}(t)$, $j = 0, 1, .., k-1$, $\delta^{(0)} = \delta(t)$, e^{At} is the fundamental matrix of the solution of the equation (1) and $\mathbf{1}(t) = 1$ for $t > 0$, while $\mathbf{1}(t) = 0$ for $t < 0$.

Let $\Delta x(0) = x(0^+) - x(0^-)$, where $x(0^+)$ is the right-hand side limit for $t = 0$. From (4) it is seen that the control (3) transfers the state x at time $t = 0$ by the jump

$$\Delta x(0) = A^k b_i \qquad (5)$$

The formula (5) can be also derived by the method described in [5].

Thus we can formulate the following

Lemma 1 *The control $u(t)$ defined by (3) causes the jump-wise transfer of the state x at time $t = 0$ in the direction determined by the vector (5). At the same time in the state transient (4) there appear the distributions ("impulses") in the form of Dirac function $\delta(t)$ and its derivatives $\delta^{(j)}(t)$, $j = 0, 1, .., k-1$ in the directions determined by the vectors $A^{k-1}b_i$, $A^{k-2}b_i$, Ab_i, b_i, respectively.*

4 A Linear Transformation of State

Let O_u and O_x be the subspace of R^r and R^n, respectively, such that

$$O_u = \{u : u^T H u = 0\}\,,\; O_x = \{x : x^T Q x = 0\} \qquad (6)$$

Let $u = P\bar{u}$ be the linear transformation such that $P^T HP = \bar{H}$, where in the last l rows and last l columns of $\bar{H}$, zeros appear. Let p_i, $i = 1, 2, .., r$ be the i-th column of the matrix P. Let $\bar{u}^T = [v^T, e^T], v^T = [\bar{u}_1, \bar{u}_2, .., \bar{u}_d]$, $e^T = [\bar{u}_{d+1}, \bar{u}_{d+2}, .., \bar{u}_r]$, dim $e = l$, $d + l = r$. From the assumption concerning H it results that the zero-costs are assigned to the components of the vector e.

Let us take into account the l following sequences of the vectors

$$Bp_i, ABp_i, .., A^{m_i-1}Bp_i, i = d+1, d+2, .., d+l = r \tag{7}$$

where m_i is determined by two following relations

$$A^m Bp_i \in O_x \text{ for } m = 0, 1, .., m_i - 2 \tag{8}$$

and

$$A^{m_i-1}Bp_i \notin /O_x \tag{9}$$

as well as $m_i \leq n$. Among the vectors (7) of these l sequences we choose the maximal number of linearly independent vectors in accordance with the following scheme. We start with the vectors $Bp_i, i = d+1, d+2, .., r$, and then $ABp_i, i = d+1, d+2, .., r$, and then $A^2Bp_i, i = d+1, d+2, .., r$, and so forth, until the maximal number, say h of linearly independent vectors is chosen. Note that if a vector, say ABp_{d+2} is skipped because of linear dependence on the previously chosen vectors then the vectors of the form $A^k Bp_{d+2}$, for $k \geq 2$ are also skipped. By this way, from each of the i-th sequence (7) we choose, say n_i first vectors $(0 < n_i \leq m_i)$ so that all the chosen vectors are linearly independent and $h = n_{d+1} + n_{n+2}, , +n_r$. The chosen vectors we write in the following order

$$Bp_{d+1}, ABp_{d+1}, .., A^{n_{d+1}-1}Bp_{d+1}, .., BP_r, ABp_r, .., A^{n_r-1}Bp_r \tag{10}$$

and denote appropriately by

$$w_{g+1}, w_{g+2}, .., w_{g+n_{d+1}}, .., w_{n-n_r+1}, w_{n-n_r+2}, .., w_n \tag{11}$$

where $g = n - h$.

Let $w_1, w_2, .., w_g$ be some n-dimensional vectors mutually independent and independent of the vectors (11). Thus, the matrix $W = [w_1, w_2, .., w_n]$ is non-singular. The formula $x = W\bar{x}$ in which $\bar{x}$ is a new n-dimensional state determines the state transformation.

5 The Transformed Equations

Applying to (1) and (2) the transformations $x = W\bar{x}$ and $u = P\bar{u}$ we obtain

$$\dot{\bar{x}} = W^{-1}AW\bar{x} + W^{-1}BP\bar{u} \tag{12}$$

$$I = \int_0^{t_f} (\bar{x}^T W^T Q W \bar{x} + 2\bar{x}^T W^T G P \bar{u} + \bar{u}^T P^T H P \bar{u})\, dt \tag{13}$$

Let $\bar{x} = [z^T, y^T]^T$, $z^T = [\bar{x}_1, \bar{x}_2, .., \bar{x}_g]$, $y^T = [\bar{x}_{g+1}, \bar{x}_{g+2}, .., \bar{x}_n]$, $dim y = h$, $g + h = n$. Let us notice that the quadratic form $\bar{x}^T W^T Q W \bar{x}$ of the vector $\bar{x} = [z^T, y^T]^T$, has zero-coefficients related to these components of y for which the corresponding vector-column w_i (11) of the matrix W fulfils the relation (8). This property results from the determination (6) of the subspace O_x. Let s be the q-dimensional vector $(0 \le q \le l)$ composed of all these components of y which correspond to the vector-columns w_j (11) fulfilling the relation (9). Thus we have

$$\bar{x}^T W^T Q W \bar{x} + 2\bar{x}^T W^T G P \bar{u} + \bar{u}^T P^T H P \bar{u} = z^T Q_{gg} z + 2z^T Q_{gq} s + s^T Q_{qq} s +$$

$$+ 2z^T G_{gd} v + 2s^T G_{qd} v + v^T H_{dd} v \tag{14}$$

where the right hand side of (14) results form deleting in the matrices $W^T Q W$, $W^T G P$, $P^T H P$ the zero-rows and zero-columns. For example, the bilinear form $2\bar{x}^T W^T G P \bar{u}$ has zero coefficients corresponding to the vector e. Really, in the opposite case the quadratic form (14) would take negative values for some e since the quadratic form of e disappears in (14). The indices of the introduced matrices determine the dimensions of the matrices, eg. the matrix Q_{gq} has the dimension $g \times q$.

Theorem 1 *The transformations $x = W\bar{x}$, $u = P\bar{u}$ applied to the equation (1) give the equation (12) in which the matrices $\bar{A} = W^{-1}AW$ and $\bar{B} = W^{-1}BP$ takes the form*

$$\bar{A} = \underbrace{\left[\begin{array}{c|c|c|c} X \cdots X & 0, \cdots, 0, X & & 0, \cdots, 0, X \\ \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ X \cdots X & 0, \cdots, 0, X & & 0, \cdots, 0, X \\ \hline X \cdots X & 0, \cdots, 0, X & & 0, \cdots, 0, X \\ X \cdots X & 1, \cdots, 0, X & \cdots & 0, \cdots, 0, X \\ \cdots & \cdots\cdots & & \cdots\cdots \\ X \cdots X & 0, \cdots, 1, X & & 0, \cdots, 0, X \\ \hline \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \hline X \cdots X & 0, \cdots, 0, X & & 0, \cdots, 0, X \\ X \cdots X & 0, \cdots, 0, X & \cdots & 1, \cdots, 0, X \\ \cdots & \cdots\cdots & & \cdots\cdots \\ X \cdots X & 0, \cdots, 0, X & & 0, \cdots, 1. X \end{array}\right]}_{g \quad n_{d+1} \quad n_r} \begin{array}{l} \}\, g \\ \}\, n_{d+1} \\ \}\, n_r \end{array},$$

$$
\bar{B} = \left[\begin{array}{cccc}
X\cdots X & 0 & & 0 \\
\cdots & \cdot\cdot & \cdot\cdot & \cdot\cdot \\
X\cdots X & 0 & & 0 \\
\hline
X\cdots X & 1 & & 0 \\
X\cdots X & 0 & & 0 \\
\cdots & \cdot\cdot & \cdot\cdot & \cdot\cdot \\
X\cdots X & 0 & & 0 \\
\hline
\cdots & \cdot\cdot & \cdot\cdot & \cdot\cdot \\
\hline
X\cdots X & 0 & & 1 \\
X\cdots X & 0 & \cdot\cdot & 0 \\
\cdots\cdot & & \cdot\cdot & \cdot\cdot \\
X\cdots X & 0 & & 0
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}X\\X\\X\end{array}}\right\} g \\
\left.\vphantom{\begin{array}{c}X\\X\\X\\X\end{array}}\right\} n_{d+1} \\
\\
\left.\vphantom{\begin{array}{c}X\\X\\X\\X\end{array}}\right\} n_r
\end{array}
\qquad (15)
$$

$$\underbrace{\qquad\qquad}_{d} \qquad \underbrace{\qquad\qquad}_{l}$$

where in the places of "X" some non zero elements can appear. The last h columns of $\bar{A}$ similarly as the vectors (11) are divided into l groups each of which contains appropriately n_i columns, $i = d+1, d+2, .., r$. For these groups of columns for which $n_i < m_i$ in the first g rows also in the places of "X" appear zeros.

Proof. The formulas (15) of the theorem 1 results directly from the form of the state transformation matrix W and from the application of the formula (5) to the equation (12) excited successively by the controls $\bar{u} = [v^T, \bar{u}_{d+1}, \bar{u}_{d+2}, .., \bar{u}_r]^T$, $v = 0$, $\bar{u}_i = \delta^{(k)}(t)$, $\bar{u}_j = 0$, $j \neq i$, $k = 0, 1, .., n_i - 1$, $i, j = d+1, d+2, .., r$. Additionally, if for some i we have $n_i < m_i$, then for $\bar{u}_i = \delta^{(n_i)}(t)$ from (5) we obtain $\Delta x(0) = A^{n_i} B p_i$. The last vector is linearly dependent on the vectors (10) which results from the method of choosing of (10). Thus, in the last "X" column of the group for which $n_i < m_i$, in the first g rows zeros must appear. ∎

Corollary 1 *The matrices $\bar{A}$ and $\bar{B}$ in the last h and l columns, respectively, are similar to those of the Luneberger-Brunowský [9,2] controllable canonical form.*

6 Construction of the Transformed LQ Problem

If $h < n$, then taking into account (12), (15) we have

$$\bar{z} = A_{gg} z + A_{gq} s + B_{gd} v \qquad (16)$$

where the matrix A_{gg} contains the elements appearing in the first g rows and first g columns of the matrix $\bar{A}$, the matrix B_{gd} - the elements of the first g rows and first d columns of $\bar{B}$, while the matrix A_{gq} is composed of the q columns which can have non zero elements in the first g rows and last h columns of $\bar{A}$.

At the same time the index (13) with accounting (14) can be written in the form

$$I = \int_0^{t_f} (z^T Q_{gg} z + 2z^T Q_{gq} s + s^T Q_{qq} s + 2z^T G_{gd} v + 2s^T G_{qd} v + v^T H_{dd} v) dt \quad (17)$$

In the formulas (16) and (17) the vector e disappears, and only these components of the vector y can appear which create the vector s. From (12) and (15) it is seen that each component of the vector s can be appropriately varied by means of appropriate variation of the related component of the vector e. Thus, the remaining state equations containing $\dot{y}$ can be neglected in the problem description. By this means we have obtained the transformed LQ problem (16), (17) with the reduced $(n-h)$-dimensional state z and the $(d+g)$-dimensional control $\begin{bmatrix} v \\ s \end{bmatrix}$, which can be written in the form (1), (2) with the matrices determined by

$$A = A_{gg}, \quad B = [B_{gd}, A_{gq}], \quad Q = Q_{gg}$$

$$G = [G_{gd}, \quad Q_{gq}], \quad H = \begin{bmatrix} H_{dd}, G_{qd}^T \\ G_{qd}, Q_{qq} \end{bmatrix} \quad (18)$$

It should be noticed that both the problems LQ, the original (1), (2) and transformed (16), (17) are equivalent in the sense that appearing in these problems variables are related by the introduced transformations of state and control.

Let (ij), $j = 1, 2, .., q$, $0 \le q \le l$, be all the indices appearing among $d+1, d+2, .., r$, for which $n_{(ij)} = m_{(ij)}$, i.e. for $i = (ij)$ the relation (9) is fulfilled. Let

$$P_d = [p_1, p_2, .., p_d],$$
$$W_q = [A^{m_{(i1)}-1} B p_{(i1)}, \ A^{m_{(i2)}-1} B p_{(i2)}, .., \ A^{m_{(iq)}-1} B p_{(iq)}],$$

$$H_{\bar{r}\bar{r}} = \begin{bmatrix} W_q^T Q W_q \ , \ W_q^T G P_d \\ P_d^T G^T W_q \ , \ P_d^T H P_d . \end{bmatrix} \quad (19)$$

be $r \times d$, $n \times q$ and $\bar{r} \times \bar{r}$, $(\bar{r} = d + q)$ -dimensional matrices, respectively. Let us introduce the following assumption.

Assumption 1 *The matrix $H_{\bar{r}\bar{r}}$ determined by (19) is non-singular.*

Theorem 2 *The Transformed LQ problem (16), (17) with the reduced* $(n-h)$ *- dimensional state* z *and the* $(d+q)$*-dimensional control* $\begin{bmatrix} v \\ s \end{bmatrix}$ *is non-singular.*

Proof. The non singularity of the matrix H determined by (18) results directly from Assumption 1. ■

7 Solution to the Transformed LQ Problem

The Optimal Control Law (OCL) for the non-singular LQ problem (16), (17) is determined by the dependence

$$\begin{bmatrix} v \\ s \end{bmatrix} = -Lz = -\begin{bmatrix} L_{dg} \\ L_{qg} \end{bmatrix} z \tag{20}$$

where the matrices L, L_{dg}, L_{qg} are $(d+q) \times g$, $d \times g$, $q \times g$ - dimensional, respectively, and on the basis of the conventional theory [3] we have

$$L = H^{-1}(B^T S + G^T) \tag{21}$$

$$-\dot{S} = Q + A^T S + SA - (SB + G)H^{-1}(B^T S + G^T),\ S(t_f) = 0 \tag{22}$$

with the matrices determined by (18).

The needed variation of the components of the vector s in accordance with (20) can be realised by an appropriate variation of the corresponding components of the vector e and related to it the components of the vector y.

The components of the vector y, similarly as the vectors (11) are divided into l groups numbered by the indices $i = d+1, d+2, .., d+l = r$. One can notice that the components of these groups which have the indices $(i1), (i2), .., (iq)$ are varied appropriately to the components of the vector s. The components of the control vector e which create the remaining groups can be varied freely. Let $\bar{n} = n_{(i1)} + n_{(i2)} .. + n_{(iq)}$.

Let $\bar{y}$ be the $\bar{n}$-dimensional vector composed of these components of y which belong to the groups numbered by the indices $(i1), (i2), .., (iq)$. Let y^* be the $(l-q) = q^*$-dimensional vector composed of the last components of the remaining groups of y (numbered by the indices taken among $d+1, d+2, .., r$ which are different from $(i1), (i2), .., (iq)$).

In the space $\{\bar{x}, t\}$ there is a set S called the singular strip [1] on which lie the trajectories corresponding to the solution of the non-singular problem (16), (17). The set S and the OCL determining e are described by

$$S = \{\bar{x}, t : \bar{y} + L_{\bar{n}g} z + F_{\bar{n}g^*} y = 0\} \tag{23}$$

$$e = -\bar{L}_{lg} z - \bar{F}_{lq^*} y^* \tag{24}$$

where the matrices $L_{\bar{n}g}, \bar{F}_{\bar{n}q^*}, \bar{L}_{lg}, \bar{F}_{lq^*}$ can be derived by using (19), (12), (15). The time t appears in (23) since the matrix $L_{\bar{n}g}$ depends on t which results from (20)-(22).

Generally the OCL (20), (24) are valid for $0 < t \leq t_f$. At time $t = 0$ some appropriate "impulses" should occur in the control e in order to establish the point $(\bar{x}(0^+), t = 0)$ lying on S. The possibility of the jump-wise control results from the following Theorem 3.

By the sentence "the state jumps with zero costs" we will mean a jump-wise transfer performed in zero time interval (e.g. of the form (5)) caused by some control "impulses" form which the performance index takes the value equal to zero. Taking into account the Lemma 1, Assumption 1, the transformed problem (16), (17) and the relations (6)-(11) one can prove the following.

Theorem 3 *The h-dimensional subspace J_x spanned by the basis (11) contains the vectors determining all the possible state-jumps with zero-costs.*

Algorithm. The considered problem can be solved in accordance with the following scheme (valid also in the case when Assumption 1 is not fulfilled):

1. Determine the set $O_u(6)$ and related transformation matrix P.
2. Determine the set $O_x(6)$, create the sequences (7), choose the linearly independent (of the previously chosen and mutually) vectors (10) and adjoin them to the previously chosen ones.
3. Check, if the number of chosen vectors is equal to n. If yes, the optimal control has the form of appropriate "impulses" transferring the state to zero in jump-wise manner.
4. Determine the indices (ij), $j = 1, 2, .., q$ for the lastly chosen vectors (10).
5. Check, if the Assumption 1 is fulfilled. If yes, construct the state transformation $x = W\bar{x}$. If not, repeat the points 1-4 with H replaced by that determined by (18).
6. Derive the transformed formulas (12), (13), construct the non-singular problem (15), (17) with using the indices (ij), $j = 1, 2, .., q$, apply the formulas (20)-(22) and derive (23), (24).

It is important that the Assumption 1 can be checked at the time of creation of the state transformation, without calculation of the transformed equations (12), (15). It is seen that if Assumption 1 is not fulfilled then proposed state transformation is created recurrently, but is applied only one time. The transformed equation (12) has then the matrix $\bar{A}$ in which the part of the first g columns of (15) has the forms similar to those appearing in the last h columns of (15).

8 Example

Let us consider the problem (1), (2) with the matrices

$$A = \begin{bmatrix} 0, & 0, & 0 \\ 1, & -1, & 0 \\ 0, & 1, & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1, & 0 \\ 0, & 1 \\ 0, & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 1 \end{bmatrix}, \; H = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix}, \; G = O \qquad (25)$$

$x = [x_1, x_2, x_3]^T, \;\; u = [u_1, u_2]^T$.
We have $v = [u_1]$, $e = [u_2]$ and the vectors (7) take the form $b_2 = [0, 1, 0]^T \in O_x$, $Ab_2 = [0, -1, 1]^T \notin O_x$. Thus we have

$$W = \begin{bmatrix} 1, & 0, & 0 \\ 0, & 1, & -1 \\ 0, & 0, & 1 \end{bmatrix}, \; W^{-1}AW = \begin{bmatrix} 0, & 0, & 0 \\ 1, & 0, & -2 \\ 0, & 1, & -3 \end{bmatrix},$$

$$W^{-1}BP = B, \; W^TQW = Q \qquad (26)$$

$P^THP = H$, $\bar{x} = [\bar{x}_1, \bar{x}_2, \bar{x}_3]^T$, $z = [\bar{x}_1]$, $y = [\bar{x}_2, \bar{x}_3]$, $s = [\bar{x}_3]$, and the problem (16), (17) takes the form

$$\dot{z} = v \;, \; I = \int_0^{t_f} (z^2 + s^2 + v^2) \, dt \qquad (27)$$

The LQ problem (27) is nonsingular and can be written in the form (1), (2) with the matrices $A = 0$, $B = [1, 0]$, $G = 0$, $Q = 1$, $H = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}$. The OCL (19), (20) takes the form 28

$$\begin{bmatrix} v \\ s \end{bmatrix} = -H^{-1}B^TS(t)z = -\begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} S(t)z = \begin{bmatrix} -S(t)z \\ 0 \end{bmatrix} \qquad (28)$$

where in accordance with (21) $S(t)$ fulfils the equation: $-\dot{S}(t) = 1 - S^2(t), S(t_f) = 0$. From the transformed state equations: $\dot{\bar{x}}_3 = \bar{x}_2 - 3\bar{x}_3$ and $\dot{\bar{x}}_2 = \bar{x}_1 - 2\bar{x}_3 + u_2$, substituting in them $\bar{x}_3 = s = 0$, we obtain $\bar{x}_2 = 0$ and $u_2 = -\bar{x}_1 = -z$. Thus the singular strip and the OCL on it takes the form

$$S = \{\bar{x}_1, \bar{x}_2, \bar{x}_3 \; : \bar{x}_2 = 0, \bar{x}_3 = 0\}, \; \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} S(t) \\ 1 \end{bmatrix} \bar{x}_1 \qquad (29)$$

If the initial state x_0 is such that $\bar{x}_2(0) = \bar{x}_{20} \neq 0$ and $\bar{x}_3(0) = \bar{x}_{30} \neq 0$ then from (5) it results that the control $u_2 = -\bar{x}_{20}\delta(t) - \bar{x}_{30}\delta^{(1)}(t)$ brings the state onto the singular strip on which the OCL (29) is applied for $0 < t \leq t_f$.

One can notice, that using the method of Clements and Anderson [2] for the considered example the transformation described by them must be applied successively two times in order to obtain the non-singular problem.

9 Final Conclusions

The applied in the paper abstract notions of the distributions in the form of Dirac function and its derivatives have led to usable results. By this way the introduced and rather abstract subspace J_x of the state-jumps with zero-costs have led to usable and novel state transformation which in one step converts the considered singular LQ problem to the non-singular one. In other methods, generally, the appropriate transformations must be applied successively several times in order to obtain a non-singular LQ problem [4].

The convenient form of the transformed state equation makes it possible to derive the exact formulas determining the singular strip. In general case this task is very difficult to solve by means of other known methods, which is exactly stated in [4].

For the initial state belonging to the singular strip the OCL of the considered problem results form the known formulas and the optimal control has the form of bounded continuous functions of time. For the initial states not belonging to the singular strip the solution of the problem exists among the distributions and the appropriate "impulses" appear at time $t = 0$. This means that in the case of bounded controls the solution must take into account the bounds of the control.

It is worthwhile to notice that the proposed state transformation can be exploited also in the case of other LQ control problems.

References

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ISBN 83-85847-34-0