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ON k-MONOTONICITY PROPERTY

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Let $(E, \|.\|)$ be a separable real Banach space. Let $f(x)$ be a real valued convex continuous function defined on an open convex subset $\Omega \subset E$. Mazur [2] proved that in this case there is a subset A_G of the first category such that on the set $\Omega \setminus A_G$ the function f is Gateaux differentiable. Asplund [1] shown that if additionally the space E has the separable dual, then there is a subset A_F of the first category such that on the set $\Omega \setminus A_F$ the function f is Fréchet differentiable.

In the papers [4], [5], [6] (see also the book [3]) those results were extended on metric spaces with the classical convexity replaced by Φ-convexity. For obtaining this generalization we need to assume that the class of functions Φ is a group with respect to addition and it has a certain additional property called k-monotonicity property. The aim of this note is to investigate more precisely this notion.

Let a set X be given. Let Φ be a family of real-valued functions defined on X.

Let $f(x)$ be a real-valued function. If

$$f(x) = \sup\{\phi(x) + c : \phi \in \Phi, c \in \mathbb{R}, \phi + c \leq f\}$$

we say that the function f is Φ-*convex*.

The function $\phi(x) \in \Phi$ will be called a Φ-*subgradient* of the function $f(x)$ at a point x_0 if

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0)$$

for all $x \in X$.

The set of all Φ-subgradients of the function f at a point x_0 we shall call Φ-*subdifferential* of the function f at a point x_0 and we shall denote it by $\partial_\Phi f|_{x_0}$.

Let (X, d_X) be a metric space. We shall say that a real-valued function $f(x)$ defined on X is *Fréchet Φ-differentiable* at a point x_0 if there are a function

$\phi_{x_0} \in \Phi$ (called *Fréchet Φ-gradient*) and a function $\gamma(t)$ mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that

$$\lim \frac{\gamma(t)}{t} = 0$$

and such that

$$|[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]| \leq \gamma(d_X(x, x_0)).$$

We say that a real-valued function f, $f : X \to \mathbb{R}$, is *Lipschitzian with a constant K* (or f is a *Lipschitz function with a constant K*, or f satisfies the *Lipschitz condition with a constant K*) if

$$|f(x_1) - f(x_2)| \leq K d_X(x_1, x_2). \tag{1}$$

for all $x_1, x_2 \in X$.

Let $\mathcal{L}$ be the space of all Lipschitzian functions defined on X. We define on $\mathcal{L}$ a quasinorm

$$\|\phi\|_L = \sup_{\substack{x_1, x_2 \in X, \\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{d_X(x_1, x_2)}. \tag{2}$$

Observe that, if $\|\phi_1 - \phi_2\|_L = 0$, then the difference of ϕ_1 and ϕ_2 is a constant function, i.e. $\phi_1(x) = \phi_2(x) + c$. Thus we consider the quotient space $\tilde{\mathcal{L}} = \mathcal{L}/_{\mathbb{R}}$. The quasinorm $\|\phi\|_L$ induces the norm in the space $\tilde{\mathcal{L}}$. Since this will not lead to any misunderstanding, this norm we shall also denote by $\|\phi\|_L$. It can be shown that the space $(\tilde{\mathcal{L}}, \|\phi\|_L)$ is a Banach space.

Let ϕ be a Lipschitz function. Let a constant k, $0 < k \leq 1$, be given. We say that ϕ has the *k-monotonicity property* (*weak k-monotonicity property*) if for all $x \in X$ and all $t > 0$, there is a $y \in X$ such that $0 < d_X(x, y) < t$ and

$$\phi(y) - \phi(x) \geq k\|\phi\|_L d_X(y, x), \tag{3}$$

(resp.

$$|\phi(y) - \phi(x)| \geq k\|\phi\|_L d_X(y, x)). \tag{4}$$

Of course, every function having k-monotonicity property has also weak k-monotonicity property. Just from the definition it follows that if on X there is a Lipschitz function having k-monotonicity property (weak k-monotonicity property), then the space X does not contain isolated points. If X is a compact space, then there is no Lipschitz function defined on X having k-monotonicity property. It trivially follows of the fact that each continuous function attains its maximum on X. There are however compact sets on which there are functions having weak k-monotonicity property.

It is obvious that the linear continuous functional over an open set in a Banach space has the k-monotonicity property for every k, $0 < k \leq 1$. If the space is reflexive every linear continuous functional over an open set has the 1-monotonicity property.

Let a constant k, $0 < k \leq 1$, be given. Let Φ be a family of real-valued Lipschitz functions defined on X. We say that the family Φ has the *k-monotonicity property* (*weak k-monotonicity property*), if each function $\phi \in \Phi$ has the k-monotonicity property (resp. weak k-monotonicity property). Of course if a family Φ has k-monotonicity property (weak k-monotonicity property), then each its subfamily $\Phi_0 \subset \Phi$ also has k-monotonicity property (weak k-monotonicity property).

If a family Φ has k-monotonicity property (weak k-monotonicity property), then it has k_1-monotonicity property (weak k_1-monotonicity property) for all $k_1 \leq k$.

Having a notion of k-monotonicity property we can formulate the extension of the Mazur [2] and Asplund [1] theorems

Theorem 1 *(Rolewicz [6], see also the book [3]). Let (X, d_X) be a metric space. Let Φ be a family of Lipschitz functions having k-monotonicity property and being an additive group* [1]. *Assume that Φ is separable in the metric d_L. Let $f(x)$ be a Φ-convex function having at each point a Φ-subgradient. Then there is a set A of the first category such that on the set $B = X \setminus A$ the subdifferential $\partial_\Phi f\big|_x$ is single-valued, $\partial_\Phi f\big|_x = \{\phi_x\}$, ϕ_x is a Fréchet Φ-gradient of the function f at the point x and moreover it is continuous in the metric d_L.*

In the sequel we shall investigate more precisely k-monotonicity property.

Proposition 1 *Let (X, d_X) be a metric space. Let Y be a dense set in X. Let Φ be a family of Lipschitz functions defined on X having k-monotonicity property (weak k-monotonicity property). Then the family $\Phi|_Y$ of Lipschitz functions being the restrictions of functions $\phi \in \Phi$ to the set Y also have k-monotonicity property (resp. weak k-monotonicity property).*

Proof. It is a trivial consequence of the fact that the Lipschitz functions are continuous. □

Proposition 1 cannot be reversed, more precisely there are a metric space (X, d_X), Y being a dense set in X, a family Φ of Lipschitz functions defined on X, such that Φ does not have k-monotonicity property (weak k-monotonicity property) and the restriction of Φ to the set Y, $\Phi|_Y$ has k-monotonicity property (weak k-monotonicity property). Even by adding of one point can destroy k-monotonicity property and weak k-monotonicity property.

Example 1 *Let $Y = (-1, 0) \cup (0, 1)$ and let $X = Y \cup \{0\} = (-1, 1)$. Let $\phi(x) = |x|$. It is easy to see that ϕ has k-monotonicity property on Y for*

[1] In the papers [4], [5], [6] linearity of family Φ is assumed, but the proofs are going without any changes for the case when Φ is an additive group.

every k, $0 < k \leq 1$, and it is obvious that if then ϕ extended on X does not have k-monotonicity property for any k, $0 < k \leq 1$ (it has however weak k-monotonicity property for every k, $0 < k \leq 1$).

Example 2 *Let $Y = (-1, 0) \cup (0, 1)$ and let $X = Y \cup \{0\} = (-1, 1)$. Let function ϕ be defined as below. Let $0 \leq \phi(t_0) \leq t_0^2$. If $0 < \phi(t_0) < t_0^2$ then $\phi(t)$ is continuously differentiable in the neighbourhood of t_0 and $\left|\frac{d\phi}{dt}|_{t_0}\right| = 1$. If either $\phi(t_0) = 0$ or $\phi(t_0) = t_0^2$ then $\phi(t)$ is left-hand side and right-hand side differentiable at the point t_0 and moreover*

$$\lim_{t \to 0-0} \frac{d\phi}{dt}|_t = - \lim_{t \to 0+0} \frac{d\phi}{dt}|_t.$$

It is easy to see that Φ has weak k-monotonicity property on Y for every k, $0 < k \leq 1$. On the other hand, Φ extended on X does not have weak k-monotonicity property for any k, $0 < k \leq 1$, since in this case the function ϕ is differentiable at 0 and its derivative at this point is equal 0.

Proposition 2 *Let (X, d_X) be a metric space. Let ϕ be a Lipschitz function defined on X having weak k-monotonicity property. Then there is a dense subset Y of X such that the restriction of functions ϕ to the set Y have k-monotonicity property.*

Proof. Suppose that the thesis does not hold. Then there is an open set $U \in X$ such that for all $x_1, x_2 \in U$

$$\phi(x_1) - \phi(x_2) \leq k\|\phi\|_L d_Y(x_1, x_2).$$

Replacing the role of x_1 and x_2 we obtain that

$$|\phi(x_1) - \phi(x_2)| \leq k\|\phi\|_L d_Y(x_1, x_2).$$

for all $x_1, x_2 \in U$.

Since U is open it implies that the function ϕ does not have weak k-monotonicity property. □

Corollary 1 *Let (X, d_X) be a metric space. Let Φ be a finite family of Lipschitz functions defined on X having weak k-monotonicity property. Then there is a dense set Y of X such that the restrictions of functions $\phi \in \Phi$ to the set Y have k-monotonicity property.*

Proof. Let $\Phi = \{\phi_1, ..., \phi_n\}$. By Proposition 4 there is a set X_1 dense in X such that ϕ_1 has k-monotonicity property on X_1. Once more, by Proposition 4 we get that there is a set $X_2 \subset X_1$ dense in X_1 such that ϕ_2 has k-monotonicity property on X_2. Observe that the set X_2 is dense in X. Since $X_2 \subset X_1$, ϕ_1 also has k-monotonicity property on X_2.

Repeating this considerations we can show that there is a set X_n dense in X such that $\phi_1, ..., \phi_n$ have k-monotonicity property on X_n. □

If Φ is an infinite family Corollary 5 may not hold.

Example 3 *Let (X, d_X) be a metric space. Let Φ be a family of Lipschitz functions defined in the following way $\Phi = \{\phi_z : \phi_z(x) = -d_X(x, z), z \in X\}$. It is easy to see that the family Φ has weak k-monotonicity property. On the other hand for each subset Y of X which contains a non-isolated point z_0 the restriction of the function ϕ_{z_0} to the set Y does not have k-monotonicity property. Thus the family $\Phi|_Y$ obtained as the restrictions of functions $\phi \in \Phi$ to the set Y does not have k-monotonicity property.*

Proposition 3 *Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $\alpha(x)$ be a Lipschitz homeomorphism of the space (X, d_X) onto the space (Y, d_Y), i.e. there are $K, M > 0$ such that*

$$d_Y(\alpha(x), \alpha(y)) \leq K d_X(x, y) \tag{5}$$

and

$$d_X(x, y) \leq M d_Y(\alpha(x), \alpha(y)). \tag{6}$$

Let Φ be a family of Lipschitz functions defined on Y having k-monotonicity property (weak k-monotonicity property). Then the family Ψ of Lipschitz functions defined on X in the following way

$$\Psi = \{\psi(x) : \psi(x) = \phi(\alpha(x)), \phi \in \Phi\}$$

has $\frac{k}{KM}$-monotonicity property (resp. weak $\frac{k}{KM}$-monotonicity property).

Proof. At the beginning we shall estimate $\|\psi\|_L$.

$$\|\psi\|_L = \sup_{\substack{x_1, x_2 \in X, \\ x_1 \neq x_2}} \frac{|\psi(x_1) - \psi(x_2)|}{d_X(x_1, x_2)} = \sup_{\substack{x_1, x_2 \in X, \\ x_1 \neq x_2}} \frac{|\phi(\alpha(x_1)) - \phi(\alpha(x_2))|}{d_X(x_1, x_2)}. \tag{7}$$

But

$$\frac{|\phi(\alpha(x_1)) - \phi(\alpha(x_2))|}{d_X(x_1, x_2)} = \frac{|\phi(\alpha(x_1)) - \phi(\alpha(x_2))|}{d_Y(\alpha(x_1), \alpha(x_2))} \frac{d_Y(\alpha(x_1), \alpha(x_2))}{d_X(x_1, x_2)}. \tag{8}$$

Our hypotheses (5) and (6) imply

$$\frac{1}{M} \leq \frac{d_Y(\alpha(x_1), \alpha(x_2))}{d_X(x_1, x_2)} \leq K. \tag{9}$$

Since α is mapping X onto Y, it implies that

$$\frac{1}{M}\|\phi\|_L \leq \|\psi\|_L \leq K\|\phi\|_L. \tag{10}$$

Suppose that the family Φ have k-monotonicity property. It means that for each $\phi \in \Phi$

$$\phi(y_1) - \phi(y_2) \geq k\|\phi\|_L d_Y(y_1, y_2). \tag{11}$$

Let x_1, x_2 be such that $y_1 = \alpha(x_1),\ y_2 = \alpha(x_2)$.

$$\psi(x_1) - \psi(x_2) = \phi(\alpha(x_1)) - \phi(\alpha(x_2)) \geq k\|\phi\|_L d_Y(\alpha(x_1), \alpha(x_2))$$

$$\geq k\frac{1}{K}\|\psi\|_L \frac{1}{M} d_X(x_1, x_2). \tag{12}$$

Then the family $\Psi = \{\psi(x) : \psi(x) = \phi(\alpha(x)), \phi \in \Phi\}$ has $\frac{k}{KM}$-monotonicity property.

The proof for weak k-monotonicity property is the same. □

This result can be extended also on the case when we do not have Lipschitz homeomorphism, but a mapping of one metric space onto the second one. More precisely

Proposition 4 *Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $\alpha(x)$ be a Lipschitz mapping of the space (X, d_X) onto the space (Y, d_Y), i.e. there is $K > 0$ such that*

$$d_Y(\alpha(x), \alpha(y)) \leq K d_X(x, y) \tag{5}$$

Suppose that there is $M > 0$ such that

$$\inf_{\substack{\{x:\alpha(x)\in u\},\\ \{y:\alpha(y)\in v\}}} d_X(x, y) \leq M d_Y(u, v)). \tag{6a}$$

Let Φ be a family of Lipschitz functions defined on Y having k-monotonicity property (weak k-monotonicity property). Then the family Ψ of Lipschitz functions defined on X in the following way

$$\Psi = \{\psi(x) : \psi(x) = \phi(\alpha(x)), \phi \in \Phi\}$$

has $\frac{k}{KM}$-monotonicity property (resp. weak $\frac{k}{KM}$-monotonicity property).

Proof. At the beginning we shall estimate $\|\psi\|_L$.

$$\|\psi\|_L = \sup_{\substack{x_1, x_2 \in X,\\ x_1 \neq x_2}} \frac{|\psi(x_1) - \psi(x_2)|}{d_X(x_1, x_2)} = \sup_{\substack{x_1, x_2 \in X,\\ x_1 \neq x_2}} \frac{|\phi(\alpha(x_1)) - \phi(\alpha(x_2))|}{d_X(x_1, x_2)}. \tag{7}$$

But

$$\frac{|\phi(\alpha(x_1)) - \phi(\alpha(x_2))|}{d_X(x_1, x_2)} = \frac{|\phi(\alpha(x_1)) - \phi(\alpha(x_2))|}{d_Y(\alpha(x_1), \alpha(x_2))} \frac{d_Y(\alpha(x_1), \alpha(x_2))}{d_X(x_1, x_2)}$$

$$\leq \frac{|\phi(\alpha(x_1)) - \phi(\alpha(x_2))|}{d_Y(\alpha(x_1), \alpha(x_2))} \frac{d_Y(u,v)}{\inf\limits_{\substack{\{x:\alpha(x)\in u\},\\ \{y:\alpha(y)\in v\}}} d_X(x,y)}.$$

Thus by hypotheses (5) and (6a)

$$\|\psi\|_L \leq M\|\phi\|_L$$

and

$$\frac{1}{M}\|\phi\|_L \geq \|\psi\|_L. \tag{10a}$$

Suppose that the family Φ have k-monotonicity property. It means that for each $\phi \in \Phi$

$$\phi(y_1) - \phi(y_2) \geq k\|\phi\|_L d_Y(y_1, y_2). \tag{11}$$

Let x_1, x_2 be arbitrary such that $y_1 = \alpha(x_1),\ y_2 = \alpha(x_2)$.

$$\psi(x_1) - \psi(x_2) = \phi(\alpha(x_1)) - \phi(\alpha(x_2)) \geq k\|\phi\|_L d_Y(\alpha(x_1), \alpha(x_2))$$

$$\geq k\frac{1}{K}\|\psi\|_L \frac{1}{M} d_X(x_1, x_2). \tag{12}$$

Then the family $\Psi = \{\psi(x) : \psi(x) = \phi(\alpha(x)), \phi \in \Phi\}$ has $\frac{k}{KM}$-monotonicity property.

The proof for weak k-monotonicity property is the same. □

By Propositions 1, 7, 8 and by observation that a linear continuous functional over an open set in a normed space have the k-monotonicity property for every k, $0 < k \leq 1$, we obtain a lot of examples of families Φ having k-monotonicity property. We do not know any example of a family Φ having k-monotonicity property and being an additive group, which is not obtained in this way. More precisely

Conjecture 1 *Let (X, d_X) be a metric space. Let Φ be a family of Lipschitz functions defined on X having k-monotonicity property and being an additive group. Then there are*

1. *a normed space $(E, \|\cdot\|)$*
2. *a subset $Y \subset E$*
3. *a family Ψ of continuous linear functionals restricted to Y being an additive group*
4. *a Lipschitz mapping $\alpha(x)$ of the space (X, d_X) onto Y and $K, M > 0$ such that*

$$d_Y(\alpha(x), \alpha(y)) \leq K d_X(x,y) \tag{5}$$

$$\inf_{\substack{\{x:\alpha(x)\in u\},\\ \{y:\alpha(y)\in v\}}} d_X(x,y) \leq M d_Y(u,v)) \tag{6a}$$

such that

$$\Phi = \{\phi(x) : \phi(x) = \psi(\alpha(x)), \psi \in \Psi\}.$$

It would be interesting to know which sets Y in a Banach space E have this property that a linear continuous functional over Y have the k-monotonicity property. If it holds we say that Y is *k-monotonicity set.* From Proposition 1 follows that every dense set Y of an open set U of a Banach space is a k-monotonicity set. There are however k-monotonicity sets being different form that described above.

Proposition 5 *Let X be a subset of a Banach space E, such that for each $x \in X$ and all $t > 0$, the cone induced by $X_{x,t} = X \cap \{y : 0 < \|y - x\| < t\}$ is dense in E. Then the set X is a k-monotonicity set for all k ,$0 < k < 1$.*

Proof. Take any continuous linear functional $f \in E^*$. Without loss of generality we may assume that $\|f\| = 1$. Let ε be an arbitrary positive number. Let $x_f \in E$ be such an element of norm one that

$$1 - \varepsilon < f(x_f) \leq 1. \tag{13}$$

Since the cone induced by $X_{x,t} = X \cap \{y : 0 < \|y - x\| < t\}$ is dense in E, there is $h \in X_{x,t}$ such that

$$\|x_f - \frac{h}{\|h\|}\| < \varepsilon. \tag{14}$$

By (13) and (14) we have

$$1 - 2\varepsilon < f(\frac{h}{\|h\|}) \qquad (15).$$

Let $y = x + h$. Thus $h = y - x$. Since $h \in X_{x,t}$, we have that $0 < \|y - x\| < t$ and by (15)

$$f(h) = f(y) - f(x) > (1 - 2\varepsilon)\|h\| = (1 - 2\varepsilon)\|y - x\|.$$

The arbitrariness of ε implies that the set X is a k-monotonicity set for all k ,$0 < k < 1$. □

Corollary 2 *Let X be an open subset of a Banach space E. Then the set X is a k-monotonicity set for all k ,$0 < k < 1$.*

Corollary 3 *Let X be an open subset of a Banach space E. Let T be a continuous linear operator mapping the space E into a Banach space F. Then the set $Y = T(X) \subset F$ is a k-monotonicity set for all k ,$0 < k < 1$.*

Proof. Without loss of generality we may assume that $T(E)$ is dense in F. It is easy to see that for each $x \in X$ and all $t > 0$ the cone induced by $Y_{x,t} = T(X) \cap \{y : 0 < \|y - x\| < t\}$ is dense in F. □

Corollary 12 gives a lot of examples of k-monotonicity sets. For example the set

$$A = \{x \in \ell^2 : \sum_{n=1}^{\infty} n x_n^2 < 1\}$$

is a k-monotonicity set. There also examples which are not of the form, which follows from Corollary 12.

Example 4 *Let $E = R^2$ with the standard euclidean norm. Let $Y = \{(x,y) : |y| > x^2\} \cup \{(0,0)\}$. It is easy to see that Y is a monotonicity set.*

In a similar way we can obtain an example of a set, which is not a k_0-monotonicity, but which is a k-monotonicity set for $k < k_0$.

Example 5 *Let $E = R^2$ with the standard euclidean norm. Let*

$$Y = \{(x,y) : y < \alpha|x|\} \cup \{(0,0)\}.$$

By simple calculation we obtain that the functional $f_0(x,y) = -y$ restricted to the set Y is a k-monotonicity function if $k < k_0 = \frac{1}{\sqrt{1+\alpha^2}}$ and it is not a k_0-monotonicity function. It is also easy to see that any functional $f_1(x,y) \neq f_0(x,y)$ is a k_0-monotonicity function.

Therefore Y is not a k_0-monotonicity set, but it is a k-monotonicity set for $k < k_0$.

Basing on Example 14 we can construct set Y, which is a k_0-monotonicity, but which is not a k-monotonicity set for $k > k_0$.

Example 6 *Let $E = R^2$ with the standard euclidean norm. Let*

$$Y = \{(x,y) : y < \min_{1 \le n < \infty} (\alpha + \frac{1}{n})|x - n|\} \cup \bigcup_{1 \le n < \infty} \{(n,0)\}.$$

By similar consideration as in Example 6 we obtain that Y is a k-monotonicity set for all k, such that $k < k_n = \frac{1}{\sqrt{1+(\alpha+\frac{1}{n})^2}}$ for all $n = 1, 2, \dots$. Thus Y is a k_0-monotonicity set. It is trivial observation that Y is not a k-monotonicity set for any $k > k_0$.

The set constructed in Example 15 is unbounded, but a similar construction can give a bounded set with the requested property.

Example 7 *Let $E = R^2$ with the standard euclidean norm. Let*

$$Y = \{(x,y) : y < \min_{1 \le n < \infty} (\alpha + \frac{1}{n})|x - \frac{1}{n}|, |x| < 1, |y| < 1\} \cup \bigcup_{1 \le n < \infty} \{(\frac{1}{n}, 0)\}.$$

By similar consideration as in Example 10 we obtain that Y is a k_0-monotonicity set and it is not a k-monotonicity set for any $k > k_0$.

Moreover it is easy to see that Y is bounded.

If we have two equivalent norms on E, then obviously there is a constant $0 < M < 1$, such that any k-monotonicity function with respect to the first norm is kM-monotonicity function with respect to the second one. This implies that any k-monotonicity set with respect to the first norm is kM-monotonicity set with respect to the second one.

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