Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
www.ibspan.waw.pl

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The intuitionistic fuzzy sets as constructive objects

Krassimir Atanassov
Dept. of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria
krat@bas.bg

Dedicated to 90-th anniversary of Prof. Lotfi Zadeh

Abstract

It is shown that the intuitionistic fuzzy sets and in particular – the fuzzy sets, can be interpreted as constructive objects. A criterion for constructiveness is introduced. It is proved that all operations and operator over intuitionistic fuzzy sets are also constructive objects.

Keywords: constructivism, intuitionistic fuzzy set, operation, operator.

1 Introduction

I will start with a recollection of my first acquaintance with Prof. Lotfi Zadeh. It was in 2001 in Villa Real, Portugal, where Prof. Pedro Melo-Pinto organized a school on fuzzy sets. Prof. Zadeh was invited for a 3-hour lecture, which he concluded with presentation of slides with articles by Samuel Kleene, Kurt Gödel and other luminaries of mathematical logic, who have written against the fuzzy sets. The fact that the sublime mathematician and logician Gödel had sometimes made slips in his judgments can be confirmed by the cosmologists, yet I was astonished

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by his opinion. Of course, nowadays, when we are aware of the enormous number of publication in the field of fuzzy sets, as well as of the various impressive applications of these, it is easy to say that Gödel had mistaken. However, I have been long tormented by the question why these mathematicians had opposed the fuzzy sets while they did not have anything against the three- and multi-valued logics of Jan Lukasiewicz. Thus I reached the conclusion that the reason for the then negative attitude towards fuzzy sets was hidden in the presence of the $[0, 1]$ interval as the set of the fuzzy sets' membership function (see, e.g, [4, 10, 11]). Indeed, the values of the membership function do belong to the $[0, 1]$ interval, yet it does not mean that this function obtains all possible values in this interval! If an expert or a group of experts evaluated, for instance, the chances of a political party to win the elections, it is slightly ever probable (if not absurd) for them to use estimations like $\frac{1}{e}$ or $\sqrt{2} - 1$. For any unbiased man it is clear that the experts would not use anything more complex than decimal fractions with one or two digits after the decimal point, i.e. rational numbers. Rational and even integer numbers are those which we use to measure the sizes of objects, the daily temperature or the speed of the vehicles (like locomotives), which are often described by fuzzy sets. Yes, the contemporary mathematics is the mathematics of multiple integrals, topological spaces, arithmetic functions, yet all these objects are abstractions of objects, existing in reality, which in the end of the day are measured, i.e. certain mathematical estimations are constructed for them, hence these estimations are constructive! Fuzzy sets, which use far from trivial mathematical apparatus, in general are based on constructive objects. It is well known that these sets contain ordered pairs (in the case of intuitionistic fuzzy sets, ordered triplets; see [1]), whose first component is a constructive object, hence following Per Matrin-Löf [5], it constitutes a finite configuration of symbols. As we saw, the following (and eventually the third) component is also a constructive object.

It follows that the fuzzy and intuitionistic fuzzy sets are constructive objects. Therefore, anyone who had been in trouble that the fuzzy sets introduced deconstructivism in science, had no occasion for fear. However, were they right to like the multivalued logics? Let us fix a certain natural number, say 11, and let us use an 11-valued scale for estimation of the statement that the political party P would win 30% of the election votes. According to the 11-valued scale, this estimation will be given the form $\frac{3}{10}$, which is perfectly correct. But what shall we do, if we have chosen a 6-valued scale, without knowing in advance that the estimation would be 30%? Now, for our estimation we have to choose between the values $\frac{1}{5}$ and $\frac{2}{5}$ and in both cases we will allow mistake of $\frac{1}{10}$ which will hardly make us happy! In other words, the multivalued logics that use rational numbers, despite being obviously constructive objects, are not always appropriate.

On the other hand, it is clear that if we use an n-valued estimation scale, we may easily say that we work with a fuzzy set, because that scale would be a subset of the [0, 1] interval.

2 The Intuitionistic Fuzzy Sets as Constructive Objects

Let the universe E be a (recursively) enumerable set, i.e., its members are no more than the set of natural numbers, whose cardinality is marked by $\aleph_0$.

Here we will use the notation from [1, 2].

Let

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

be an IFS and let the values of its functions μ_A and ν_A be rational numbers, i.e. for each $x \in E$:

$$\mu_A(x) = \frac{p(x)}{q(x)}$$

and

$$\nu_A(x) = \frac{r(x)}{s(x)},$$

where $p(x), q(x), r(x), s(x)$ are natural numbers, $p(x), r(x) \geq 0$, $q(x), s(x) > 0$ and

$$0 \leq \frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} \leq 1.$$

We will juxtapose to A the (natural) number

$$range(A) = LCM(\{q(x), s(x)|x \in E\}) = \underset{x\in E}{LCM} \;(LCM(q(x), s(x))),$$

where LCM is the lowest common multiple of all natural numbers $q(x), s(x)$ for all $x \in E$.

We will call the number $range(A)$ a "range" of IFS A.

Now it is clear that the IFS A will be a constructive object (in the present case – Constructive IFS (CIFS)) if

$$range(A) < \infty$$

while, in the opposite case it will not be a constructive object, because some of its parameters will not be rational numbers.

If for every $x \in E$: $\mu_A(x), \nu_A(x) \in \{0, 1\}$, then we will put

$$range(A) = 1.$$

We see immediately that if A and B are two CIFSs, then the IFSs

$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\};$

$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}; vsv$

$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle \mid x \in E\};$

$A.B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)\rangle \mid x \in E\};$

$A@B = \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\rangle | x \in E\};$

$A|B = \{\langle \min(\mu_A(x), \nu_B(x)), \max(\mu_B(x), \nu_A(x))\rangle | x \in E\};$

$A - B = \{\langle x, \mu_{A-B}(x), \nu_{A-B}(x)\rangle | x \in E\};$

$A : B = \{\langle x, \mu_{A:B}(x), \nu_{A:B}(x)\rangle | x \in E\};$

$n.A = \{\langle x, 1 - (1 - \mu_A(x))^n, (\nu_A(x))^n\rangle \mid x \in E\};$

$A^n = \{\langle x, (\mu_A(x))^n, 1 - (1 - \nu_A(x))^n\rangle \mid x \in E\},$

where n is a natural number and

$$\mu_{A-B}(x) = \begin{cases} \dfrac{\mu_A(x) - \mu_B(x)}{1 - \mu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \\ & \text{and } \nu_B(x) > 0 \\ & \text{and } \nu_A(x)\pi_B(x) \leq \pi_A(x)\nu_B(x) \\ 0, & \text{otherwise} \end{cases},$$

$$\nu_{A-B}(x) = \begin{cases} \dfrac{\nu_A(x)}{\nu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \\ & \text{and } \nu_B(x) > 0 \\ & \text{and } \nu_A(x)\pi_B(x) \leq \pi_A(x)\nu_B(x) \\ 1, & \text{otherwise} \end{cases}$$

and

$$\mu_{A:B}(x) = \begin{cases} \dfrac{\mu_A(x)}{\mu_B(x)}, & \text{if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \\ & \text{and } \mu_B(x) > 0 \\ & \text{and } \mu_A(x)\pi_B(x) \leq \pi_A(x)\mu_B(x) \\ 0, & \text{otherwise} \end{cases},$$

$$\nu_{A:B}(x) = \begin{cases} \dfrac{\nu_A(x) - \nu_B(x)}{1 - \nu_B(x)}, & \text{if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \\ & \text{and } \mu_B(x) > 0 \\ & \text{and } \mu_A(x)\pi_B(x) \leq \pi_A(x)\mu_B(x) \\ 1, & \text{otherwise} \end{cases}$$

also are CIFSs. For them it will be valid the equality

$$range(A \cup B) = range(A \cap B) = range(A|B)$$

$$= LCM(range(A), range(B)).$$

$$range(A + B) = range(A.B) = range(A).range(B),$$

$$range(A@B) = LCM(range(A), range(B), 2),$$

$$range(A^n) = range(n.A) = (range(A))^n,$$

$$range(A - B)$$

$$= LCM(range(A), range(B), \underset{x \in E}{LCM}\ (r_B(x)), \underset{x \in E}{LCM}\ (q_B(x) - p_B(x))),$$

$$range(A : B)$$

$$= LCM(range(A), range(B), \underset{x \in E}{LCM}\ (p_B(x)), \underset{x \in E}{LCM}\ (s_B(x) - r_B(x))),$$

where for each $x \in E$:

$$\mu_B(x) = \frac{p_B(x)}{q_B(x)}$$

and

$$\nu_B(x) = \frac{r_B(x)}{s_B(x)}.$$

For the CIFS A the following equalities

$$range(\Box A) = range(\Diamond A) = range(C(A)) = range(I(A))$$

$$range(C_\mu(A)) = range(C_\nu(A)) = range(I_\mu(A)) = range(I_\nu(A))$$

$$range(C^*_\mu(A)) = range(I^*_\nu(A)) = range(A),$$

where

$$\begin{aligned}
&\Box A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in E\},\\
&\Diamond A = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\},\\
&C(A) = \{\langle x, K, L\rangle | x \in E\},\\
&I(A) = \{\langle x, k, l\rangle | x \in E\},\\
&C_\mu(A) = \{\langle x, K, \min(1-K, \nu_A(x))\rangle | x \in E\},\\
&C_\nu(A) = \{\langle x, \mu_A(x), L\rangle | x \in E\},\\
&I_\mu(A) = \{\langle x, k, \nu_A(x)\rangle | x \in E\},\\
&I_\nu(A) = \{\langle x, \min(1-l, \mu_A(x)), l\rangle | x \in E\},\\
&C^*_\mu(A) = \{\langle x, \min(K, 1-\nu_A(x)), \min(1-K, \nu_A(x))\rangle | x \in E\}\\
&I^*_\nu(A) = \{\langle x, \min(1-l, \mu_A(x)), \min(l, 1-\mu_A(x))\rangle | x \in E\},
\end{aligned}$$

where

$$K = \sup_{y \in E} \mu_A(y),$$

$$L = \inf_{y \in E} \nu_A(y),$$

$$k = \inf_{y \in E} \mu_A(y),$$

$$l = \sup_{y \in E} \nu_A(y)$$

hold.

Let $\alpha = \frac{a_1}{a_2}$ and $\beta = \frac{b_1}{b_2}$ be rational numbers for which $a_1, b_1 \geq 0$ and $a_2, b_2 > 0$. Then the IFSs

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\},$$

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}$$
where $\alpha + \beta \leq 1$,

$$G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$

$$H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},$$

$$H^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\},$$

$$J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$

$$J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\},$$

$$d_\alpha(A) = \{\langle x, \nu_A(x) + \alpha.\pi_A(x), \mu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\},$$

$$f_{\alpha,\beta}(A) = \{\langle x, \nu_A(x) + \alpha.\pi_A(x), \mu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},$$
where $\alpha + \beta \leq 1$,

$$g_{\alpha,\beta}(A) = \{\langle x, \alpha.\nu_A(x), \beta.\mu_A(x)\rangle | x \in E\},$$

$$h_{\alpha,\beta}(A) = \{\langle x, \alpha.\nu_A(x), \mu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},$$

$$h^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\nu_A(x), \mu_A(x) + \beta.(1 - \alpha.\nu_A(x) - \mu_A(x))\rangle | x \in E\},$$

$$j_{\alpha,\beta}(A) = \{\langle x, \nu_A(x) + \alpha.\pi_A(x), \beta.\mu_A(x)\rangle | x \in E\},$$

$$j^*_{\alpha,\beta}(A) = \{\langle x, \nu_A(x) + \alpha.(1 - \nu_A(x) - \beta.\mu_A(x)), \beta.\mu_A(x)\rangle$$

$$P_{\alpha,\beta}(A) = \{\langle x, \max(\alpha, \mu_A(x)), \min(\beta, \nu_A(x))\rangle | x \in E\},$$

$$Q_{\alpha,\beta}(A) = \{\langle x, \min(\alpha, \mu_A(x)), \max(\beta, \nu_A(x))\rangle | x \in E\}$$

are CIFSs and for them

$$range(D_\alpha(A)) = range(d_\alpha(A)) = LCM(a_2, range(A)),$$

$$range(F_{\alpha,\beta}(A)) = range(G_{\alpha,\beta}(A)) = range(H_{\alpha,\beta}(A)) = range(H^*_{\alpha,\beta}(A))$$
$$= range(J_{\alpha,\beta}(A)) = range(J^*_{\alpha,\beta}(A)) = range(f_{\alpha,\beta}(A)) = range(g_{\alpha,\beta}(A))$$
$$= range(h_{\alpha,\beta}(A)) = range(h^*_{\alpha,\beta}(A)) = range(j_{\alpha,\beta}(A)) = range(j^*_{\alpha,\beta}(A))$$
$$= range(P_{\alpha,\beta}(A)) = range(Q_{\alpha,\beta}(A)) = LCM(a_2, b_2, range(A)).$$

Let it be valid for $a, b, c, d, e, f \in [0, 1]$ and

$$a + e - e.f \leq 1,$$

$$b + d - b.c \leq 1$$

that $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$, $c = \frac{c_1}{c_2}$, $d = \frac{d_1}{d_2}$, $e = \frac{e_1}{e_2}$, $f = \frac{f_1}{f_2}$. Let a, b, c, d, e, f be rational numbers for which $a_1, b_1, c_1, d_1, e_1, f_1 \geq 0$ and $a_2, b_2, c_2, d_2, e_2, f_2 > 0$. Then, the IFSs

$$X_{a,b,c,d,e,f}(A) = \{\langle x, a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)),$$
$$d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle | x \in E\}$$

and

$$x_{a,b,c,d,e,f}(A) = \{\langle x, a.\nu_A(x) + b.(1 - \nu_A(x) - c.\mu_A(x)),$$
$$d.\mu_A(x) + e.(1 - f.\nu_A(x) - \mu_A(x))\rangle | x \in E\}$$

are CIFSs and for them

$$range(X_{a,b,c,d,e,f}(A)) = range(x_{a,b,c,d,e,f}(A))$$
$$= LCM(a_2, b_2, c_2, d_2, e_2, f_2, range(A)).$$

When A and B are CIFSs, then the IFSs

$$F_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle | \; x \in E\},$$

$$G_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_B(x).\nu_A(x)\rangle \; | \; x \in E\},$$

$$H_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle \; | \; x \in E\},$$

$$H^*_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_B(x).(1 - \mu_B(x).\mu_A(x) - \nu_A(x))\rangle | \; x \in E\},$$

$$J_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_B(x).\nu_A(x)\rangle \; | \; x \in E\},$$

$$J^*_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).(1 - \mu_A(x) - \nu_B(x).\nu_A(x)), \nu_B(x).\nu_A(x)\rangle | \; x \in E\},$$

$$f_B(A) = \{\langle x, \nu_A(x) + \mu_B(x).\pi_A(x), \mu_A(x) + \nu_B(x).\pi_A(x)\rangle | \; x \in E\},$$

$$g_B(A) = \{\langle x, \mu_B(x).\nu_A(x), \nu_B(x).\mu_A(x)\rangle \; | \; x \in E\},$$

$$h_B(A) = \{\langle x, \mu_B(x).\nu_A(x), \mu_A(x) + \nu_B(x).\pi_A(x)\rangle \; | \; x \in E\},$$

$$h^*_B(A) = \{\langle x, \mu_B(x).\nu_A(x), \mu_A(x) + \nu_B(x).(1 - \mu_B(x).\nu_A(x) - \nu_A(x))\rangle | \; x \in E\},$$

$$j_B(A) = \{\langle x, \nu_A(x) + \mu_B(x).\pi_A(x), \nu_B(x).\mu_A(x)\rangle \; | \; x \in E\},$$

$$j^*_B(A) = \{\langle x, \nu_A(x) + \mu_B(x).(1 - \nu_A(x) - \nu_B(x).\nu_A(x)), \nu_B(x).\mu_A(x)\rangle | \; x \in E\},$$

$$P_B(A) = \{\langle x, \max(\mu_B(x), \mu_A(x)), \min(\nu_B(x), \nu_A(x))\rangle \; | \; x \in E\},$$

$$Q_B(A) = \{\langle x, \min(\mu_B(x), \mu_A(x)), \max(\nu_B(x), \nu_A(x))\rangle \; | \; x \in E\}$$

are also CIFSs for which

$$range(F_B(A)) = range(G_B(A)) = range(H_B(A)) = range(H^*_B(A))$$

$$= range(J_B(A)) = range(J^*_B(A)) = range(f_B(A)) = range(g_B(A))$$

$$= range(h_B(A)) = range(h^*_B(A)) = range(j_B(A)) = range(j^*_B(A))$$

$$= range(P_B(A)) = range(Q_B(A)) = LCM(range(B), range(A)).$$

For the CIFS A, the IFSs

$$\neg_1 A = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\},$$

$$\neg_2 A = \{\langle x, \overline{sg}(\mu_A(x)), \mathrm{sg}(\mu_A(x))\rangle | x \in E\},$$

$$\neg_3 A = \{\langle x, \nu_A(x), \mu_A(x).\nu_A(x) + \mu_A(x)^2\rangle | x \in E\},$$

$$\neg_4 A = \{\langle x, \nu_A(x), 1 - \nu_A(x)\rangle | x \in E\},$$

$$\neg_5 A = \{\langle x, \overline{sg}(1 - \nu_A(x)), \mathrm{sg}(1 - \nu_A(x))\rangle | x \in E\},$$

$$\neg_6 A = \{\langle x, \overline{sg}(1 - \nu_A(x)), \mathrm{sg}(\mu_A(x))\rangle | x \in E\},$$

$$\neg_7 A = \{\langle x, \overline{sg}(1 - \nu_A(x)), \mu_A(x)\rangle | x \in E\},$$

$$\neg_8 A = \{\langle x, 1 - \mu_A(x), \mu_A(x)\rangle | x \in E\},$$

$$\neg_9 A = \{\langle x, \overline{sg}(\mu_A(x)), \mu_A(x)\rangle | x \in E\},$$

$$\neg_{10} A = \{\langle x, \overline{sg}(1 - \nu_A(x)), 1 - \nu_A(x)\rangle | x \in E\},$$

$$\neg_{11} A = \{\langle x, \mathrm{sg}(\nu_A(x)), \overline{sg}(\nu_A(x))\rangle | x \in E\},$$

$$\neg_{12} A = \{\langle x, \nu_A(x).(\nu_A(x) + \mu_A(x)),$$

$$\mu_A(x).(\nu_A(x)^2 + \mu_A(x) + \nu_A(x).\mu_A(x))\rangle | x \in E\},$$

$$\neg_{13} A = \{\langle x, \mathrm{sg}(1 - \mu_A(x)), \overline{sg}(1 - \mu_A(x))\rangle | x \in E\},$$

$$\neg_{14} A = \{\langle x, \mathrm{sg}(\nu_A(x)), \overline{sg}(1 - \mu_A(x))\rangle | x \in E\},$$

$$\neg_{15} A = \{\langle x, \overline{sg}(1 - \nu_A(x)), \overline{sg}(1 - \mu_A(x))\rangle | x \in E\},$$

$$\neg_{16} A = \{\langle x, \overline{sg}(\mu_A(x)), \overline{sg}(1 - \mu_A(x))\rangle | x \in E\},$$

$$\neg_{17} A = \{\langle x, \overline{sg}(1 - \nu_A(x)), \overline{sg}(\nu_A(x))\rangle | x \in E\},$$

$$\neg_{18} A = \{\langle x, \min(\nu_A(x), \mathrm{sg}(\mu_A(x))), \min(\mu_A(x), \mathrm{sg}(\nu_A(x)))\rangle | x \in E\},$$

$$\neg_{19} A = \{\langle x, \min(\nu_A(x), \mathrm{sg}(\mu_A(x))), 0\rangle | x \in E\},$$

$$\neg_{20}A = \{\langle x, \nu_A(x), 0\rangle | x \in E\},$$
$$\neg_{21}A = \{\langle x, \min(1-\mu_A(x), \text{sg}(\mu_A(x))), \min(\mu_A(x), \text{sg}(1-\mu_A(x)))\rangle | x \in E\},$$
$$\neg_{22}A = \{\langle x, \min((1-\mu_A(x)), \text{sg}(\mu_A(x))), 0\rangle | x \in E\},$$
$$\neg_{23}A = \{\langle x, 1-\mu_A(x), 0\rangle | x \in E\},$$
$$\neg_{24}A = \{\langle x, \min(\nu_A(x), \text{sg}(1-\nu_A(x))), \min(1-\nu_A(x), \text{sg}(\nu_A(x)))\rangle | x \in E\},$$
$$\neg_{25}A = \{\langle x, \min(\nu_A(x), \text{sg}(1-\nu_A(x))), 0\rangle | x \in E\},$$
$$\neg_{26}A = \{\langle x, \nu_A(x), \mu_A(x).\nu_A(x) + \overline{sg}(1-\mu_A(x))\rangle | x \in E\},$$
$$\neg_{27}A = \{\langle x, 1-\mu_A(x), \mu_A(x).(1-\mu_A(x)) + \overline{sg}(1-\mu_A(x))\rangle | x \in E\},$$
$$\neg_{28}A = \{\langle x, \nu_A(x), (1-\nu_A(x)).\nu_A(x) + \overline{sg}(\nu_A(x))\rangle | x \in E\},$$
$$\neg_{29}A = \{\langle x, \nu_A(x).\mu_A(x) + \overline{sg}(1-\nu_A(x)),$$
$$\mu_A(x).(\nu_A(x).\mu_A(x) + \overline{sg}(1-\nu_A(x))) + \overline{sg}(1-\mu_A(x))\rangle | x \in E\},$$
$$\neg_{30}A = \{\langle x, \nu_A(x).\mu_A(x), \mu_A(x).(\nu_A(x).\mu_A(x) + \overline{sg}(1-\nu_A(x)))$$
$$+\overline{sg}(1-\mu_A(x))\rangle | x \in E\},$$
$$\neg_{31}A = \{\langle x, (1-\mu_A(x)).\mu_A(x) + \overline{sg}(\mu_A(x)), \mu_A(x).((1-\mu_A(x)).\mu_A(x)$$
$$+\overline{sg}(\mu_A(x))) + \overline{sg}(1-\mu_A(x))\rangle | x \in E\},$$
$$\neg_{32}A = \{\langle x, (1-\mu_A(x)).\mu_A(x), \mu_A(x).((1-\mu_A(x)).\mu_A(x) + \overline{sg}(\mu_A(x)))$$
$$+\overline{sg}(1-\mu_A(x))\rangle | x \in E\},$$
$$\neg_{33}A = \{\langle x, \nu_A(x).(1-\nu_A(x)) + \overline{sg}(1-\nu_A(x)), (1-\nu_A(x)).(\nu_A(x).((1-\nu_A(x))$$
$$+\overline{sg}(1-\nu_A(x)) + \overline{sg}(\nu_A(x))\rangle | x \in E\},$$
$$\neg_{34}A = \{\langle x, \nu_A(x).(1-\nu_A(x)), (1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x)) + \overline{sg}(1-\nu_A(x)))$$
$$+\overline{sg}(\nu_A(x))\rangle | x \in E\},$$

are CIFSs for which

$$range(\neg_2 A) = range(\neg_5 A) = range(\neg_6 A) = range(\neg_7 A) = range(\neg_{13} A)$$
$$= range(\neg_{14} A) = range(\neg_{15} A) = range(\neg_{16} A) = range(\neg_{17} A) = 1,$$
$$range(\neg_1 A) = range(\neg_4 A) = range(\neg_7 A) = range(\neg_8 A) = range(\neg_9 A)$$
$$= range(\neg_{10} A) = range(\neg_{11} A) = range(\neg_{18} A) = range(\neg_{19} A)$$
$$= range(\neg_{20} A) = range(\neg_{21} A) = range(\neg_{22} A) = range(\neg_{23} A)$$

$$= range(\neg_{24} A) = range(\neg_{25} A) = range(A),$$

$$range(\neg_3 A) = range(\neg_{26} A) = range(\neg_{27} A) = range(\neg_{28} A) = range(A)^2,$$

$$range(\neg_{12} A) = range(\neg_{29} A) = range(\neg_{30} A) = range(\neg_{31} A)$$

$$= range(\neg_{32} A) = range(\neg_{33} A) = range(\neg_{34} A) = range(A)^3.$$

When A and B are CIFSs, then the IFSs

$\rightarrow_1$	$\{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))),$ $\min(\mu_A(x), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_2$	$\{\langle x, \overline{sg}(\mu_A(x) - \mu_B(x)), \nu_B(x).\text{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_3$	$\{\langle x, 1 - (1 - \mu_B(x)).\text{sg}(\mu_A(x) - \mu_B(x)))),$ $\nu_B(x).\text{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_4$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_5$	$\{\langle x, \min(1, \nu_A(x) + \mu_B(x)),$ $\max(0, \mu_A(x) + \nu_B(x) - 1)\rangle \| x \in E\}$
$\rightarrow_6$	$\{\langle x, \nu_A(x) + \mu_A(x)\mu_B(x), \mu_A(x)\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_7$	$\{\langle x, \min(\max(\nu_A(x), \mu_B(x)), \max(\mu_A(x), \nu_A(x)),$ $\max(\mu_B(x), \nu_B(x))),\ \max(\min(\mu_A(x), \nu_B(x)),$ $\min(\mu_A(x), \nu_A(x)), \min(\mu_B(x), \nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_8$	$\{\langle x, 1 - (1 - \min(\nu_A(x), \mu_B(x))).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\max(\mu_A(x), \nu_B(x)).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\text{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$
$\rightarrow_9$	$\{\langle x, \nu_A(x) + \mu_A(x)^2\mu_B(x), \mu_A(x)\nu_A(x) + \mu_A(x)^2\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_{10}$	$\{\langle x, \mu_B(x).\overline{sg}(1 - \mu_A(x))$ $+\text{sg}(1 - \mu_A(x)).(\overline{sg}(1 - \mu_B(x)) + \nu_A(x).\text{sg}(1 - \mu_B(x))),$ $\nu_B(x).\overline{sg}(1 - \mu_A(x)) + \mu_A(x).\text{sg}(1 - \mu_A(x))$ $.\text{sg}(1 - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{11}$	$\{\langle x, 1 - (1 - \mu_B(x)).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\nu_B(x).\text{sg}(\mu_A(x) - \mu_B(x)).\text{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{12}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), 1 - \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{13}$	$\{\langle x, \nu_A(x) + \mu_B(x) - \nu_A(x).\mu_B(x), \mu_A(x).\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_{14}$	$\{\langle x, 1 - (1 - \mu_B(x)).\text{sg}(\mu_A(x) - \mu_B(x))$ $-\nu_B(x).\overline{sg}(\mu_A(x) - \mu_B(x)).\text{sg}(\nu_B(x) - \nu_A(x)),$ $\nu_B(x).\text{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{15}$	$\{\langle x, 1 - \text{sg}(\mu_A(x) - \mu_B(x)).\text{sg}(\nu_B(x) - \nu_A(x))$ $\text{sg}(\overline{sg}(\mu_A(x) - \mu_B(x)) + \overline{sg}(\nu_B(x) - \nu_A(x)))$
$\rightarrow_{16}$	$\{\langle x, \max(\overline{sg}(\mu_A(x)), \mu_B(x)), \min(\text{sg}(\mu_A(x)), \nu_B(x))\rangle \| x \in E\}$

$\rightarrow_{17}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)),$ $\min(\mu_A(x).\nu_A(x)+\mu_A(x)^2, \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{18}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), \min(1-\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{19}$	$\{\langle x, \max(1-\text{sg}(\text{sg}(\mu_A(x))+\text{sg}(1-\nu_A(x))), \mu_B(x)),$ $\min(\text{sg}(1-\nu_A(x)), \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{20}$	$\{\langle x, \max(\overline{sg}(\mu_A(x)), \text{sg}(\mu_B(x))), \min(\text{sg}(\mu_A(x)), \overline{sg}(\mu_B(x)))\rangle$ $\mid x \in E\}$
$\rightarrow_{21}$	$\{\langle x, \max(\nu_A(x), \mu_B(x).(\mu_B(x)+\nu_B(x))),$ $\min(\mu_A(x).(\mu_A(x)+\nu_A(x)), \nu_B(x).$ $(\mu_B(x)^2+\nu_B(x)+\mu_B(x).\nu_B(x)))\rangle \mid x \in E\}$
$\rightarrow_{22}$	$\{\langle x, \max(\nu_A(x), 1-\nu_B(x)), \min(1-\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{23}$	$\{\langle x, 1-\min(\text{sg}(1-\nu_A(x)), \overline{sg}(1-\nu_B(x))),$ $\min(\text{sg}(1-\nu_A(x)), \overline{sg}(1-\nu_B(x)))\rangle \mid x \in E\}$
$\rightarrow_{24}$	$\{\langle x, \overline{sg}(\mu_A(x)-\mu_B(x)).\overline{sg}(\nu_B(x)-\nu_A(x)),$ $\text{sg}(\mu_A(x)-\mu_B(x)).\text{sg}(\nu_B(x)-\nu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{25}$	$\{\langle x, \max(\nu_A(x), \overline{sg}(\mu_A(x)).\overline{sg}(1-\nu_A(x)),$ $\mu_B(x).\overline{sg}(\nu_B(x)).\overline{sg}(1-\mu_B(x))),$ $\min(\mu_A(x), \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{26}$	$\{\langle x, \max(\overline{sg}(1-\nu_A(x)), \mu_B(x)),$ $\min(\text{sg}(\mu_A(x)), \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{27}$	$\{\langle x, \max(\overline{sg}(1-\nu_A(x)), \text{sg}(\mu_B(x))),$ $\min(\text{sg}(\mu_A(x)), \overline{sg}(1-\nu_B(x)))\rangle \mid x \in E\}$
$\rightarrow_{28}$	$\{\langle x, \max(\overline{sg}(1-\nu_A(x)), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{29}$	$\{\langle x, \max(\overline{sg}(1-\nu_A(x)), \overline{sg}(1-\mu_B(x))),$ $\min(\mu_A(x), \overline{sg}(1-\nu_B(x)))\rangle \mid x \in E\}$
$\rightarrow_{30}$	$\{\langle x, \max(1-\mu_A(x), \min(\mu_A(x), 1-\nu_B(x))),$ $\min(\mu_A(x), \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{31}$	$\{\langle x, \overline{sg}(\mu_A(x)+\nu_B(x)-1), \nu_B(x).\text{sg}(\mu_A(x)+\nu_B(x)-1)\rangle \mid x \in E\}$
$\rightarrow_{32}$	$\{\langle x, 1-\nu_B(x).\text{sg}(\mu_A(x)+\nu_B(x)-1),$ $\nu_B(x).\text{sg}(\mu_A(x)+\nu_B(x)-1)\rangle \mid x \in E\}$
$\rightarrow_{33}$	$\{\langle x, 1-\min(\mu_A(x), \nu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \mid x \in E\}$
$\rightarrow_{34}$	$\{\langle x, \min(1, 2-\mu_A(x)-\nu_B(x)),$ $\max(0, \mu_A(x)+\nu_B(x)-1)\rangle \mid x \in E\}$
$\rightarrow_{35}$	$\{\langle x, 1-\mu_A(x).\nu_B(x), \mu_A(x).\nu_B(x)\rangle \mid x \in E\}$
$\rightarrow_{36}$	$\{\langle x, \min(1-\min(\mu_A(x), \nu_B(x)),$ $\max(\mu_A(x), (1-\mu_A(x)), \max(1-\nu_B(x), \nu_B(x))),$ $\max(\min(\mu_A(x), \nu_B(x)), \min(\mu_A(x), 1-\mu_A(x)),$ $\min(1-\nu_B(x), \nu_B(x)))\rangle \mid x \in E\}$

$\rightarrow_{37}$	$\{\langle x, 1-\max(\mu_A(x),\nu_B(x)).\text{sg}(\mu_A(x)+\nu_B(x)-1),$ $\max(\mu_A(x),\nu_B(x)).\text{sg}(\mu_A(x)+\nu_B(x)-1)\rangle \| x\in E\}$
$\rightarrow_{38}$	$\{\langle x, 1-\mu_A(x)+(\mu_A(x)^2.(1-\nu_B(x))),$ $\mu_A(x).(1-\mu_A(x))+\mu_A(x)^2.\nu_B(x)\rangle \| x\in E\}$
$\rightarrow_{39}$	$\{\langle x, (1-\nu_B(x)).\overline{sg}(1-\mu_A(x))$ $+\text{sg}(1-\mu_A(x)).(\overline{sg}(\nu_B(x))+(1-\mu_A(x)).\text{sg}(\nu_B(x))),$ $\nu_B(x).\overline{sg}(1-\mu_A(x))+\mu_A(x).\text{sg}(1-\mu_A(x))$ $.\text{sg}(\nu_B(x))\rangle \| x\in E\}$
$\rightarrow_{40}$	$\{\langle x, 1-\text{sg}(\mu_A(x)+\nu_B(x)-1), 1-\overline{sg}(\mu_A(x)+\nu_B(x)-1)\rangle \| x\in E\}$
$\rightarrow_{41}$	$\{\langle x, \max(\overline{sg}(\mu_A(x)),(1-\nu_B(x))),$ $\min(\text{sg}(\mu_A(x)),\nu_B(x))\rangle \| x\in E\}$
$\rightarrow_{42}$	$\{\langle x, \max(\overline{sg}(\mu_A(x)),\text{sg}(1-\nu_B(x))),$ $\min(\text{sg}(\mu_A(x)),\overline{sg}(1-\nu_B(x)))\rangle \| x\in E\}$
$\rightarrow_{43}$	$\{\langle x, \max(\overline{sg}(\mu_A(x)),$ $1-\nu_B(x)),\min(\text{sg}(\mu_A(x)),\nu_B(x))\rangle \| x\in E\}$
$\rightarrow_{44}$	$\{\langle x, \max(\overline{sg}(\mu_A(x)),1-\nu_B(x)),\min(\mu_A(x),\nu_B(x))\rangle \| x\in E\}$
$\rightarrow_{45}$	$\{\langle x, \max(\overline{sg}(\mu_A(x)),\overline{sg}(\nu_B(x))),$ $\min(\mu_A(x),\overline{sg}(1-\nu_B(x)))\rangle \| x\in E\}$
$\rightarrow_{46}$	$\{\langle x, \max(\nu_A(x),\min(1-\nu_A(x),\mu_B(x))),$ $1-\max(\nu_A(x),\mu_B(x))\rangle \| x\in E\}$
$\rightarrow_{47}$	$\{\langle x, \overline{sg}(1-\nu_A(x)-\mu_B(x)),$ $(1-\mu_B(x)).\text{sg}(1-\nu_A(x)-\mu_B(x))\rangle \| x\in E\}$
$\rightarrow_{48}$	$\{\langle x, 1-(1-\mu_B(x)).\text{sg}(1-\nu_A(x)-\mu_B(x)),$ $(1-\mu_B(x)).\text{sg}(1-\nu_A(x)-\mu_B(x))\rangle \| x\in E\}$
$\rightarrow_{49}$	$\{\langle x, \min(1,\nu_A(x)+\mu_B(x)),$ $\max(0,1-\nu_A(x)-\mu_B(x))\rangle \| x\in E\}$
$\rightarrow_{50}$	$\{\langle x, \nu_A(x)+\mu_B(x)-\nu_A(x).\mu_B(x),$ $1-\nu_A(x)-\mu_B(x)+\nu_A(x).\mu_B(x)\rangle \| x\in E\}$
$\rightarrow_{51}$	$\{\langle x, \min(\max(\nu_A(x),\mu_B(x)),$ $\max(1-\nu_A(x),\nu_A(x)),\max(\mu_B(x),1-\mu_B(x))),$ $\max(1-\max(\nu_A(x),\mu_B(x)),\min(1-\nu_A(x),\nu_A(x)),$ $\min(\mu_B(x),1-\mu_B(x)))\rangle \| x\in E\}$
$\rightarrow_{52}$	$\{\langle x, 1-(1-\min(\nu_A(x),\mu_B(x))).\text{sg}(1-\nu_A(x)-\mu_B(x)),$ $1-\min(\nu_A(x),\mu_B(x)).\text{sg}(1-\nu_A(x)-\mu_B(x))\rangle \| x\in E\}$
$\rightarrow_{53}$	$\{\langle x, \nu_A(x)+(1-\nu_A(x))^2.\mu_B(x)),$ $(1-\nu_A(x)).\nu_A(x)+(1-\nu_A(x))^2.(1-\mu_B(x))\rangle \| x\in E\}$

$\rightarrow_{54}$	$\{\langle x, \mu_B(x).\overline{sg}(\nu_A(x))$ $+\text{sg}(\nu_A(x)).(\overline{sg}(1-\mu_B(x)) + \nu_A(x).\text{sg}(1-\mu_B(x))),$ $(1-\mu_B(x)).\overline{sg}(\nu_A(x)) + (1-\nu_A(x))$ $.\text{sg}(\nu_A(x)).\text{sg}(1-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{55}$	$\{\langle x, 1 - \text{sg}(1-\nu_A(x)-\mu_B(x)),$ $1 - \overline{sg}(1 - \nu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{56}$	$\{\langle x, \max(\overline{sg}(1 - \nu_A(x)), \mu_B(x)),$ $\min(\text{sg}(1 - \nu_A(x)), 1 - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{57}$	$\{\langle x, \max(\overline{sg}(1 - \nu_A(x)), \text{sg}(\mu_B(x))),$ $\min(\text{sg}(1 - \nu_A(x)), \overline{sg}(\mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{58}$	$\{\langle x, \max(\overline{sg}(1 - \nu_A(x)), \overline{sg}(1 - \mu_B(x))),$ $1 - \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{59}$	$\{\langle x, \max(\overline{sg}(1 - \nu_A(x)), \mu_B(x)),$ $(1 - \max(\nu_A(x), \mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{60}$	$\{\langle x, \max(\overline{sg}(1 - \nu_A(x)), \overline{sg}(1 - \mu_B(x))),$ $\min(1 - \nu_A(x), \overline{sg}(\mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{61}$	$\{\langle x, \max(\mu_B(x), \min(\nu_B(x), \nu_A(x))),$ $\min(\nu_B(x), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{62}$	$\{\langle x, \overline{sg}(\nu_B(x)-\nu_A(x)), \mu_A(x).\text{sg}(\nu_B(x)-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{63}$	$\{\langle x, 1 - (1 - \nu_A(x)).\text{sg}(\nu_B(x) - \nu_A(x)),$ $\mu_A(x).\text{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{64}$	$\{\langle x, \mu_B(x) + \nu_B(x).\nu_A(x), \nu_B(x).\mu_A(x)\rangle \| x \in E\}$
$\rightarrow_{65}$	$\{\langle x, 1-(1-\min(\mu_B(x), \nu_A(x))).\text{sg}(\nu_B(x)-\nu_A(x)),$ $\max(\nu_B(x), \mu_A(x)).\text{sg}(\nu_B(x)-\nu_A(x))$ $.\text{sg}(\mu_A(x)-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{66}$	$\{\langle x, \mu_B(x) + \nu_B(x)^2.\nu_A(x),$ $\nu_B(x).\mu_B(x) + \nu_B(x)^2.\mu_A(x)\rangle \| x \in E\}$
$\rightarrow_{67}$	$\{\langle x, \nu_A(x).\overline{sg}(1-\nu_B(x)) + \text{sg}(1-\nu_B(x)).(\overline{sg}(1-\nu_A(x))$ $+\mu_B(x).\text{sg}(1 - \nu_A(x))),$ $\mu_A(x).\overline{sg}(1 - \nu_B(x)) + \nu_B(x).\text{sg}(1 - \nu_B(x))$ $.\text{sg}(1 - \nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{68}$	$\{\langle x, 1 - (1 - \nu_A(x)).\text{sg}(\nu_B(x) - \nu_A(x)),$ $\mu_A(x).\text{sg}(\nu_B(x) - \nu_A(x)).\text{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{69}$	$\{\langle x, 1 - (1 - \nu_A(x)).\text{sg}(\nu_B(x) - \nu_A(x))$ $-\mu_A(x).\overline{sg}(\nu_B(x) - \nu_A(x)).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\mu_A(x).\text{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{70}$	$\{\langle x, \max(\overline{sg}(\nu_B(x)), \nu_A(x)),$ $\min(\text{sg}(\nu_B(x)), \mu_A(x))\rangle \| x \in E\}$

$\rightarrow_{71}$	$\{\langle x, \max(\mu_B(x), \nu_A(x)),$ $\min(\nu_B(x).\mu_B(x)+\nu_B(x)^2, \mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{72}$	$\{\langle x, \max(\mu_B(x), \nu_A(x)),$ $\min(1-\mu_B(x), \mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{73}$	$\{\langle x, \max(1-\max(\text{sg}(\nu_B(x)), \text{sg}(1-\mu_B(x))), \nu_A(x)),$ $\min(\text{sg}(1-\mu_B(x)), \mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{74}$	$\{\langle x, \max(\overline{sg}(\nu_B(x)), \text{sg}(\nu_A(x))),$ $\min(\text{sg}(\nu_B(x)), \overline{sg}(\nu_A(x)))\rangle \mid x \in E\}$
$\rightarrow_{75}$	$\{\langle x, \max(\mu_B(x), \nu_A(x).(\nu_A(x)+\mu_A(x))),$ $\min(\nu_B(x).(\nu_B(x)+\mu_B(x)), \mu_A(x).(\nu_A(x)^2+\mu_A(x))$ $+\nu_A(x).\mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{76}$	$\{\langle x, \max(\mu_B(x), 1-\mu_A(x)), \min(1-\mu_B(x), \mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{77}$	$\{\langle x, (1-\min(\text{sg}(1-\mu_B(x)), \overline{sg}(1-\mu_A(x)))),$ $\min(\text{sg}(1-\mu_B(x)), \overline{sg}(1-\mu_A(x)))\rangle \mid x \in E\}$
$\rightarrow_{78}$	$\{\langle x, \max(\overline{sg}(1-\mu_B(x)), \nu_A(x)),$ $\min(\text{sg}(\nu_B(x)), \mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{79}$	$\{\langle x, \max(\overline{sg}(1-\mu_B(x)), \text{sg}(\nu_A(x))),$ $\min(\text{sg}(\nu_B(x)), \overline{sg}(1-\mu_A(x)))\rangle \mid x \in E\}$
$\rightarrow_{80}$	$\{\langle x, \max(\overline{sg}(1-\mu_B(x)), \nu_A(x)), \min(\nu_B(x), \mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{81}$	$\{\langle x, \max(\overline{sg}(1-\mu_B(x)), \overline{sg}(1-\nu_A(x))),$ $\min(\nu_B(x), \overline{sg}(1-\mu_A(x)))\rangle \mid x \in E\}$
$\rightarrow_{82}$	$\{\langle x, \max(1-\nu_B(x), \min(\nu_B(x), 1-\mu_A(x))),$ $\min(\nu_B(x), \mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{83}$	$\{\langle x, \overline{sg}(\nu_B(x)+\mu_A(x)-1),$ $\mu_A(x).\text{sg}(\nu_B(x)+\mu_A(x)-1)\rangle \mid x \in E\}$
$\rightarrow_{84}$	$\{\langle x, 1-\mu_A(x).\text{sg}(\nu_B(x)+\mu_A(x)+1),$ $\mu_A(x).\text{sg}(\nu_B(x)+\mu_A(x)+1)\rangle \mid x \in E\}$
$\rightarrow_{85}$	$\{\langle x, 1-\nu_B(x)+\nu_B(x)^2.(1-\mu_A(x)),$ $\nu_B(x).(1-\nu_B(x))+\nu_B(x)^2\rangle \mid x \in E\}$
$\rightarrow_{86}$	$\{\langle x, (1-\mu_A(x)).\overline{sg}(1-\nu_B(x))$ $+\text{sg}(1-\nu_B(x)).\overline{sg}(\mu_A(x)+\min(1-\nu_B(x), \text{sg}(\mu_A(x)))),$ $\mu_A(x).\overline{sg}(1-\nu_B(x))+\nu_B(x).\text{sg}(1-\nu_B(x))$ $.\text{sg}(\mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{87}$	$\{\langle x, \max(\overline{sg}(\nu_B(x)), 1-\mu_A(x)),$ $\min(\text{sg}(\nu_B(x)), \mu_A(x))\rangle \mid x \in E\}$
$\rightarrow_{88}$	$\{\langle x, \max(\overline{sg}(\nu_B(x)), \text{sg}(1-\mu_A(x))),$ $\min(\text{sg}(\nu_B(x)), \overline{sg}(1-\mu_A(x)))\rangle \mid x \in E\}$
$\rightarrow_{89}$	$\{\langle x, \max(\overline{sg}(\nu_B(x)), 1-\mu_A(x)), \min(\nu_B(x), \mu_A(x))\rangle \mid x \in E\}$

$\rightarrow_{90}$	$\{\langle x, \max(\overline{sg}(\nu_B(x)), \overline{sg}(\mu_A(x))),$ $\min(\nu_B(x), \overline{sg}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{91}$	$\{\langle x, \max(\mu_B(x), \min(1-\mu_B(x), \nu_A(x)),$ $1-\max(\mu_B(x), \nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{92}$	$\{\langle x, \overline{sg}(1-\mu_B(x)-\nu_A(x)),$ $\min(1-\nu_A(x), \text{sg}(1-\mu_B(x)-\nu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{93}$	$\{\langle x, 1-\min(1-\nu_A(x), \text{sg}(1-\mu_B(x)-\nu_A(x))),$ $\min(1-\nu_A(x), \text{sg}(1-\mu_B(x)-\nu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{94}$	$\{\langle x, \mu_B(x)+(1-\mu_B(x))^2.\nu_A(x),$ $(1-\mu_B(x)).\mu_B(x)+(1-\mu_B(x))^2.(1-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{95}$	$\{\langle x, \min(\nu_A(x), \overline{sg}(\mu_B(x)))+(\text{sg}(\mu_B(x)).(\overline{sg}(1-\nu_A(x))$ $+\min(\mu_B(x), \text{sg}(1-\nu_A(x))))),$ $(\min(1-\nu_A(x), \overline{sg}(\mu_B(x)))+\min(\min(1-\mu_B(x),$ $\text{sg}(\mu_B(x))), \text{sg}(1-\nu_A(x))))\rangle \| x \in E\}$
$\rightarrow_{96}$	$\{\langle x, \max(\overline{sg}(1-\mu_B(x)), \nu_A(x)),$ $\min(\text{sg}(1-\mu_B(x)), 1-\nu_A(x))\rangle \| x \in E\}$
$\rightarrow_{97}$	$\{\langle x, \max(\overline{sg}(1-\mu_B(x)), \text{sg}(\nu_A(x))),$ $\min(\text{sg}(1-\mu_B(x)), \overline{sg}(\nu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{98}$	$\{\langle x, \max(\overline{sg}(1-\mu_B(x)), \nu_A(x)),$ $(1-\max(\mu_B(x), \nu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{99}$	$\{\langle x, \max(\overline{sg}(1-\mu_B(x)), \overline{sg}(1-\nu_A(x))),$ $\min(1-\mu_B(x), \overline{sg}(\nu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{100}$	$\{\langle x, \max(\min(\nu_A(x), \text{sg}(\mu_A(x))), \mu_B(x)),$ $\min(\min(\mu_A(x), \text{sg}(\nu_A(x))), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{101}$	$\{\langle x, \max(\min(\nu_A(x), \text{sg}(\mu_A(x))), \min(\mu_B(x), \text{sg}(\nu_B(x)))),$ $\min(\min(\mu_A(x), \text{sg}(\nu_A(x))), \min(\nu_B(x),$ $\text{sg}(\mu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{102}$	$\{\langle x, \max(\nu_A(x), \min(\mu_B(x), \text{sg}(\nu_B(x)))),$ $\min(\mu_A(x), \min(\nu_B(x), \text{sg}(\mu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{103}$	$\{\langle x, \max(\min(1-\mu_A(x), \text{sg}(\mu_A(x))), (-\nu_B(x)),$ $\min(\min(\mu_A(x), \text{sg}(1-\mu_A(x))), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{104}$	$\{\langle x, \max(\min(1-\mu_A(x), \text{sg}(\mu_A(x))),$ $\min(1-\nu_B(x), \text{sg}(\nu_B(x)))),$ $\min(\min(\mu_A(x), \text{sg}(1-\mu_A(x))),$ $\min(\nu_B(x), \text{sg}(1-\nu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{105}$	$\{\langle x, \max(1-\mu_A(x), \min(1-\nu_B(x), \text{sg}(\nu_B(x)))),$ $\min(\mu_A(x), \min(\nu_B(x), \text{sg}(1-\nu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{106}$	$\{\langle x, \max(\min(\nu_A(x), \text{sg}(1-\nu_A(x))), \mu_B(x)),$ $\min(\min(1-\nu_A(x), \text{sg}(\nu_A(x))), 1-\mu_B(x))\rangle \| x \in E\}$

$\rightarrow_{107}$	$\{\langle x, \max(\min(\nu_A(x), \mathrm{sg}(1-\nu_A(x))),$ $\min(\mu_B(x), \mathrm{sg}(1-\mu_B(x)))),$ $\min(\min(1-\nu_A(x), \mathrm{sg}(\nu_A(x))),$ $\min(1-\mu_B(x), \mathrm{sg}(\mu_B(x)))))\rangle \| x \in E\}$
$\rightarrow_{108}$	$\{\langle x, \max(\nu_A(x), \min(\mu_B(x), \mathrm{sg}(1-\mu_B(x)))),$ $\min(1-\nu_A(x), \min(1-\mu_B(x), \mathrm{sg}(\mu_B(x)))))\rangle \| x \in E\}$
$\rightarrow_{109}$	$\{\langle x, \nu_A(x) + \min(\overline{sg}(1-\mu_A(x)), \mu_B(x)),$ $\mu_A(x).\nu_A(x) + \min(\overline{sg}(1-\mu_A(x)), \nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{110}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)),$ $\min(\mu_A(x).\nu_A(x) + \overline{sg}(1-\mu_A(x)), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{111}$	$\{\langle x, \max(\nu_A(x), \mu_B(x).\nu_B(x) + \overline{sg}(1-\mu_B(x))),$ $\min(\mu_A(x).\nu_A(x) + \overline{sg}(1-\mu_A(x)), \nu_B(x).(\mu_B(x).\nu_B(x)$ $+\overline{sg}(1-\mu_B(x)))) + \overline{sg}(1-\nu_B(x)))))\rangle \| x \in E\}$
$\rightarrow_{112}$	$\{\langle x, \nu_A(x) + \mu_B(x) - \nu_A(x).\mu_B(x),$ $\mu_A(x).\nu_A(x) + \overline{sg}(1-\mu_A(x)).\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_{113}$	$\{\langle x, \nu_A(x) + (\mu_B(x).\nu_B(x)) - \nu_A(x).$ $(\mu_B(x).\nu_B(x) + \overline{sg}(1-\mu_B(x))),$ $(\mu_A(x).\nu_A(x) + \overline{sg}(1-\mu_A(x))).(\nu_B(x).(\mu_B(x).\nu_B(x)$ $+\overline{sg}(1-\mu_B(x))) + \overline{sg}(1-\nu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{114}$	$\{\langle x, ((1-\mu_A(x)) + \min(\overline{sg}(1-\mu_A(x)), (1-\nu_B(x)))),$ $(\mu_A(x).(1-\mu_A(x)) + \min(\overline{sg}(1-\mu_A(x)), \nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{115}$	$\{\langle x, 1 - \min(\mu_A(x), \nu_B(x)),$ $\min((\mu_A(x).(1-\mu_A(x)) + \overline{sg}(1-\mu_A(x))), \nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{116}$	$\{\langle x, \max(1-\mu_A(x), (1-\nu_B(x)).\nu_B(x) + \overline{sg}(\nu_B(x)),$ $\min(\mu_A(x).(1-\mu_A(x)) + \overline{sg}(1-\mu_A(x)),$ $\nu_B(x).((1-\nu_B(x)).\nu_B(x)$ $+\overline{sg}(\nu_B(x))) + \overline{sg}(1-\nu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{117}$	$\{\langle x, 1 - \mu_A(x).\nu_B(x)$ $(\mu_A(x).(1-\mu_A(x)) + \overline{sg}(1-\mu_A(x))).\nu_B(x)\rangle \| x \in E\}$
$\rightarrow_{118}$	$\{\langle x, (1-\mu_A(x)).\mathrm{sg}(\nu_B(x)) + \mu_A(x).\nu_B(x).(1-\nu_B(x)),$ $(\mu_A(x) - \mu_A(x)^2 + \overline{sg}(1-\mu_A(x))).((1-\nu_B(x)).\nu_B(x)^2$ $+\overline{sg}(1-\nu_B(x))) + \overline{sg}(1-\nu_B(x))\rangle \| x \in E\}$
$\rightarrow_{119}$	$\{\langle x, \nu_A(x) + \min(\overline{sg}(\nu_A(x)), \mu_B(x)),$ $(1-\nu_A(x)).\nu_A(x) + \min(\overline{sg}(\nu_A(x)), 1-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{120}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)),$ $\min((1-\nu_A(x)).\nu_A(x) + \overline{sg}(\nu_A(x)), 1-\mu_B(x))\rangle \| x \in E\}$

$\rightarrow_{121}$	$\{\langle x, \max(\nu_A(x), (\mu_B(x).(1-\mu_B(x)) + \overline{sg}(1-\mu_B(x)))),$ $\min((((1-\nu_A(x)).\nu_A(x)) + \overline{sg}(\nu_A(x))),$ $(((1-\mu_B(x)).((\mu_B(x).$ $(1-\mu_B(x))) + \overline{sg}(1-\mu_B(x)))) + \overline{sg}(\mu_B(x))))\rangle \| x \in E\}$
$\rightarrow_{122}$	$\{\langle x, \nu_A(x) + \mu_B(x) - \nu_A(x).\mu_B(x),$ $((1-\nu_A(x)).\nu_A(x) + \overline{sg}(\nu_A(x))).(1-\mu_B(x))\rangle \| x \in E\}$
$\rightarrow_{123}$	$\{\langle x, (\nu_A(x) + \mu_B(x).(1-\mu_B(x)) - \nu_A(x).$ $(\mu_B(x).(1-\mu_B(x)) + \overline{sg}(1-\mu_B(x))))$ $(((1-\nu_A(x)).\nu_A(x) + \overline{sg}(\nu_(x))).((1-\mu_B(x)).(\mu_B(x)$ $.(1-\mu_B(x)) + \overline{sg}(1-\mu_B(x)))) + \overline{sg}(\mu_B(x)))\rangle \| x \in E\}$
$\rightarrow_{124}$	$\{\langle x, \mu_B(x) + \min(\overline{sg}(1-\nu_B(x)), \nu_A(x)),$ $\nu_B(x).\mu_B(x) + \min(\overline{sg}(1-\nu_B(x)), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{125}$	$\{\langle x, \max(\mu_B(x), \nu_A(x)),$ $\min(\nu_B(x).\mu_B(x) + \overline{sg}(1-\nu_B(x)), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{126}$	$\{\langle x, \max(\mu_B(x), \nu_A(x).\mu_A(x) + \overline{sg}(1-\nu_A(x))),$ $\min(\nu_B(x).\mu_B(x) + \overline{sg}(1-\nu_B(x)), \mu_A(x).$ $(\nu_A(x).\mu_A(x) + \overline{sg}(1-\nu_A(x))) + \overline{sg}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{127}$	$\{\langle x, \mu_B(x) + \nu_A(x) - \mu_B(x).\nu_A(x),$ $(\nu_B(x).\mu_B(x) + \overline{sg}(1-\nu_B(x))).\mu_A(x)\rangle \| x \in E\}$
$\rightarrow_{128}$	$\{\langle x, \mu_B(x) + \nu_A(x).\mu_A(x) - \mu_B(x).$ $(\nu_A(x).\mu_A(x) + \overline{sg}(1-\nu_A(x))),$ $(\nu_B(x).\mu_B(x) + \overline{sg}(1-\nu_B(x))).(\mu_A(x).(\nu_A(x).\mu_A(x)$ $+\overline{sg}(1-\nu_A(x))) + \overline{sg}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{129}$	$\{\langle x, 1-\nu_B(x) + \min(\overline{sg}(1-\nu_B(x)), 1-\mu_A(x))),$ $\nu_B(x).(1-\nu_B(x)) + \min(\overline{sg}(1-\nu_B(x)), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{130}$	$\{\langle x, 1-\min(\nu_B(x), \mu_A(x)),$ $\min(\nu_B(x).(1-\nu_B(x)) + \overline{sg}(1-\nu_B(x)), \mu_A(x))\rangle \| x \in E\}$
$\rightarrow_{131}$	$\{\langle x, \max(1-\nu_B(x), (1-\mu_A(x)).\mu_A(x) + \overline{sg}(\mu_A(x))),$ $\min(\nu_B(x).(1-\nu_B(x)) + \overline{sg}(1-\nu_B(x)), \mu_A(x).((1-\mu_A(x))$ $.\mu_A(x) + \overline{sg}(\mu_A(x))) + \overline{sg}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{132}$	$\{\langle x, 1-\mu_A(x).\nu_B(x),$ $(\nu_B(x).(1-\nu_B(x)) + \overline{sg}(1-\nu_B(x))).\mu_A(x)\rangle \| x \in E\}$
$\rightarrow_{133}$	$\{\langle x, ((1-\nu_B(x) + ((1-\mu_A(x)).\mu_A(x)))$ $-((1-\nu_B(x)).(((1-\mu_A(x)).\mu_A(x)) + \overline{sg}(\mu_A(x))))),$ $(\nu_B(x).(1-\nu_B(x)) + \overline{sg}(1-\nu_B(x))).(\mu_A(x).((1-\mu_A(x))$ $.\mu_A(x) + \overline{sg}(\mu_A(x))) + \overline{sg}(1-\mu_A(x)))\rangle \| x \in E\}$
$\rightarrow_{134}$	$\{\langle x, \mu_B(x) + \min(\overline{sg}(\mu_B(x)), \nu_A(x)),$ $(1-\mu_B(x)).\mu_B(x) + \min(\overline{sg}(\mu_B(x)), 1-\nu_A(x))\rangle \| x \in E\}$

$\rightarrow_{135}$	$\{\langle x, \max(\mu_B(x), \nu_A(x)),$ $\min((1-\mu_B(x)).\mu_B(x)+\overline{sg}(\mu_B(x)), 1-\nu_A(x))\rangle \vert x \in E\}$
$\rightarrow_{136}$	$\{\langle x, \max(\mu_B(x), \nu_A(x).(1-\nu_A(x))+\overline{sg}(1-\nu_A(x))),$ $\min((1-\mu_B(x)).\mu_B(x)+\overline{sg}(\mu_B(x)), (1-\nu_A(x))$ $.(\nu_A(x).(1-\nu_A(x))+\overline{sg}(1-\nu_A(x)))+\overline{sg}(\nu_A(x)))\rangle \vert x \in E\}$
$\rightarrow_{137}$	$\{\langle x, \mu_B(x)+\nu_A(x)-\mu_B(x).\nu_A(x),$ $((1-\mu_B(x)).\mu_B(x)+\overline{sg}(\mu_B(x))).(1-\nu_A(x))\rangle \vert x \in E\}$
$\rightarrow_{138}$	$\{\langle x, \mu_B(x)+\nu_A(x).(1-\nu_A(x))$ $-\mu_B(x).(\nu_A(x).(1-\nu_A(x))+\overline{sg}(1-\nu_A(x))),$ $((1-\mu_B(x)).\mu_B(x)+\overline{sg}(\mu_B(x))).((1-\nu_A(x)).(\nu_A(x)$ $.(1-\nu_A(x))+\overline{sg}(1-\nu_A(x))+\overline{sg}(\nu_A(x)))\rangle \vert x \in E\}$

are CIFSs for which

$$range(A \rightarrow_{15} B) = range(A \rightarrow_{20} B) = range(A \rightarrow_{23} B)$$
$$= range(A \rightarrow_{24} B) = range(A \rightarrow_{40} B) = range(A \rightarrow_{42} B)$$
$$= range(A \rightarrow_{55} B) = range(A \rightarrow_{57} B) = range(A \rightarrow_{74} B)$$
$$= range(A \rightarrow_{79} B) = range(A \rightarrow_{88} B) = range(A \rightarrow_{97} B) = 1;$$

$$range(A \rightarrow_{29} B) = range(A \rightarrow_{45} B) = range(A \rightarrow_{60} B)$$
$$= range(A \rightarrow_{62} B) = range(A \rightarrow_{63} B) = range(A \rightarrow_{68} B)$$
$$= range(A \rightarrow_{69} B) = range(A \rightarrow_{70} B) = range(A \rightarrow_{73} B)$$
$$= range(A \rightarrow_{78} B) = range(A \rightarrow_{80} B) = range(A \rightarrow_{83} B)$$
$$= range(A \rightarrow_{84} B) = range(A \rightarrow_{87} B) = range(A \rightarrow_{89} B)$$
$$= range(A \rightarrow_{92} B) = range(A \rightarrow_{93} B) = range(A \rightarrow_{96} B)$$
$$= range(A \rightarrow_{98} B) = range(A);$$

$$range(A \rightarrow_{2} B) = range(A \rightarrow_{3} B) = range(A \rightarrow_{11} B)$$
$$= range(A \rightarrow_{14} B) = range(A \rightarrow_{16} B) = range(A \rightarrow_{19} B)$$
$$= range(A \rightarrow_{26} B) = range(A \rightarrow_{27} B) = range(A \rightarrow_{31} B)$$
$$= range(A \rightarrow_{32} B) = range(A \rightarrow_{41} B) = range(A \rightarrow_{43} B)$$
$$= range(A \rightarrow_{47} B) = range(A \rightarrow_{48} B) = range(A \rightarrow_{54} B)$$

$$= range(A \rightarrow_{56} B) = range(A \rightarrow_{77} B) = range(A \rightarrow_{81} B)$$
$$= range(A \rightarrow_{90} B) = range(A \rightarrow_{99} B) = range(B);$$

$$range(A \rightarrow_{1} B) = range(A \rightarrow_{4} B) = range(A \rightarrow_{5} B)$$
$$= range(A \rightarrow_{8} B) = range(A \rightarrow_{10} B) = range(A \rightarrow_{12} B)$$
$$= range(A \rightarrow_{18} B) = range(A \rightarrow_{22} B) = range(A \rightarrow_{25} B)$$
$$= range(A \rightarrow_{28} B) = range(A \rightarrow_{30} B) = range(A \rightarrow_{33} B)$$
$$= range(A \rightarrow_{34} B) = range(A \rightarrow_{35} B) = range(A \rightarrow_{36} B)$$
$$= range(A \rightarrow_{37} B) = range(A \rightarrow_{39} B) = range(A \rightarrow_{44} B)$$
$$= range(A \rightarrow_{46} B) = range(A \rightarrow_{49} B) = range(A \rightarrow_{51} B)$$
$$= range(A \rightarrow_{52} B) = range(A \rightarrow_{58} B) = range(A \rightarrow_{59} B)$$
$$= range(A \rightarrow_{61} B) = range(A \rightarrow_{65} B) = range(A \rightarrow_{67} B)$$
$$= range(A \rightarrow_{72} B) = range(A \rightarrow_{76} B) = range(A \rightarrow_{82} B)$$
$$= range(A \rightarrow_{86} B) = range(A \rightarrow_{91} B) = range(A \rightarrow_{95} B)$$
$$= range(A \rightarrow_{100} B) = range(A \rightarrow_{101} B) = range(A \rightarrow_{102} B)$$
$$= range(A \rightarrow_{103} B) = range(A \rightarrow_{104} B) = range(A \rightarrow_{105} B)$$
$$= range(A \rightarrow_{106} B) = range(A \rightarrow_{107} B) = range(A \rightarrow_{108} B)$$
$$= range(A \rightarrow_{109} B) = range(A \rightarrow_{110} B) = LCM(range(A), range(B)).$$

$$range(A \rightarrow_{6} B) = range(A \rightarrow_{13} B) = range(A \rightarrow_{50} B)$$
$$= range(A \rightarrow_{64} B) = range(A \rightarrow_{112} B) = range(A).range(B);$$

$$range(A \rightarrow_{66} B) = range(A \rightarrow_{85} B) = range(A).range(B)^2;$$

$$range(A \rightarrow_{9} B) = range(A \rightarrow_{17} B) = range(A \rightarrow_{38} B)$$
$$= range(A \rightarrow_{53} B) = range(A)^2.range(B);$$

$$range(A \rightarrow_{71} B) = range(A \rightarrow_{94} B) = range(A \rightarrow_{119} B)$$

$$= range(A \to_{124} B) = range(A \to_{125} B) = range(A \to_{127} B)$$
$$= range(A \to_{129} B) = range(A \to_{130} B) = range(A \to_{132} B)$$
$$= range(A \to_{134} B) = range(A \to_{135} B) = range(A \to_{137} B)$$
$$= LCM(range(A), range(B)^2);$$

$$range(A \to_{114} B) = range(A \to_{115} B) = range(A \to_{117} B)$$
$$= range(A \to_{120} B) = range(A \to_{122} B) = LCM(range(A)^2, range(B));$$

$$range(A \to_{21} B) = range(A \to_{111} B) = range(A \to_{113} B)$$
$$= range(A \to_{116} B) = range(A \to_{118} B) = range(A \to_{121} B)$$
$$= range(A \to_{123} B) = LCM(range(A)^2, range(B)^3);$$

$$range(A \to_{75} B) = range(A \to_{126} B) = range(A \to_{128} B)$$
$$= range(A \to_{131} B) = range(A \to_{133} B) = range(A \to_{136} B)$$
$$= range(A \to_{138} B) = LCM(range(A)^3, range(B)^2).$$

In the same way we can check that all other operators, defined over CIFSs and having as parameters rational numbers, are also CIFSs.

Therefore, all operations and operators, applied over CIFSs, as a result keep the property of constructiveness.

When universe E is a finite set, its elements can be enumerated by some rational numbers in interval $[0, 1]$ and now, we can apply the results from [7, 8, 9].

3 On the Intuitionistic Fuzzy Degrees of Provability

In [6] Pyotr Novikov introduced the concept for provable assertion (D-assertion), following an idea of Gödel, published in [3]. On logical level we can assert that a given assertion is provable or unprovable. Now, from IF-point of view we can speak about degree of provability of a given assertion, degree of provability of the contrary assertion and for degree of unprovability. These three estimations will be calculated constructively in all cases, whenever the assertion is proved constructively.

We will give the following example that uses Dirichlet's principle. Let us have m boxes and n balls placed in them. Let us assertion be:
$T(m,n,s) =$ "there exists at least one box that contains at least s balls."

We can see easy that the IF-estimation function for this assertion will give the pair of values

$$V(T(m,n,s)) = \langle \frac{n - m(s-1)}{n+1}, \frac{s}{n+1} \rangle.$$

Obviously, number $\frac{n-m(s-1)}{n+1}$ corresponds to the number of cases for which the assertion is valid, number $\frac{s}{n+1}$ corresponds to the number of cases for which the assertion is not valid, and

$$n + 1 - (n - m(s-1) + s) = ms - m - s + 1$$

is the number of cases, when the assertion is uncertain.

Therefore, we can define the intuitionistic fuzzy degree of provability.

For example, if we have $m = 10$ boxes and $n = 0, 1$ or 2 balls, we can prove that there is no box containing $s = 3$ or more balls. If we have $n = 21, 22, ...$ balls, we can prove that we always have at least one box containing 3 balls. When we have $n = 3, 4, ..., 20$ balls, we cannot prove that there exists at least one box with 3 balls – this assertion may be true, but it may also be false. Hence, we are in a situation of uncertainly.

All this is valid in the particular case for fuzzy estimations.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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