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FUZZY LOGIC = COMPUTING WITH WORDS[*, 1]

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To Professor Roman Kulikowski for his anniversary

As its name suggests, computing with words, CW, is a methodology in which words are used in place of numbers for computing and reasoning. The point of this note is that fuzzy logic plays a pivotal role in CW and vice-versa. Thus, as an approximation, fuzzy logic may be equated to CW.

There are two major imperatives for computing with words. First, computing with words is a necessity when the available information is too imprecise to justify the use of numbers. And second, when there is a tolerance for imprecision which can be exploited to achieve tractability, robustness, low solution cost and better rapport with reality. Exploitation of the tolerance for imprecision is an issue of central importance in CW.

In CW, a words is viewed as a label of a granule, that is, a fuzzy set of points drawn together by similarity, with the fuzzy set playing the role of a fuzzy constraint on a variable. The premises are assumed to be expressed as propositions in a natural language. For purposes of computation, the propositions are expressed as canonical forms which serve to place in evidence the fuzzy constraints that are implicit in the premises. Then, the rules of inference in fuzzy logic are employed to propagate the constraints from premises to conclusions.

At this juncture, the techniques of computing with words underlie -- in one way or another -- almost all applications of fuzzy logic. In coming years,

computing with words is likely to evolve into a basic methodology in its own right with wide-ranging ramifications on both basic and applied levels.

1. Introduction

Fuzzy logic has come of age. Its foundations have become firmer, its applications have grown in number and variety; and its influence within the basic sciences -- and especially in mathematical and physical sciences -- has become more visible and more substantive.

And yet, there are two questions that are still frequently raised: (a) What is fuzzy logic?; and (b) What can be done with fuzzy logic that cannot be done equally well with other methodologies, e.g., predicate logic, probability theory, neural network theory, Bayesian networks and classical control.

The title of this note is intended to suggest a succinct answer: The main contribution of fuzzy logic is a methodology for computing with words (CW). No other methodology serves this purpose.

What follows is an elaboration on this suggestion. A fuller exposition of the methodology of CW will appear in a forthcoming paper.

Needless to say, there is more to fuzzy logic than a methodology for CW. Thus, strictly speaking, the equality in the title of this note should be an inclusion; using the equality serves to accentuate the importance of computing with words as a branch of fuzzy logic.

2. What is Computing with Words?

In its traditional sense, computing involves for the most part manipulation of numbers and symbols. By contrast, humans employ mostly words in computing and reasoning, arriving at conclusions expressed as words from premises expressed in a natural language or having the form of mental perceptions. As used by humans, words have fuzzy denotations. The same applies to the role played by words in computing with words (CW).

The concept of CW is rooted in several papers starting with my 1973 paper "A New Approach to the Analysis of Complex Systems and Decision Processes," in which the concepts of a linguistic variable and granulation were introduced. The concepts of a fuzzy constraint and fuzzy constraint propagation were introduced in "The Calculus of Fuzzy Restrictions," (1974), and developed more fully in "A Theory of Approximate Reasoning," (1979) and "Outline of a Computational Approach to Meaning and Knowledge Representation Based on a Concept of a Generalized Assignment Statement," (1986). Application of fuzzy logic to meaning representation and its role in test-score semantics are discussed in "PRUF - A Meaning Representation Language for Natural Languages," (1978),

and "Test-Score Semantics for Natural Languages and Meaning-Representation via PRUF" (1981).

Although the foundations of computing with words were laid some time ago, its evolution into a distinct methodology in its own right reflects many advances in our understanding of fuzzy logic and soft computing -- advances which took place within the past few years. (See References and Related Papers.) A key aspect of CW is that it involves a fusion of natural languages and computation with fuzzy variables. It is this fusion that is likely to result in an evolution of CW into a basic methodology in its own right, with wide ranging ramifications and applications.

We begin our exposition of CW with a few definitions. It should be understood that the definitions are dispositional, that is, admit of exceptions.

The point of departure in CW is the concept of a granule. In essence, a granule is a fuzzy set of points having the form of a clump of elements drawn together by similarity. A word, w, is a label of a granule, g, and conversely g is the denotation of w. A word may be atomic, as in *young*, or composite, as in *not very young*. Unless stated to the contrary, a word will be assumed to be composite. The denotation of a word may be a higher order predicate, as in a Montague grammar (Partee 1976).

In CW, a granule, g, which is the denotation of a word, w, is viewed as a fuzzy constraint on a variable. A pivotal role in CW is played by fuzzy constraint propagation from premises to conclusions. It should be noted that, as a basic technique, constraint propagation plays important roles in many methodologies, especially in mathematical programming, constraint programming and logic programming. (See References and Related Papers.)

As a simple illustration, consider the proposition *Mary is young*. In this case, *young* is the label of a granule *young*. (Note that for simplicity the same symbol is used both for a word and its denotation.) The fuzzy set *young* plays the role of a fuzzy constraint on the age of Mary.

As a further example consider the propositions

p_1 = *Carol lives near Mary*

and

p_2 = *Mary lives near Pat.*

In this case, the words *lives near* in p_1 and p_2 play the role of fuzzy constraints on the distances between the residences of Carol and Mary, and Mary and Pat, respectively. If the query is: How far is Carol from Pat?, an answer yielded by fuzzy constraint propagation might be expressed as p_3, where

p_3 = *Carol lives not far from Pat*

More about fuzzy constraint propagation will be said at a later point.

A basic assumption in CW is that information is conveyed by constraining the values of variables. Furthermore, information is assumed to consist of a collection of propositions expressed in natural or synthetic language.

A basic generic problem in CW is the following:

We are given a collection of propositions expressed in a natural language which constitute the *initial data set*, or IDS for short.

From the initial data set we wish to infer an answer to a query expressed in a natural language. The answer, also expressed in a natural language, is referred to as the *terminal data set*, or TDS for short. The problem is to derive TDS from IDS.

A few problems will serve to illustrate these concepts. At this juncture, the problems will be formulated but not solved:

1. Assume that a function, f, f: $U \rightarrow V$, $X \in U$, $Y \in V$, is described in words by the fuzzy if-then rules:

f: if X is small then Y is small
         if X is medium then Y is large
         if X is large then Y is small

What this implies is that *f* is approximated to by the fuzzy graph f^* (Fig. 1), where
$f^* = small \times small + medium \times large + large \times small$
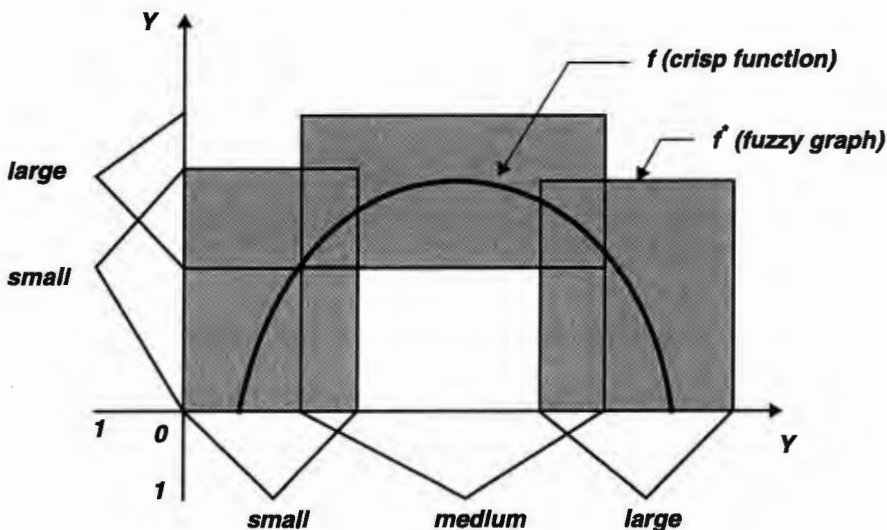

Fig. 1. f^* is a fuzzy graph which approximates a function f

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In f^*, + and × denote, respectively, the disjunction and cartesian product. An expression of the form $A \times B$, where A and B are words, will be referred to as a *cartesian granule*. In this sense, a fuzzy graph may be viewed as a disjunction of cartesian granules. In essence, a fuzzy graph serves as an approximation to a function or a relation (Zadeh 1974,1994).
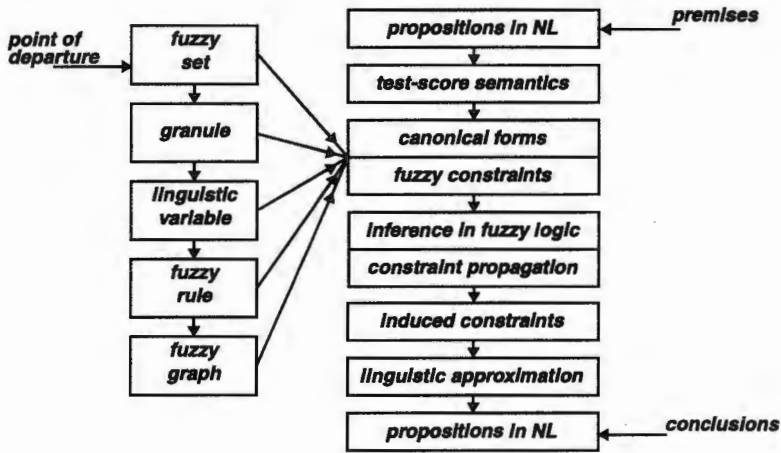
(a)
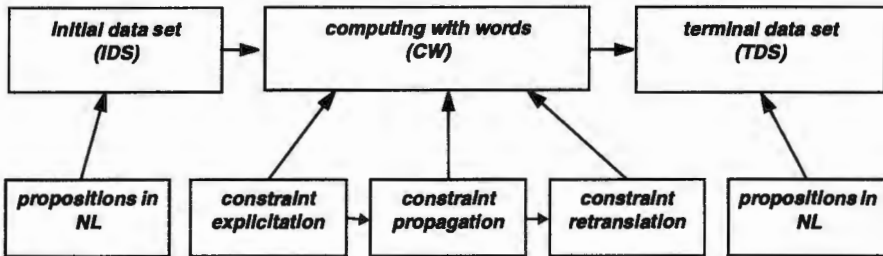
(b)

Fig. 2. Maximization of a crisp function; an interval-valued function; and a fuzzy graph

In the example under consideration, the IDS consists of the fuzzy rule-set f. The query is: What is the maximum value of f^* (cf. Fig. 2)?

More broadly, the problem is: How can one compute an attribute of a function, f, e.g., its maximum value or its area or its roots if f is described in words as a collection of fuzzy if-then rules?

2. A box contains ten balls of various sizes of which several are large and a few are small. What is the probability that a ball drawn at random is neither large nor small? In this case, the IDS is a verbal description of the contents of the box: The TDS is the desired probability.

3. A less simple example of computing with words is the following.

Let X and Y be independent random variables taking values in a finite set $V=\{v_1,...,v_n\}$ with probabilities $p_1,..., p_n$ and $q_1,..., q_n$, respectively. For simplicity of notation, the same symbols will be used to denote X and Y and their generic values, with p and q denoting the probabilities of X and Y, respectively.

Assume that the probability distributions of X and Y are described in words through the fuzzy if-then rules:

P: if X is small then p is small
              if X is medium then p is large
              if X is large then p is small

and

Q: if Y is small then q is large
              if Y is medium then q is small
              if Y is large then q is large

where the granules small, medium and large are the values of the linguistic variables X and Y in their respective universes of discourse. In the example under consideration, those rule-sets constitute the IDS. Note that small in P need not have the same meaning as small in Q, and likewise for medium and large.

The query is: How can we describe in words the joint probability distribution of X and Y? This probability distribution is the TDS.

For convenience, the probability distributions of X and Y may be represented as fuzzy graphs:

  P: *small* $\times$ *small* + *medium* $\times$ *large* + *large* $\times$ *small*

  Q: *small* $\times$ *large* + *medium* $\times$ *large* + *large* $\times$ *large*

with the understanding that the underlying numerical probabilities must add up to unity.

Since X and Y are independent random variables, their joint probability distribution (P,Q) is the product of P and Q. In words, the product may be expressed as (Zadeh 1994):

(P,Q): small $\times$ *small* $\times$ *(small * large) + small* $\times$ *medium* $\times$ *(small * small) + small* $\times$ *large* $\times$ *(small * large) + ... + large* $\times$ *large* $\times$ *(small * large)*

where * is the arithmetic product in fuzzy arithmetic (Kaufmann and Gupta 1986, Mares 1994). In this example, what we have done, in effect, amounts to a derivation of a linguistic characterization of the joint probability distribution of X and Y starting with linguistic characterizations of the probability distribution of X and the probability distribution of Y.

A few comments are in order. In linguistic characterizations of variables and their dependencies, words serve as the values of variables and play the role of fuzzy constraints. In this perspective, the use of words may be viewed as a form of granulation, which in turn may be regarded as a form of fuzzy quantization.

Granulation plays a key role in human cognition. For humans, it serves as a way of achieving data compression. This is one of the pivotal advantages accruing through the use of words in human, machine and man-machine communication.

In the final analysis, the rationale for computing with words rests on two major imperatives. First, computing with words is a necessity when the available information is too imprecise to justify the use of numbers. And second, when there is a tolerance for imprecision which can be exploited to achieve tractability, robustness, low solution cost and better rapport with reality.

The conceptual structure of computing with words is schematized in Fig. 3a and Fig. 3b. Basically, CW may be viewed as a confluence of two related streams: fuzzy logic and test-score semantics, with the latter based on fuzzy logic. The point of contact is the collection of canonical forms of the premises, which are assumed to be propositions expressed in a natural language (NL). The function of canonical forms is to explicitate the implicit fuzzy constraints which are resident in the premises. With canonical forms as the point of departure, fuzzy constraint propagation leads to conclusions in the form of induced fuzzy constraints. Finally, the induced constraints are translated into a natural language through the use of linguistic approximation (Zadeh 1973, Mamdani and Gaines 1981).

In computing with words, there are two core issues that arise. First is the issue of representation of fuzzy constraints. More specifically, the question is: How can the fuzzy constraints which are implicit in propositions expressed in a natural language be made explicit. And second is the issue of fuzzy constraint propagation, that is, the question of how can fuzzy constraints in premises be propagated to conclusions.

These are the issues which are addressed in the following.

3. Presentation of fuzzy constraints and canonical forms

Our approach to the representation of fuzzy constraints is based on test-score semantics (Zadeh 1978, 1981). In outline, in this semantics, a proposition, p, in a natural language is viewed as a network of fuzzy (elastic) constraints. Upon aggregation, the constraints which are embodied in p result in an overall fuzzy constraint which can be represented as an expression of the form X is R
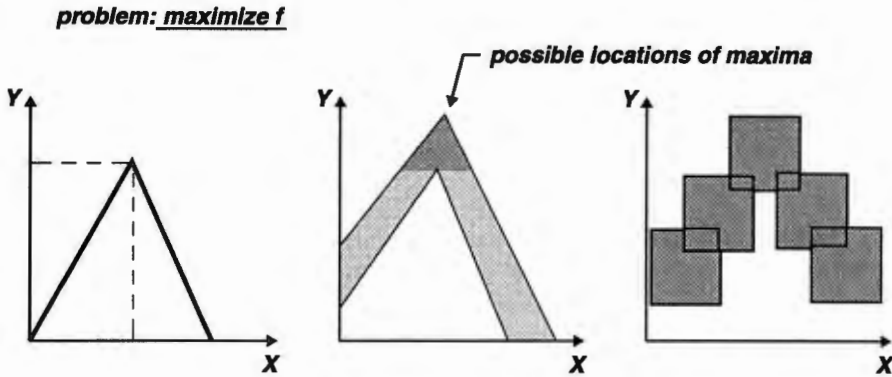

Fig. 3. Conceptual structure of CW.
Figure 3b is a more detailed version of Figure 3a.

where R is a constraining fuzzy relation and X is the constrained variable. The expression in question is called a *canonical form*. The function of a canonical form is to place in evidence the fuzzy constraint which is implicit in p. This is represented schematically as $p \rightarrow X$ is R

in which the arrow $\rightarrow$ denotes explicitation. The variable X may be vector-valued and/or conditioned.

In this perspective, the meaning of p is defined by two procedures. The first procedure acts on a so-called explanatory database, ED, and returns the constrained variable, X. The second procedure acts on ED and returns the constraining relation, R.

An explanatory database is a collection of relations in terms of which the meaning of p is defined. The relations are empty, that is, they consist of relation names, relations attributes and attribute domains, with no entries in the relations. When there are entries in ED, ED is said to be *instantiated* and is denoted EDI. EDI may be viewed as a description of a possible world in possible world semantics (Cresswell 1973), while ED defines a collection of possible worlds, with each possible world in the collection corresponding to a particular instantiation of ED.

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As a simple illustration, consider the proposition

p = *Mary is not young*

Assume that the explanatory database is chosen to be

ED = POPULATION[Name; Age] + YOUNG[Age; μ]

in which POPULATION is a relation with arguments Name and Age; YOUNG is a relation with arguments Age and μ; and + is the disjunction. In this case, the constrained variable is the age of Mary, which in terms of ED may be expressed as

X = Age(Mary) =$_{Age}$ POPULATION[Name=Mary]

This expression specifies the procedure which acts on ED and returns X. More specifically, in this procedure, Name is instantiated to Mary and the resulting relation is projected on Age, yielding the age of Mary.

The constraining relation, R, is given by

$R = (^2 \text{YOUNG})$

which implies that the intensifier *very* is interpreted as a squaring operation, and the negation *not* as the operation of complementation.

Equivalently, R may be expressed as

R = YOUNG [Age; $1\text{-}\mu^2$]

As a further example, consider the proposition

p = *Carol lives in a small city near San Francisco*

and assume that the explanatory database is

ED = POPULATION[Name;Residence]

+SMALL[City; μ] +NEAR[City1; City2; μ]

In this case

X = Residence(Carol) =$_{Residence}$ POPULATION[Name=Carol]

and

R =   SMALL[City; μ] ∩ $_{City1}$ NEAR[City2=San Francisco]

In R, the first constituent is the fuzzy set of small cities; the second constituent is the fuzzy set of cities which are near San Francisco; and ∩ denotes the intersection of these sets.

So far we have confined our attention to constraints of the form

X is R

In fact, constraints can have a variety of forms. In particular, a constraint -- expressed as a canonical form -- can be conditional, that is, of the form

if X is R then Y is S

which may also be written as

Y is S if X is R

The constraints in question will be referred to as *basic*.

For purposes of meaning representation, the richness of natural languages necessitates a wide variety of constraints in relation to which the basic constraints form an important though special class. The so-called generalized constraints (Zadeh 1986) contain the basic constraints as a special case and are defined as follows.

A generalized constraint is represented as

X isr R

where i_{sr}, pronounced "ezar," is a variable copula which defines the way in which R constrains X. More specifically, the role of R in relation to X is defined by the value of the discrete variable r. The values of r and their interpretations are defined below:

e: equal   (abbreviated to =)
d: disjunctive (possibilistic)   (abbreviated to blank)
c: conjunctive
p: probabilistic
λ: probability value
u: usuality
rs: random set
rsf: random fuzzy set
fg: fuzzy graph
ps   :   rough set (Pawlak Set)
.:   ...

As an illustration, when $r=e$, the constraint is an equality constraint and is abbreviated to =. When r takes the value d, the constraint is *disjunctive* (possibilistic) and is abbreviated to is, leading to the expression

X is R

in which R is a fuzzy relation which constrains X by playing the role of the possibility distribution of X. More specifically, if X takes values in a universe of discourse, $U=\{u\}$, then $Poss\{X=u\} = \mu_R(u)$, where μ_R is the membership function of R, and Π_X is the possibility distribution of X, that is, the fuzzy set of its possible values. In schematic form:

X is R $\rightarrow$ $\Pi_X = R$

$Poss\{X=u\} = \mu_R(u)$

Similarly, when R takes the value c, the constraint is *conjunctive*. In the case,

X isc R

means that if the grade of membership of u in R is μ, then $X = u$ has truth value μ. For example, a canonical form of the proposition

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p= John is proficient in English, French and German

may be expressed as

   Proficiency(John) isc (1/English + 0.7/French + 0.6/German)

in which 1.0, 0.7 and 0.6 represent, respectively, the truth values of the propositions *John is proficient in English, John is proficient in French* and *John is proficient in German.*

When $r = p$, the constraint is *probabilistic*. In this case,

   X isp r

means that R is the probability distribution of X. For example

   X isp $N(m, \sigma^2)$

means that X is normally distributed with mean m and variance σ^2. Similarly,

   X isp (0.2\a + 0.5\b + 0.3\c)

means that X is a random variable which takes the values, a, b and c with respective probabilities 0.2, 0.5 and 0.3.

  The constraint

   X isu R

is an abbreviation for

   usually(X is R)

which in turn means that

   Prob{X is R} is *usually*

In this expression X is R is a fuzzy event and *usually* is its fuzzy probability, that is, the possibility distribution of its crisp probability.

  The constraint

   X isrs P

is a random set constraint. This constraint is a combination of probabilistic and possibilistic constraints. More specifically, in a schematic form, it is expressed as

$$\begin{array}{c} X \ \text{isp } P \\ \underline{(X,Y) \text{ is } Q} \\ Y \text{ isrs } R \end{array}$$

where Q is a joint possibilitistic constraint on X and Y, and R is a random set. It is of interest to note that the Dempster-Shafer theory of evidence is, in essence, a theory of random set constraints.

  In computing with words, the starting point is a collection of propositions which play the role of premises. In most cases, the canonical forms of these propositions are constraints of the basic, disjunctive type. In a more general setting, the constraints are of the generalized type, implying that the explicitation of a proposition, p, may be represented as

   $p \rightarrow X$ isr R

where X isr R is the canonical form of p.

As in the case of basic constraints, the canonical form of a proposition may be derived through the use of *test-score semantics*. In this context, the depth of *p* is, roughly, a measure of the effort that is needed to explicitate *p*, that is, to translate *p* into its canonical form. In this sense, the proposition X isr R is a surface constraint (depth=zero), with the depth of explicitation increasing in the downward direction (Fig. 4). Thus a proposition such as *Mary is young* is shallow, whereas a proposition *it is not very likely that there will be a substantial increase in the price of oil in the near future*, is not.
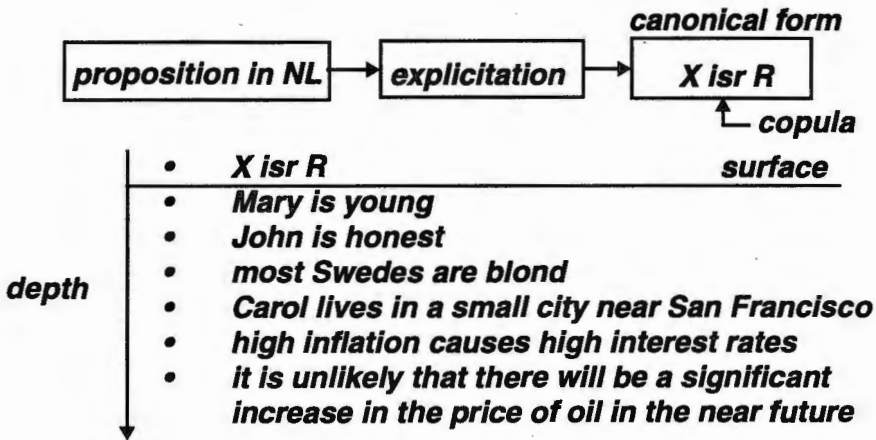

Fig. 4. Depth of expliciation

Once the propositions in the initial data set are expressed in their canonical forms, the groundwork is laid for fuzzy constraint propagation. This is a basic part of CW which is discussed in the following section.

4. Fuzzy constraint propagation and the rules of inference in fuzzy logic

The rules governing fuzzy constraint propagation are, in effect, the rules of inference in fuzzy logic. In addition to these rules, it is helpful to have rules governing fuzzy constraint modification, The latter rules will be discussed at a later point in this section.

In a summarized form, the rules governing fuzzy constraint propagation are the following [A and B are fuzzy relations. Disjunction and conjunction are defined, respectively, as max and min, with the understanding that, more generally, they could be defined via *t*-norms and *s*-norms (Klir and Yuan 1995).]:

conjunctive rule 1

$$X \text{ is } A$$
$$X \text{ is } B$$
$$\overline{X \text{ is } A \cap B}$$

conjunctive rule 2 $(X \in U, Y \in B, A \subset U, B \subset V)$

$$X \text{ is } A$$
$$Y \text{ is } B$$
$$\overline{(X, Y) \text{ is } A \times B}$$

disjunctive rule 1

$$X \text{ is } A$$
$$X \text{ is } B$$
$$\overline{X \text{ is } A \cup B}$$

disjunctive rule 2 $(A \subset U, B \subset V)$

$$X \text{ is } A$$
$$Y \text{ is } B$$
$$\overline{(X, Y) \text{ is}}$$
$$A \times V \cup U \times B$$

where A×V and U×B are cylindrical extensions of A and B, respectively.

conjunctive rule for isc

$$X \text{ isc } A$$
$$X \text{ isc } B$$
$$\overline{X \text{ isc } A \cup B}$$

disjunctive rule for isc

$$X \text{ isc } A$$
or
$$X \text{ isc } B$$
$$\overline{X \text{ isc } A \cap B}$$

projective rule

$$\underline{(X,Y) \text{ is } A}$$
$$Y \text{ is } proj_v A$$

where $proj_v = sup_u A$,

surjective rule

$$\underline{X \text{ is } A}$$
$$(X,Y) \text{ is } A \times V$$

Derived rules

compositional rule

$$X \text{ is } A$$
$$(X,Y) \text{ is } B$$
$$\overline{Y \text{ is } A \circ B}$$

where $A \circ B$ denotes the composition of A and B.

extension principle (mapping rule)

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$$\frac{X \text{ is } A}{f(X) \text{ is } f(A)}$$

where $f: U \to V$, and $f(A)$ is defined by

$$\mu_{f(A)} = \sup_{u \,|\, v = f(u)} \mu_A(u)$$

inverse mapping rule

$$\frac{f(X) \text{ is } A}{X \text{ is } f^{-1}(A)}$$

where $\mu_{f^{-1}(A)}(u) = \mu_A(f(u))$,
generalized modus ponens

$$\frac{\begin{array}{l} X \text{ is } A \\ \textit{If } X \text{ is } B \textit{ then } Y \\ \textit{is } C \end{array}}{\begin{array}{l} Y \textit{ is} \\ A^\circ((\neg B) \oplus C) \end{array}}$$

where the bounded sum $\neg B \oplus C$ represents £ukasiewicz's definition of implication,
generalized extension principle

$$\frac{f(X) \textit{ is } A}{\begin{array}{l} q(X) \textit{ is } q(f \\ {}^{1}(A)) \end{array}}$$

where $\mu_q(v) = \sup_{u \,|\, v = f(u)} \mu_A((q(u)))$.

The generalized extension principle plays a pivotal role in fuzzy constraint propagation.

syllogistic rule (Zadeh 1981)

$$\frac{\begin{array}{l} Q_1 \textit{ A's are B's} \\ Q_2 \textit{ (A and B)'s are C's} \end{array}}{\begin{array}{l} (Q_1 \otimes Q_2) \textit{ A's are (B and} \\ C)\text{'s} \end{array}}$$

where Q_1 and Q_2 are fuzzy quantifiers; A, B, and C are fuzzy relations; and $Q_1 \otimes Q_2$ is the product of Q_1 and Q_2 in fuzzy arithmetic,
constraint modification rules (Zadeh 1972,1979)

$$X \textit{ is } mA \to X \textit{ is } f(A)$$

where m is a modifier such as *not, very, more or less*, etc. and $f(A)$ defines the way in which m modifies A. Specifically,

if m = not *then* $f(A) = A'$ *(complement)*

if m = very *then* $f(A) = {}^2A$ *(left square)*

where $\mu^2_A(u)=[\mu_A(u)]^2$. This rule is a convention and should not be construed as a realistic approximation to the way in which the modifier *very* functions in a natural language.

probability qualification rule *(Zadeh 1979)*

$$(X \; is \; A) \; is \; \Lambda \rightarrow P \; is \; \Lambda$$

where X is a random variable taking values in U with probability density $p(u)$; Λ is a linguistic probability expressed in words like *likely, not very likely*, etc.; and P is the probability of the fuzzy event X is A, expressed as

$$P = \int_U \mu_A(u)p(u)du$$

The primary purpose of this summary is to underscore the coincidence of the principal rules governing fuzzy constraint propagation with the principal values of inference in fuzzy logic. Of necessity, the summary is not complete and there are many specialized rules which are not included. Furthermore, most of the rules in the summary apply to constraints which are of the basic, disjunctive type. Further development of the rules governing fuzzy constraint propagation will require an extension of the rules of inference to generalized constraints.

As was alluded to in the summary, the principal rule governing constraint propagation is the generalized extension principle which in a schematic form may be represented as

$$\frac{f(X_1,...,X_n) \; is \; A}{q(X_1,...,X_n) \; is \; q(f^{-1}(A))}$$

In this expression, $X_1,...,X_n$ are database variables; the term above the line represents the constraint induced by the IDS; and the term below the line is the TDS expressed as a constraint on the query $q(X_1,...,X_n)$. In the latter constraint, by $f^{-1}(A)$ we denote the preimage of the fuzzy relation A under the mapping $f: U \rightarrow V,$, where A is a fuzzy subset of V and U is the domain of $f(X_1,...,X_n)$.

Expressed in terms of the membership functions of A and $q(f^{-1}(A))$, the generalized extension principle reduces the derivation of the TDS to the solution of the constrained maximization problem

$$\mu q(X_1,...,X_n)(v) = \sup_{(u_1,...,u_n)}(\mu_A(f(u_1,...,u_n)))$$

in which $u_1,...,u_n$ are constrained by

$$v = q(u_1,...,u_n)$$

The generalized extension principle is simpler than it appears. An illustration of its use is provided by the following example.

The IDS is:

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most Swedes are tall

The query is:

What is the average height of Swedes?

The explanatory database consists of a population of N Swedes, $Name_1, ...,$ $Name_N$. The database variables are $h_1, ..., h_N$, where h_i is the height of $Name_i$, and the grade of membership of $Name_i$ in *tall* is $\mu_{tall}(h_i)$, $i = 1, ..., n$.

The proportion of Swedes who are tall is given by

$$\frac{1}{N} \sum_{i=1}^{N} \mu_{tall}(h_i)$$

from which it follows that the constraint on the database variables induced by the IDS is

$$\frac{1}{N} \sum_{i=1}^{N} \mu_{tall}(h_i) \text{ is most}$$

In terms of the database variables, $h_1, ..., h_N$, the average height of Swedes is given by

$$h_{ave} = \frac{1}{N} \sum_{i=1}^{N} h_i$$

Since the IDS is a fuzzy proposition, h_{ave} is a fuzzy set whose determination reduces to the constrained maximization problem

$$\mu_{h_{ave}}(v) = \sup_{h_1,...,h_N} \left(\mu_{most} \left(\frac{1}{N} \sum_{i=1}^{N} \mu_{tall}(h_i) \right) \right)$$

subject to the constraint

$$v = \frac{1}{N} \sum_{i=1}^{N} h_i$$

It is possible that approximate solutions to problems of this type might be obtainable through the use of genetic-algorithm-based methods.

A key point which is brought out by this example and the preceding discussion is that explicitation and constraint propagation play pivotal roles in CW. What is important to recognize is that there is a great deal of computing with numbers in CW which takes place behind a curtain, unseen by the user. Thus, what matters is that in CW the IDS is allowed to be expressed in a natural language. No other methodology offers this facility. As an illustration of this point, consider the following problem.

A box contains ten balls of various sizes of which several are large and a few are small. What is the probability that a ball drawn at random is neither large nor small?

To be able to answer this question it is necessary to be able to define the meanings of *large, small, several large balls* and *neither large nor small*. This is a problem in semantics which falls outside of probability theory, neurocomputing and other methodologies.

There are two observations which are in order. First, in using fuzzy constraint propagation rules in computing with words, application of the extension principle reduces, in general, to the solution of a nonlinear program. What we need - and do not have at present - are approximate methods of solving such programs which are capable of exploiting the tolerance for imprecision. Without such methods, the cost of solutions may be excessive in relation to the imprecision which is intrinsic in the use of words. In this connection, an intriguing possibility is to use genetic-algorithms-based methods to arrive at approximate solutions to constrained maximization problems.

Second, given a collection of premises expressed in a natural language, we can in principle express them in their canonical forms and thereby explicitate the implicit fuzzy constraints. For this purpose, we have to employ test-score semantics. However, in test-score semantics we do not have at present effective algorithms for the derivation of canonical forms without human intervention. This is a problem that remains to be addressed.

5. Concluding remarks

The main purpose of this note is to draw attention to the centrality of the role of fuzzy logic in computing with words and vice-versa.

In coming years, computing with words is likely to emerge as a major field in its own right. In a reversal of long-standing attitudes, the use of words in place of numbers is destined to gain respectability. This is certain to happen because it is becoming abundantly clear that in dealing with real-world problems there is much to be gained by exploiting the tolerance for imprecision. In the final analysis, it is the exploitation of the tolerance for imprecision that is the prime motivation for CW. The role model for CW is the human mind.

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